Ridesharing options for secondary payloads to be delivered to regions beyond geostationary altitude are increasingly available with propulsive EELV Secondary Payload Adapter (ESPA) rings. However, mission design for secondary payloads faces certain challenges. A significant mission constraint for a secondary payload is the dropoff orbit orientation, as it is dependent on the primary mission. In this analysis, assume that the dropoff orbit is an Earth-centered Geosynchronous Transfer Orbit (GTO). Then, efficient transfers to orbits near the Sun-Earth L1 Lagrange point are constructed from a range of GTO orientations. Dynamical structures, such as stable invariant manifolds, associated with periodic and quasi-periodic orbits near Sun-Earth L1 are leveraged to identify and summarize types of transfer opportunities.

INTRODUCTION

Ridesharing opportunities for smallsats offer increased mission capabilities with lower launch costs. Over the past two decades, an increasing number of smallsats, especially cubesats, have been launched into Low Earth Orbit (LEO) or Geosynchronous Orbit (GEO) by exploiting ridesharing opportunities. Additionally, the potential to place smallsats into regions beyond GEO is facilitated by the introduction of propulsive Evolved Expendable Launch Vehicle (EELV) Secondary Payload Adapter (ESPA) rings. In a ridesharing configuration, an ESPA ring is utilized to mount a series of smallsats alongside the primary payload inside the launch vehicle. In this layout, the ESPA ring essentially holds and releases one or more secondary payloads, i.e., the smallsats, at some point along the trajectory. Propulsive ESPA rings carry an independent propulsion system that provides the necessary energy to place a secondary payload into regions beyond the primary payload orbit. The Lunar Crater and Observation and Sensing Satellite (LCROSS) spacecraft was a secondary payload, launched alongside the Lunar Reconnaissance Orbiter (LRO), and leveraged a propulsive ESPA ring to impact the Moon.1 Propulsive ESPA rings allow secondary payloads access to regions beyond GEO, however, variable orbit geometries and shifting launch dates present significant trajectory design challenges. In a ridesharing scenario, a primary payload is released from a launch vehicle into a dropoff orbit, also denoted as the initial secondary payload orbit, designed to meet the primary mission requirements. The geometry and orbital orientation of the dropoff orbit presents trajectory design challenges for secondary payloads enroute to other specified destinations.

There are nearly twenty-five yearly launches to GEO that provide opportunities for ridesharing smallsats. Missions to GEO utilize a Geosynchronous Transfer Orbit (GTO) to reach the desired 35,786 km altitude. A GTO is defined as an eccentric orbit, within the context of the two-body problem, with a fixed apoapsis altitude at the GEO altitude. A satellite enroute to GEO is released from a GTO, the dropoff orbit in this investigation, from a launch vehicle and a series of maneuvers near apoapsis allows entry into a circular orbit at GEO. In a ridesharing scenario, a secondary payload is released along a GTO and, depending on the propulsive capabilities of the payload, may access regions beyond GEO. Gershman et al. investigates...
transfers from GTO to Mars and Venus with a lunar flyby.\textsuperscript{2} Fujiwara et al. examines transfers to the Moon for a spacecraft placed in a GTO with a hybrid rocket kick motor.\textsuperscript{3} Additionally, Stender et al. presents approximate maneuver magnitudes for mission to the Sun-Earth L\textsubscript{1} vicinity as well as for Low Lunar Orbits (LLO).\textsuperscript{4} In this investigation, the propulsive ESPA rings provide the necessary energy to place a secondary payload, i.e., smallsat, beyond GEO and toward Lagrange point orbits.

The Sun-Earth collinear Lagrange points are regions of scientific interest for secondary payloads. Sun-Earth L\textsubscript{1}, located between the Sun and the Earth, is an ideal region to investigate the Solar environment while also offering favorable thermal conditions, eclipse avoidance, and continuous communications with the Earth. However, transfers to the Sun-Earth L\textsubscript{1} region must avoid communications interference caused by the Sun, defined as a Solar Exclusion Zone (SEZ) region.\textsuperscript{5} The International Sun-Earth Explorer-3 (ISEE-3), later renamed the International Cometary Explorer (ICE), was the first orbiter placed near L\textsubscript{1}.\textsuperscript{6} The success of the ICE mission led to the following L\textsubscript{1} orbiters: the Advanced Composition Explorer (ACE), the Solar Heliospheric Observatory (SOHO), and the International Physics Laboratory (WIND).\textsuperscript{5} In this investigation, transfers from a GTO to the Sun-Earth L\textsubscript{1} vicinity are constructed for smallsats leveraging a propulsive ESPA ring.

BACKGROUND

Preliminary transfer design to Sun-Earth Lagrange points is facilitated by leveraging the motion within the Circular Restricted Three Body Model (CRTBP)\textsuperscript{7} model. The motion in the CRTBP can be generally categorized into four types: equilibrium points, periodic solutions, quasi-periodic solutions, and chaotic.\textsuperscript{8} Periodic orbits and quasi-periodic orbits near Lagrange points, i.e., equilibrium points in the CRTBP, and dynamical structures associated with these orbits, such as hyperbolic invariant manifolds, are leveraged to construct efficient and flexible transfers to Sun-Earth Lagrange points.

Dynamical Model

Insightful flow information observed in the CRTBP model is analyzed via methods from Dynamical Systems Theory (DST). The CRTBP model describes the motion of a spacecraft, P\textsubscript{3}, with negligible mass, m\textsubscript{3}, subject to the gravitational force of two larger bodies, P\textsubscript{1} and P\textsubscript{2}, with masses m\textsubscript{1} and m\textsubscript{2}, respectively. Additionally, P\textsubscript{1} and P\textsubscript{2} are assumed to be in a circular orbit about their system barycenter, O. The system is non-dimensionalized with a characteristic length, l\textsuperscript{*}, defined as the constant distance between the primary bodies, P\textsubscript{1} and P\textsubscript{2}, and a characteristic time, t\textsuperscript{*}, defined as: t\textsuperscript{*} = \sqrt{\frac{(l\textsuperscript{*})\textsuperscript{3}}{G(m\textsubscript{1}+m\textsubscript{2})}}, where G is the universal gravitational constant. The non-dimensional scalar equations of motion for the CRTBP are written as,

\[ \dot{x} - 2\dot{y} = \frac{\partial U^*}{\partial x}, \quad \dot{y} + 2\dot{x} = \frac{\partial U^*}{\partial y}, \quad \ddot{z} = \frac{\partial U^*}{\partial z}, \]  

(1)

where \( \partial \) indicates a partial derivative, \( \mu \) is the mass parameter of the system defined as \( \mu = \frac{m_2}{m_1+m_2} \), and \( U^* \) is the pseudo-potential function denoted as,

\[ U^* = \frac{1-\mu}{r_{13}} + \frac{\mu}{r_{23}} + \frac{1}{2}(x^2 + y^2). \]  

(2)

In Equation (2), the distances between the spacecraft and P\textsubscript{1} and P\textsubscript{2}, i.e., \( r_{13} \) and \( r_{23} \), respectively, are defined as:

\[ r_{13} = \sqrt{(x + \mu)^2 + y^2 + z^2} \quad \text{and} \quad r_{23} = \sqrt{(x - 1 + \mu)^2 + y^2 + z^2}. \]

Note that the reference frame of the system is a rotating frame with basis unit vectors \( \{\hat{x}, \hat{y}, \hat{z}\} \) defined such that \( \hat{x} \) is directed from P\textsubscript{1} towards P\textsubscript{2}, \( \hat{z} \) is in the direction of the angular momentum vector for the P\textsubscript{1}-P\textsubscript{2} circular orbit, and \( \hat{y} = \hat{z} \times \hat{x} \). The state of the spacecraft is denoted as \( \vec{z} = [x, y, z, \dot{x}, \dot{y}, \dot{z}]^T \) where the superscript, \( ^T \), indicates a matrix transpose. Additionally, the position and velocity vectors for P\textsubscript{3}, in the rotating frame, are defined as \( \vec{r} = [x, y, z]^T \) and \( \vec{v} = [\dot{x}, \dot{y}, \dot{z}]^T \), respectively. The non-dimensional positions of the primary bodies P\textsubscript{1} and P\textsubscript{2} are fixed at \( \vec{r}_1 = [-\mu, 0, 0]^T \) and \( \vec{r}_2 = [1 - \mu, 0, 0]^T \), respectively. Five equilibrium solutions (Lagrange points) exist in the \( \hat{x}\hat{y} \) plane of the rotating frame of the CRTBP model. Three collinear Lagrange points, \( L_{1-3} \), lie on the \( \hat{x} \) axis and two triangular Lagrange points, \( L_{4-5} \), lie on vertices of an equilateral triangle created with \( \vec{r}_1 \)
An important insight from the CRTBP is the existence of an integral of the motion, termed the Jacobi Constant. The Jacobi Constant is evaluated as $C = 2U^* - v^2$, where $v = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$ is the rotating speed of the spacecraft. Insights from the CRTBP facilitate the search for efficient transfers in the Sun-Earth system.

**Periodic Orbits**

Periodic motion near the Lagrange points facilitates the construction of feasible transfers in the Sun-Earth system. Periodic orbits in the CRTBP exist in one parameter families with planar Lyapunov orbits, vertical orbits, and spatial out-of-plane halo orbits, which emanate from Lyapunov orbit families, associated with the collinear Lagrange points, $L_{1-3}$. Additionally, maps, in particular stroboscopic maps, are powerful tools for assessing the stability characteristics associated with periodic orbits. A stroboscopic map transforms a continuous time flow into a discrete time system at constant time intervals, or orbital periods for examination of stability analysis of a periodic orbit. A periodic orbit on a stroboscopic map, taken at time intervals equal to the orbit period, $T$, appears as a single point, $\bar{x}^*$, termed a fixed point. Linear stability analysis in the vicinity of this fixed point reveals insightful observations concerning the flow near the periodic orbit. A monodromy matrix, $\Phi(T, 0)$, is defined as a State Transition Matrix (STM) that relates the downstream variations in the state to the initial state variations after one orbit period. The stability characteristics associated with the periodic orbit are observed through the eigenvalues and eigenvectors of the monodromy matrix. The eigenvalues, $\lambda$, associated with periodic orbits in the CRTBP occur in reciprocal pairs, where the underline refers to eigenvalues corresponding to periodic orbits. A pair of eigenvalues are equal to unity and the remaining eigenvalues describe the local flow near the fixed point, $\bar{x}^*$, corresponding to a periodic orbit. The existence of a center manifold associated with a periodic orbit is revealed by identifying complex eigenvalues on the unit circle, i.e., $|\lambda_1| = 1$. Complementary real eigenvalues reveal the existence of stable and unstable manifold structures defined with $|\lambda_1| < 1$ and $|\lambda_1| > 1$, respectively. Trajectories along the surface of the global representations of the stable and unstable manifold structures are frequently leveraged in preliminary mission design and serve as low-cost options that asymptotically approach, along the stable manifold, and depart, along the unstable manifold, a periodic orbit. Additionally, the existence of quasi-periodic orbits within the center manifold of periodic orbits offers more trajectory options to construct flexible transfers.

**Quasi-Periodic Orbits**

Quasi-periodic motion observed in the vicinity of the Sun-Earth Lagrange points offer complex geometries that facilitate the construction of efficient transfers for a range of mission design objectives. Dynamical structures associated with periodic orbits are regularly exploited, but the behavior associated with quasi-periodic orbits offers additional transfer opportunities and geometries. Several authors have offered strategies to generate quasi-periodic orbits in the CRTBP. Olikara et al. categorize equilibrium points, periodic orbits, and quasi-periodic orbits as $p$-dimensional invariant tori. The equilibrium points, i.e., Lagrange points, are represented as 0-dimensional tori, periodic orbits as 1-dimensional tori, and quasi-periodic orbit as $q$-dimensional tori with $q \geq 2$. The flow for a quasi-periodic orbit densely covers the surface of the $q$-dimensional torus after infinite time. In this investigation, quasi-periodic motion is represented as flow on the surface of a $q$-dimensional torus, with $q = 2$, and a quasi-periodic orbit is constructed via a numerical corrections scheme. The method for generating families of quasi-periodic orbits, in this analysis, originates with Gómez and Mondelo and includes enhancements by Olikara and Scheeres. A two-dimensional torus is characterized by two fundamental frequencies, $\theta_1$ and $\theta_2$, and a six-dimensional state vector on the surface of the torus, $\bar{u}$, is parametrized by the angles, $\theta_1$ and $\theta_2$, such that $\bar{u}(\theta_1, \theta_2)$. In this analysis, $\theta_1$ and $\theta_2$ are termed as the longitudinal and latitudinal frequencies, respectively. The angles, $\theta_i$, are defined such that $\theta_i(t) = \dot{\theta}_i t$, where $t$ is the propagation time along the torus. From the invariant tori definition, quasi-periodic motion is observed for tori with irrational frequency ratios, that is $\frac{\theta_1}{\theta_2}$, and periodic motion is observed for tori when the frequency ratio is rational.

To numerically construct a quasi-periodic orbit, the foundation is a periodic orbit in the CRTBP Sun-Earth system represented in terms of a fixed point, $\bar{x}^*$. A stroboscopic map, generated with time $T_1$, is leveraged
to reduce the construction of the two-dimensional torus, one that characterizes the quasi-periodic orbit, to a search for an invariant curve. The periodic time for the stroboscopic map, i.e., $T_1$, is associated with the period of the longitudinal frequency, $\theta_1$, and defined as $\theta_1 = \frac{2\pi}{T_1}$. An initial state, defined as $\bar{u}(\theta_1(0), \theta_2(0))$, is propagated along the torus and the returns to the stroboscopic map represent the invariant curve associated with a specific quasi-periodic orbit. Additionally, a state on the surface of the two-dimensional torus, as formulated in the rotating frame of the Sun-Earth CRTBP, is defined with $\bar{u}(\theta_1(T_1), \theta_2(T_1))$, on the torus back to its initial state, i.e.,

$$R[-\rho] \left[ \bar{u}(\theta_1(T_1), \theta_2(T_1)) \right] = \left[ \bar{u}(\theta_1(0), \theta_2(0)) \right] = 0,$$

where $[\bar{u}]$ represents a row vector for the torus state. For the stroboscopic map, $\theta_1(0) = \theta_1(T_1)$, therefore, the first return to the map occurs at $\theta_1(0)$ and the state of the first return onto the stroboscopic map, i.e., the final state in Equation (3), is parametrized as $\bar{u}(\theta_1(0), \theta_2(0) + \rho)$, where $\rho$ represents a rotation angle along the invariant curve. The invariant curve, identified via implementation of the invariance condition in Equation (3), is discretized into $n$ torus states $\bar{u}_j$ with $j = [1, ..., n]$ by using a truncated Fourier series. Note that the torus states approximate the geometry of the invariant curve associated with a specific torus and the value of $n$ affects the accuracy of the numerical algorithm for quasi-periodic orbits. The torus states along the invariant curve are defined as $\bar{u}_j(\theta_1(0), \theta_2(0) + \rho)$, where $\theta_2(0) \in [0, 2\pi]$. Note that the torus states, $\bar{u}_j$, are six-dimensional vectors and the rotation matrix, $R[-\rho]$, is a $n \times n$ real matrix evaluated with elements of the Fourier series and the rotation angle $(-\rho)$.

Quasi-periodic orbits are numerically constructed by implementing a multiple-shooting strategy along with the invariance condition stated in Equation (3). In a multiple-shooting strategy, a single trajectory is divided into a series of smaller arcs which, in a numerical corrections process, facilitate the construction of a trajectory in a complex dynamical regime. Recall that the stroboscopic map that identifies the invariant curve is constructed at sequential times such that each interval is $T_1$. In a multiple-shooting scheme for a torus state propagated along the invariant curve, i.e., $\bar{u}_j(\theta_1(0), \theta_2(0))$. The trajectory is subdivided into $p$ segments, each with an associated propagation time $\frac{T_1}{p}$. The quasi-periodic orbit, represented by a two-dimensional torus, is now characterized by torus states, $\bar{u}_j^k$, where $k = [1, ..., p]$. The invariance condition, Equation (3), is evaluated with the initial states along the invariant curve, $\bar{u}_j^k(\theta_1(0), \theta_2(0))$ as well as the final propagated state, i.e., the state $\bar{u}_j^p(0, \theta_2(0))$ propagated by time $\frac{T_1}{p}$. Full state continuity is enforced throughout the trajectory segments. Initial conditions for the corrections process are included in Olikara. Note that a torus state along the invariant circle, $\bar{u}_j^1$, is expressed in terms of the phase-space, consistent with the CRTBP via $\bar{x}_j^Q = \bar{u}_j^1 + \bar{x}^*$, where, it is recalled that, $\bar{x}^*$ is a fixed point associated with a periodic orbit. Families of quasi-periodic orbits are constructed by enforcing constraints on the Jacobi Constant, $C$, the rotation angle, $\rho$, or the mapping time associated with the stroboscopic map, $T_1$. In this investigation, quasi-periodic orbit families at a fixed $C$ value are constructed via pseudo-arclength continuation. The approximation of the invariant curve via a truncated Fourier series is not unique so additional phasing constraints are included in the continuation process. Regions with large resonances may challenge the convergence of the multiple-shooting algorithm, especially when generating families with a fixed $C$ value, but anticipating these regions mitigates the difficulties associated with this numerical algorithm.

**Stability of Quasi-Periodic Orbits** Linear stability analysis offers observations concerning the local behavior of the flow in the vicinity of a quasi-periodic orbits. The stability properties near a fixed point associated with a periodic orbit describe the local dynamical flow. Essentially, a stroboscopic map is constructed with the period corresponding to the orbit. Similarly, the stability properties for quasi-periodic orbits are observed via a linearized stroboscopic map for the invariant curve, i.e.,

$$DP = \left( R[-\rho] \otimes I \right) \Theta,$$

where $\Theta$ represents a row vector for the invariant curve.
where $I$ is a $6 \times 6$ identity matrix, $\otimes$ is the kronecker product, and $\overline{\Phi}$ is a matrix, of size $6n \times 6n$, that includes the STM, $\Phi_j(T_1, 0)$, from the torus states, $\bar{u}_j$, along the invariant curve, such that,

$$
\overline{\Phi} = \begin{bmatrix}
\Phi_1(T_1, 0) & 0_{6\times6} & \cdots & 0_{6\times6} \\
0_{6\times6} & \Phi_2(T_1, 0) & \cdots & 0_{6\times6} \\
\vdots & \vdots & \ddots & \vdots \\
0_{6\times6} & 0_{6\times6} & \cdots & \Phi_n(T_1, 0)
\end{bmatrix},
$$

(5)

where $0_{6\times6}$ denotes a $6 \times 6$ zero matrix. The eigenvalues, $\Lambda$, of the linearized map provide the stability characteristics for the local behavior in the vicinity of a quasi-periodic orbit. The existence of a stable manifold subspace is defined via the eigenvalues with magnitudes less than unity, i.e., $\|\Lambda\| < 1$, and the existence of an unstable manifold subspace is identified via $\|\Lambda\| > 1$. The eigenvalues for each invariant subspace, either stable or unstable, occur in concentric circles in the complex plane. The flow characterized by the stable and unstable subspaces provides opportunities to construct efficient transfers into and away from specific quasi-periodic orbits.

**Invariant Manifolds for the Quasi-Periodic Orbit** Trajectories that exist within the stable subspace associated with a quasi-periodic orbit offer opportunities to construct low-cost transfers to the vicinity of Sun-Earth $L_1$. An infinite number of trajectories densely fill the surface of the hyperbolic stable manifold and, inherently, these trajectories asymptotically approach the quasi-periodic orbit in reverse time. The global representation of the stable manifold surface associated with a quasi-periodic orbit leverages an eigenvector, $\bar{\Psi}_s$, which corresponds to a real eigenvalue, $\Lambda_s$, and the stable manifold subspace, with a zero imaginary element, such that, $\text{Im}(\Lambda_s) = 0$. Recalling that eigenvalues occur in concentric circles, an eigenvalue, $\Lambda_s$, with only real parts possess a corresponding eigenvector, $\bar{\Psi}_s$, with real components. The eigenvector, $\bar{\Psi}_s$, is a vector of size $6n$ and is comprised of a set of sub-eigenvectors $\{\bar{\psi}_j^S\}$ that correspond to torus states $\bar{u}_j$ along the invariant curve. A state on the stable manifold, $\bar{x}_j^S$, is approximated as,

$$
\bar{x}_j^S = \bar{x}_j^Q + d\frac{\bar{\psi}_j^S}{\|\bar{\psi}_j^S\|},
$$

(6)

where $\bar{\psi}_j^S$ is the associated six-dimensional stable eigenvector for $\bar{u}_j$ also written as, $\bar{\psi}_j^S = [\bar{\psi}_{r_j}^S, \bar{\psi}_{v_j}^S]^T$. The three-dimensional vectors $\bar{\psi}_{r_j}^S$ and $\bar{\psi}_{v_j}^S$ are defined such that $\bar{\psi}_{r_j}^S$ corresponds to the elements associated with position and $\bar{\psi}_{v_j}^S$ isolates the velocity elements. The eigenvectors along the full torus are evaluated from the STM, $\Phi(T_1, 0)$.

The global representation of the invariant stable manifold is approximated via propagation in reverse time from the computed states in Equation (6) at a set of points around the two-dimensional torus.

**DROPOFF ORBIT ORIENTATION IN THE ROTATING FRAME**

The dropoff orbit orientation presents a significant design challenge for secondary payloads. A Geosynchronous Transfer Orbit is an eccentric periodic orbit, within the context of the two-body problem, that transports a primary payload to the GEO altitude. In this analysis, the GTO, i.e., the dropoff orbit, is the departure orbit and the arrival orbit is a desired periodic or quasi-periodic orbit near the Sun-Earth $L_1$ point. The size of the GTO is fixed, that is, the apoapsis altitude is fixed at the GEO altitude of 35,786 km, and the periapsis altitude is 185 km. Note that the periapsis altitude for this analysis is a standard GTO periapsis altitude defined by the United Launch Alliance (ULA). Higher altitudes can be assessed within the same framework. The orientation of the GTO is defined via the Keplerian orbital elements: $\Omega$, $\omega$, and $i$, defined in McClain and the departure location is the GTO periapsis. In a ridesharing scenario, the dropoff orbit orientation is dictated by the primary mission constraints, therefore, in this analysis, it is assumed that there is no a priori information about the GTO, i.e., the dropoff orbit, orientation. In this investigation, a change in GTO orientation is described as a change in the position of the GTO periapsis, i.e., the departure location. The position of the GTO periapsis, $r_{dep}$, is described with Keplerian orbital elements associated with the J2000 Earth Mean Equatorial (EME) inertial reference frame, but the target periodic and quasi-periodic orbits exist in the rotating frame of the Sun-Earth CRTBP model. The GTO departure position in the EME frame, $\hat{r}_{dep}$,
is parameterized via the traditional Keplerian orbital elements, $\Omega$, $\omega$, and $i$, see McClain;\(^{16}\) note that $i$ is the inclination of the orbit in the EME frame. In the Sun-Earth rotating frame, the GTO departure position, $\vec{r}_{dep}$, is parameterized via the angles, $\lambda$ and $\delta$, as illustrated in Figure 1(b), such that the departure position, at a fixed altitude, is represented as $\vec{r}_{dep}(\lambda, \delta)$. In this analysis, the parameters that describe the GTO departure position in the inertial frame are defined in Table 1 and a comparison of the Keplerian orbital elements in the inertial frame and the rotating frames angles, $\lambda$ and $\delta$, is plotted in Figure 2. For a range of departure epochs,

![Diagram](a) GTO periapsis departure position, expressed in the Sun-Earth rotating frame, parameterized by angles: $\lambda$ and $\delta$

![Diagram](b) GTO periapsis departure position, expressed in the Sun-Earth rotating frame, parameterized by angles: $\lambda$ and $\delta$

![Graph](a) Rotating frame inclination, $r_i$, and (b) $\delta$ angles at varying epoch dates and $\Omega$ angles. The values of $r_i$ and $\delta$ are formulated in the Sun-Earth rotating frame

![Graph](b) Rotating frame inclination, $r_i$, and (b) $\delta$ angles at varying epoch dates and $\Omega$ angles. The values of $r_i$ and $\delta$ are formulated in the Sun-Earth rotating frame

6
the GTO departure position, $\vec{r}_{\text{dep}}$, varies in the Sun-Earth rotating frame as the $i$, remains constant in the EME frame. In this investigation, the objective is the construction of transfers at a fixed departure epoch from a range of orientations. The range of orientations is described via a changing $\Omega$ angle in the EME frame with a fixed inclination, $i$, and $\omega$ (refer to Table 1). The plots in Figure 2 present the varying inclination in the rotating frame, $i$, and the angle $\delta$, defined in Figure 1(b), for a range of epochs and $\Omega$ values. The rotating inclination, $i$, values plotted in Figure 2(a) possess a range: $|i - i_{\text{sun}}| < i < |i + i_{\text{sun}}|$, where $i_{\text{sun}}$ is the inclination of the ecliptic in the EME frame, as illustrated in Figure 1(a). Note that the inclination of the ecliptic, $i_{\text{sun}}$, varies with epoch, however, the variation is small such that, in this investigation, the inclination is defined as: $i_{\text{sun}} = 23.43^\circ$. Additionally, the range of the $\delta$ values, illustrated in Figure 1(b) and plotted in Figure 2(b), is denoted: $-i_{\text{sun}} < \delta < i_{\text{sun}}$. The range of $\delta$ plotted in Figure 2(b) and presented in Table 1 is only applicable for a fixed $\omega$ value. In this investigation, optimal $\Delta V$ transfers are constructed over a range of orientations from a departure epoch of: June 2, 2022 12:00:00.000, with the variation of the rotating frame inclination, $i$, and the $\delta$ angles plotted as a red line in Figure 2. An understanding of the variable GTO departure positions, i.e., the varying orientations, facilitates the search and construction of efficient transfers to the vicinity of the Sun-Earth $L_1$ Lagrange point.

**Table 1: Comparison of variables that parameterize the GTO periapsis position in the J2000 EME inertial and Sun-Earth rotating frame**

<table>
<thead>
<tr>
<th>Inertial Frame</th>
<th>Rotating Frame</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_a = 35,786$ km</td>
<td>$r_a = 35,786$ km</td>
</tr>
<tr>
<td>$r_p = 185$ km</td>
<td>$r_p = 185$ km</td>
</tr>
<tr>
<td>$i = 27^\circ$</td>
<td>$</td>
</tr>
<tr>
<td>$0^\circ &lt; \Omega &lt; 360^\circ$</td>
<td>$-i_{\text{sun}} &lt; \delta &lt; i_{\text{sun}}$</td>
</tr>
<tr>
<td>$\omega = 0^\circ$</td>
<td>$0^\circ &lt; \lambda &lt; 360^\circ$</td>
</tr>
</tbody>
</table>

**TRANSFERS WITH A DEEP SPACE MANEUVER**

Transfers leveraging the trajectories along the surface of the stable manifold corresponding to periodic and quasi-periodic orbits and incorporating a single deep space maneuver (DSM) offer advantageous options over a range of departure positions near the Earth. Transfers with a single DSM are constructed by connecting a trajectory along the surface of the stable manifold associated with a periodic, or quasi-periodic, orbit and a bridging arc from a departure position near the Earth. Figure 3 illustrates a transfer into a periodic orbit near the Sun-Earth $L_1$ point with a stable manifold trajectory arc, plotted in cyan, and a bridging arc, plotted in black. The transfer arcs in Figure 3 are propagated in reverse time, that is, the motion of the spacecraft is from an arrival location on the periodic orbit, i.e., an injection point, to the departure position at the GTO periapsis near Earth. Note that the injection point, $x_{\text{inj}}$, into a periodic orbit can be parameterized by the
time variable: \( \tau \). For transfers into quasi-periodic orbits, that is, orbits constructed from a two-dimensional torus, the injection point, \( \vec{x}_{inj} \), is parameterized by two components: \( T_0 \), and \( \theta_2 \), where \( T_0 \) is a time along the orbit, analogous to the longitudinal time along the torus, and \( \theta_2 \) is a location along the invariant curve. Two options for the DSM, the vector \( \Delta V \) in Figure 3, are explored in this analysis: a tangent maneuver and a general maneuver, both impulsive. Recalling that in Figure 3, the spacecraft is propagated in reverse time, the DSM is an impulsive maneuver implemented at the end of the bridging arc and the initiation of the stable manifold arc. For a tangent burn DSM, the maneuver is assumed to be directed along the velocity vector at the final state along the bridging arc and can, therefore, be defined via the scalar magnitude, \( \Delta V_{tan} \). A transfer with a general impulsive DSM possesses maneuver components defined as: \( \Delta V_{gen} = [\Delta V_x, \Delta V_y, \Delta V_z]^T \), and are also introduced at the end of the bridging arc. Observe that a state along a manifold structure, either stable or unstable, associated with a periodic and quasi-periodic orbit can be parameterized by two and three variables, respectively. Recall that a state in the CRTBP model is defined as a six-dimensional vector. For stable or unstable, associated with a periodic and quasi-periodic orbit can be parameterized by two and three variables, respectively. Recall that a state in the CRTBP model is defined as a six-dimensional vector. For periodic orbits, the injection point, \( \vec{x}_{inj} \), is evaluated by propagating from an initial state, \( \vec{x}_0 \), on the orbit by time \( \tau \). From the periodic orbit injection point, a state on the stable manifold is constructed, as detailed by Bosanac, and the state is propagated in reverse time, \( T_M \). In this process, the initial state on the orbit, \( \vec{x}_0 \), is fixed and it is assumed that a state along the global representation of the stable manifold surface can be approximated. Only two variables, \( \tau \) and \( T_M \), are required to locate a state on the stable manifold structure associated with a periodic orbit and, additionally, two variables are also necessary to parameterize states along the unstable manifold structure. For quasi-periodic orbits, the injection point, \( \vec{x}_{inj} \), is computed by propagating a state along the invariance curve, \( \vec{x}_{inv}(\theta_2) \), through time \( T_0 \). The invariant curve state, \( \vec{x}_{inv}(\theta_2) \), is parameterized by the angle \( \theta_2 \) and is evaluated with the truncated Fourier series utilized in the invariance condition corrections process. For states along the stable manifold structures associated with quasi-periodic orbits, the variables \( \theta_2, T_0, \) and \( T_M \), describe the state along the global representation of the manifold. In describing an end state for a transfer, i.e., \( \vec{x}_f \), with a tangent burn into a periodic orbit, also plotted in Figure 3, only four variables are required: \( \tau, T_M, \Delta V_{tan}, T_{arc} \). These variables are termed the fundamental variables for the transfer. To address convergence issues near dynamically complex regions, the transfer arcs, i.e., the stable manifold arc and the bridging arc, are subdivided into a series of segments, that is, reformulated into a multiple-shooting problem. However, the number of fundamental variables does not change, that is, the end state along the transfer is always described with the same number of fundamental variables. The fundamental variables for single DSM transfers into periodic and quasi-periodic orbits are summarized in Table 2. In this investigation, two constraints are enforced at the end state, i.e., \( \vec{x}_f \), of the transfer: an altitude and an apsis constraint. Note that the end state along the transfer is essentially the departure state at periapsis for a GTO with a fixed size, i.e., fixed periapsis and apoapsis altitude. Therefore, the GTO departure position is parameterized by two angles: \( \lambda \) and \( \delta \), as illustrated in Figure 1(b). In this investigation, the constraints for the DSM transfers are written as:

\[
\text{Altitude Constraint : } \quad \bar{r}_f - \bar{r}_{dep}(\lambda, \delta) = 0, \tag{7}
\]

\[
\text{Apsis Constraint : } \quad (\bar{v}_f - \bar{v}_e) \cdot \bar{e} = 0,
\]

where \( \bar{r}_f \) and \( \bar{v}_f \) are the final end state position and velocity along the transfer, \( \bar{v}_e \) is the position of the Earth with respect to the Sun-Earth barycenter in the CRTBP rotating frame, and the desired GTO departure position, \( \bar{r}_{dep} \) is defined as:

\[
\bar{r}_{dep}(\lambda, \delta) = \bar{r}_e + h_{alt} \begin{bmatrix} \cos(\lambda) \cos(\delta) \\ \sin(\lambda) \cos(\delta) \\ \sin(\delta) \end{bmatrix}, \tag{8}
\]

with \( h_{alt} \) as the fixed GTO periapsis altitude such that: \( h_{alt} = 185 \) km. Two additional free variables are introduced to the transfer problem: \( \lambda \) and \( \delta \) with the formulation of the constraint conditions in Equation (7) as a four-dimensional column vector. For example, transfers into a periodic orbit with a general maneuver, i.e., scenario B from Table 2, requires six fundamental variables: \( \tau, T_M, \Delta V_x, \Delta V_y, \Delta V_z, T_{arc} \). In all transfer scenarios, only the end state apsis and altitude constraints are enforced, that is, the constraint vector for the transfer problem is two-dimensional. However, with the constraint conditions from Equation (7), the constraint is written as a four-dimensional vector with two additional free variables, \( \lambda \) and \( \delta \), included in the transfer problem. The reformulation of the constraint vector and introduction of the free variables, \( \lambda \)
and $\delta$, permits more control over the departure position. Additionally, the reformulation of the constraint condition does not affect the solution space for the DSM transfer problem, summarized in Table 2. The dimension of the solution space applicable to the DSM transfer is evaluated as the difference between the number of fundamental variables and the number of constraints. For example, transfers with a tangential DSM to a periodic orbit appear on a two-dimensional surface of solutions. Information regarding surface shape and terminal conditions, e.g., if the surface is closed, is not known a priori or not known as a closed form function. The scenarios in Table 2 are applicable for transfers leveraging either stable and unstable manifold structures and a bridging arc. Transfers in the solution space for the scenarios in Table 2 offer advantageous options for mission design in the Sun-Earth system.

**Table 2: Fundamental variables and constraints for different DSM type transfers into periodic and quasi-periodic orbits**

<table>
<thead>
<tr>
<th>Scenario</th>
<th>Target</th>
<th>Fundamental Variables</th>
<th>DSM Type</th>
<th>Departure Constraints</th>
<th>Dimension of Solution Space</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>Periodic Orbit</td>
<td>$\tau, T_M, T_{Arc}$</td>
<td>Tangent $(\Delta V_{tan})$</td>
<td>Altitude Apsis</td>
<td>2</td>
</tr>
<tr>
<td>B</td>
<td>Periodic Orbit</td>
<td>$\tau, T_M, T_{Arc}$</td>
<td>General $(\Delta V_{gen})$</td>
<td>Altitude Apsis</td>
<td>4</td>
</tr>
<tr>
<td>C</td>
<td>Quasi-Periodic Orbit</td>
<td>$T_{\theta_1}, \theta_2, T_M, T_{Arc}$</td>
<td>Tangent $(\Delta V_{tan})$</td>
<td>Altitude Apsis</td>
<td>3</td>
</tr>
<tr>
<td>D</td>
<td>Quasi-Periodic Orbit</td>
<td>$T_{\theta_1}, \theta_2, T_M, T_{Arc}$</td>
<td>General $(\Delta V_{gen})$</td>
<td>Altitude Apsis</td>
<td>5</td>
</tr>
</tbody>
</table>

**SOLAR EXCLUSION ZONE PATH CONSTRAINT**

Communications constraints, in the form of a path constraint, for transfers to the Sun-Earth $L_1$ vicinity must be included in the preliminary design process. Communications constraints during the Earth-to-$L_1$ transfer and along the $L_1$ Lagrange point orbit require that the spacecraft avoid crossing in front of the solar disk when viewed from the Earth. In the rotating frame of the Sun-Earth system, a Sun-Earth-Vehicle (SEV) angle, $\alpha$, is defined as the angle between the position of the spacecraft with respect to the Earth, $\vec{r}_{23}$, and the negative $\hat{x}$-axis direction, as illustrated in Figure 4. The vector $\vec{r}_{23}$ is defined as: $\vec{r}_{23} = \vec{r} - \vec{r}_e$, where $\vec{r}$ and $\vec{r}_e$ are the position vectors of the spacecraft and the Earth measured from the Sun-Earth barycenter, respectively, $\pm \hat{x} = [-1, 0, 0]^T$, and $\vec{r}_e = [1 - \mu, 0, 0]^T$. The SEV angle is defined:

$$\alpha = \cos^{-1} \left( \frac{\vec{r}_{23} \cdot -\hat{x}}{\|\vec{r}_{23}\|} \right),$$

where $\vec{A} \cdot \vec{B}$ is the vector dot product in this formulation. The Solar Exclusion Zone (SEZ), plotted in Figure 4, is a region defined by a right circular cone with the vertex at the Earth and a constant SEZ angle, $\alpha_{SEZ}$. The communications constraint must be enforced along the transfer path and a mathematical definition for a general path constraint is formulated as,

$$F_{path} = \sum_{i}^{N} \int_{0}^{T_i} F_p^2 \left( \vec{x}_i(t) \right) - |F_p \left( \vec{x}_i(t) \right) F_{\parallel} \left( \vec{x}_i(t) \right) | dt = 0,$$

where $\cdot$ implies an absolute value. The path constraint condition in Equation (10) is similar to the formulation from Ojeda Romero et al., but the form in Equation (10) has smooth first order partial derivative, i.e., the partial derivative is defined everywhere along the trajectory arcs. In Equation (10), the transfer is formulated as a multiple-shooting problem and applicable over $N$ nodes along the trajectory where $T_i$ is the propagation time for the $i^{th}$ arc. The path constraint formulation is versatile and may be applied at arc segments near the constraint region, i.e., the SEZ cone. The general path constraint formulation in Equation
Figure 4: Solar Exclusion Zone defined in the Sun-Earth rotating frame (red). The fixed angle $\alpha_{SEZ}$ defines the size of this region.

(10) is implemented in this investigation to prevent crossing into the SEZ for L$_1$ transfers. The function that expresses the SEZ crossing condition is defined as,

$$F_p = \sin (\alpha (\bar{x}_i(t)) - \alpha_{SEZ})$$

such that, the SEV angle, $\alpha$, and the position with respect to the Earth, $\bar{r}_{23}$, are functions of the state, $\bar{x}_i(t)$, that corresponds to the state node, $\bar{x}_i$, propagated by time $t$. A trapezoidal numerical integration scheme, consistent with the Trapz function in MATLAB, is implemented for the general path constraint in Equation (10). An important observation is that the general path constraint function in Equation (10) is compatible with any path constraint condition that can be formulated as an inequality $F_p \geq 0$.

TRANSFERS INTO PERIODIC ORBITS FROM GTO

Transfers into a halo orbit near the Sun-Earth L$_1$ Lagrange point are constructed with stable manifold trajectories and a DSM that is tangent to the path. The SOHO mission$^5$ leveraged a periodic halo orbit near the Sun-Earth L$_1$ point to avoid crossing into the SEZ. The halo orbit selected for the SOHO mission employed the following amplitudes: x-amplitude of 206, 448 km, y-amplitude of 66, 672 km, and z-amplitude of 120, 000 km.$^5$ The halo orbit is a southern halo orbit, i.e., the motion of the spacecraft as viewed by an observer at the Earth is counterclockwise, with a period of 180 days. Recall that the objective is the construction of efficient transfers from a GTO with a fixed periapsis altitude at 185 km with no a priori information about the GTO orientation. The secondary spacecraft departure state, $\bar{x}_{dep}$, is at the GTO periapsis and, from Figure 1(b), is parameterized by angles $\lambda$ and $\delta$. Transfers with a tangential DSM, i.e., scenario A in Table 2, are identified as potential transfer options into the southern halo orbit from different departure positions near the Earth. Although transfers described by both scenarios A and B are applicable for the periodic halo orbit, transfer scenario A is selected for the two-dimensional solution space. Recall that in scenario A from Table 2, the end point location is also the departure position of the GTO, $\bar{r}_{dep}$ in Figure 3, at different near-Earth positions. Additionally, the departure position is constrained to be on the $\hat{x}$-$\hat{y}$ plane of the Sun-Earth rotating frame, i.e., $\delta = 0^\circ$. The additional constraint decreases the solution space for the transfers with tangent DSMs by one, therefore, all transfer solutions into the periodic halo orbit with a departure state along the $\hat{x}$-$\hat{y}$ plane near Earth appear on a one-dimensional curve of solutions.

A curve of solutions that represents transfers into a southern L$_1$ halo orbit with a tangential DSM is constructed via a multiple-shooting corrections algorithm. The transfer scenario illustrated in Figure 3 is formulated as a multiple-shooting problem by further subdividing the stable manifold arcs and the bridging arcs. Full state continuity is enforced along the subarcs that belong to the stable manifold arc segment as well as the bridging arc segment. Recall that, in reformulating the transfer problem into a multiple-shooting problem, the dimension of the solution space does not change and the multiple-shooting formulation only aids in the corrections process. An initial guess is generated by utilizing a Poincaré map, as plotted in Figure 5. The trajectories along the surface of the stable manifold that emanates from the southern L$_1$ halo orbit are prop-
agated in reverse time to a surface of section defined as a plane, i.e., \( x = 1.48 \times 10^8 \) km, and their crossings onto the map are plotted in blue in Figure 5. The bridging arc trajectories, i.e., the red points in Figure 5, are propagated forward in time from a location near the Earth, i.e., the departure position corresponding to the GTO periapsis, towards the surface of section. The bridging arc trajectories are propagated from a location near Earth that corresponds to \( \lambda = 0^\circ \) and \( \delta = 0^\circ \). In the map in Figure 5, only two dimensions are displayed, \( y \) and \( z \), and there is no information about the velocity along the trajectories. For this scenario, an initial guess is selected by identifying intersections between the stable manifold crossings and the bridging arc crossings, i.e., the black stars in Figure 5. Recall that the solutions space for this scenario is two-dimensional and, therefore, the transfers with a tangent maneuver are contained on a two-dimensional surface. However, for simplicity the focus of the analysis is constructing transfers originating from different locations along the GTO near the Earth and initiated in the Ecliptic plane, i.e., the \( \hat{x} \)-\( \hat{y} \) plane of the Sun-Earth rotating frame. An additional departure constraint is added to scenario A in Table 2, \( \delta = 0^\circ \). The tangent maneuver transfers are now all captured along a one-dimensional curve of solutions. Figures 6(a) and 7(a) display curves of transfer solutions for a range of \( \lambda \) values.

![Figure 5: Poincaré map of stable manifold crossings from southern halo orbit (blue) and bridging arc crossing (red) onto the cross-section. Two star points indicate the chosen stable manifold arc and bridging arc to construct an initial guess.](image)

A selected range of transfers, i.e., corresponding to the box in red in Figure 6(a), is plotted in Figure 6(b). The transfers presented in Figure 7(b) correspond to the transfers in Figure 7(a). Positive \( \Delta V \) magnitudes for the DSM, presented in Figures 6-7, correspond to prograde burns and negative \( \Delta V \) values are retrograde.

![Figure 6: (a) Curve of transfer solutions into a southern halo orbit with a tangent DSM. (b) Selected transfers with the location of the DSM in red and the location of the injection points into halo orbit in magenta.](image)
TRANSFERS INTO QUASI-PERIODIC ORBITS FROM GTO

Trajectories into quasi-periodic orbits in the vicinity of the Sun-Earth L₁ Lagrange point are constructed by including a DSM along the transfer. In this example, transfers to quasi-periodic Lissajous orbits near L₁ are constructed with trajectories along the stable manifold structure associated with a Lissajous orbit and a bridging arc segment. This example is identified as either scenario C, a tangent maneuver transfer, or scenario D, a general maneuver transfer, from Table 2. Mission requirements, such as maximum/minimum SEV, α, constraints, may restrict a set of applicable quasi-periodic orbits for a secondary spacecraft.

A ‘tighter’ Lissajous orbit, similar to the operational orbits of the ACE or the Deep Space Climate Observatory (DSCVR) missions, is often selected to maintain specific science or communications requirements. In this application, a quasi-periodic orbit with characteristics similar to that designed for the ACE spacecraft is selected as the target destination. The ACE spacecraft is in an L₁ Lissajous orbit with approximate x-, y-, and z-amplitudes of 81,755 km, 264,071 km, and 154,406 km, respectively. An orbit with similar amplitudes is selected from a family of Lissajous orbits constructed employing a fixed Jacobi Constant value, \( C = 3.000876398 \), in the Sun-Earth system. In Figure 8, specific members of the fixed Jacobi Constant family, plotted in blue, are projected onto the \( \hat{y} - \hat{z} \) plane, while the selected Lissajous orbit is depicted in black. A communications constraint imposed on the ACE spacecraft requires the trajectory to remain outside the Solar Exclusion Zone, defined...
as $\alpha_{SEZ} = 5^\circ$ and illustrated in Figure 4, for this investigation. Stationkeeping (STK) strategies have been developed and implemented to maintain the Lissajous trajectory beyond the SEZ; however, the focus of this application is the transfer into a Lissajous orbit at an ideal location, i.e., an injection point, that will maximize the time outside the SEZ before an STK maneuver is necessary. The search for an injection point that maximizes time outside the SEZ is formulated as the search for an angle, $\theta_2$, along the invariant curve. An injection point along a quasi-periodic orbit is parameterized by $T_\theta_1$ and $\theta_2$, where $T_\theta_1$ is the propagation time from a point along the invariant curve and $\theta_2$ is an angle along the invariant curve, such that: $\bar{x}_{inj}(T_\theta_1, \theta_2)$. Recall that an injection point is computed by propagating a state along the invariant curve, i.e., $x_{inv}(\theta_2)$, over time, $T_\theta_1$. The state along the invariant curve is defined as: $\bar{x}_{inv}(\theta_2) = \bar{x}^* + \bar{u}(0, \theta_2)$, where $\bar{x}^*$ is the fixed point of a periodic orbit associated with the quasi-periodic family and $\bar{u}$ is a six-dimensional torus state. In the search for an ideal injection point, the propagation time is assumed to be zero, $T_\theta_1 = 0$, that is, the ideal injection point is dependent on the angle $\theta_2$. The state along the invariant curve, $x_{inv}(\theta_2)$, is propagated along the Lissajous orbit and the SEV angle, $\alpha$, is illustrated in Figure 4. The computation of a trajectory arc along a quasi-periodic orbit is consistent with the following steps:

1. Identify an initial angle, $\theta_2^0$, along the invariant curve. Create the state along the invariant curve: $\bar{x}_{inv}^0 = \bar{x}^* + \bar{u}(0, \theta_2^0)$. Recall that $\bar{x}^*$ is a fixed point associated with a periodic orbit used in the quasi-periodic orbit corrections process.

2. Propagate the current state, $\bar{x}_{inv}^0$, with the mapping time, $T_1$, associated with the quasi-periodic orbit.

3. Identify the next state along the invariant curve. The next angle along the invariant curve is: $\theta_2^1 = \theta_2^0 + \rho$, where $\rho$ is the rotation angle corresponding to the quasi-periodic orbit. Calculate the new state, $\bar{x}_{inv}^1 = \bar{x}^* + \bar{u}(0, \theta_2^1)$.

4. Propagate the new state, $\bar{x}_{inv}^1$, with the mapping time $T_1$.

5. Connect the beginning of the trajectory from $\bar{x}_{inv}^0$ to the end of the trajectory from $\bar{x}_{inv}^1$.

6. Repeat steps 3-5 for any number of revolutions around the quasi-periodic orbit.

Note that this representation of the quasi-periodic trajectory utilizes the approximation of the invariant curve from a truncated Fourier series. The time beyond the SEZ threshold for a set of revolutions on the Lissajous trajectory corresponding to a range of $\theta_2$ values with 13 revolutions is plotted in Figure 9(a). Points A and B identified in Figure 9(a) correspond to the $\theta_2$ values: 152.85° and 335.35°, respectively. The trajectories that emerge from the identified injection points with the maximum time above the SEZ threshold, $\alpha_{SEZ} = 5^\circ$, before crossing into the SEZ cone. In Figure 9(a), several points in red, i.e., trajectories along the Lissajous orbit, are defined by even longer intervals outside the SEZ than trajectories indicated by points A and B. However, trajectories corresponding to the red points initially violate the SEZ cone, therefore, these injections points are not viable for consideration. The characterization of the invariant curve is not unique, therefore the $\theta_2$ values in Figure 9(a) are dependent on the characterization of the invariant curve via the truncated Fourier series. The trajectory for point B appears in a polar plot in Figure 9(b) with the SEZ illustrated by a dashed red line. The angular dimension for the polar plot corresponds to an angle $\zeta$ defined as $\zeta = \tan^{-1}\left(\frac{z}{y}\right)$, computed with the $y$- and $z$-components of a state along a Lissajous trajectory, and the radial direction is the SEV angle $\alpha$, defined in Equation (9).

Poincaré maps are leveraged to determine an initial guess, i.e., a single DSM transfer, constructed leveraging the stable manifold structures associated with a Lissajous orbit and a bridging arc segment. The construction of a transfer into a quasi-periodic orbit, i.e., a Lissajous orbit, is essentially scenarios C and D from Table 2. In this analysis, a transfer with a single general DSM originating from a GTO periapsis position near Earth is consistent with scenario D which, from Table 2, has a five-dimensional solution space. Recall that the objective of this example is to enter into an ideal injection point along a Lissajous orbit, that is, a fixed value of $T_\theta_1$ and $\theta_2$. Therefore, the solution space consistent with this scenario is reduced to a three-dimensional surface. Additionally, for simplicity, the focus is the identification of transfers originating from different
Figure 9: (a) Time above 5° threshold. Points in red are trajectories that initially violate the threshold. (b) Trajectory for injection point corresponding to $\theta_2 = 335.35^\circ$. The red dashed line corresponds to $\alpha_{SEZ} = 5^\circ$ cone.

near-Earth positions along the ecliptic, i.e., the $\hat{x}$-$\hat{y}$ plane of the Sun-Earth rotating frame. This condition is satisfied via the introduction of the constraint $\delta = 0^\circ$. Recall that the constraint conditions in Equation (7) are reformulated for scenarios A-D in Table 2 and two additional variables, $\lambda$ and $\delta$, corresponding to the departure position, are included in the transfer problem. The fundamental variables for transfer scenario D are: $T_{\theta_1}$, $\theta_2$, $T_M$, $T_A$, $\Delta V_x$, $\Delta V_y$, $\Delta V_z$, $\lambda$, and $\delta$. Note that the added variables do not change the solution space. Finally, the solution space for transfers with a general DSM that insert into an ideal injection point along a Lissajous orbit from a GTO periapsis position along the Sun-Earth ecliptic is two-dimensional. To generate an initial guess for the single general DSM transfers, the trajectories on the surface of the stable manifold corresponding to the Lissajous orbit are propagated in reverse time towards a surface of section. The position vector of the GTO departure position is a function of $\delta$ and $\lambda$ as illustrated in Figure 1(b), with a fixed altitude, 185 km, with respect to the Earth. The GTO periapsis is situated along the Sun-Earth $\hat{x}$ line on the opposite side of the Sun and Earth; such a location corresponds to $\delta = 0^\circ$ and $\lambda = 0^\circ$. The initial guess for the velocity vector associated with the departure positions, i.e., the bridging arc segment, is constructed by applying a maneuver $\Delta V$ in a direction perpendicular to the radial direction of the GTO periapsis; note that the radial direction is with respect to the Earth. The direction of the departure velocity vector varies such that a set of bridging arc transfers is generated when the states are forward propagated towards a surface of section. A surface of section, defined as $y = 2.69 \times 10^5$ km in the rotating frame, is selected to produce the initial conditions and generate a single DSM transfer. The points in red in Figure 10(a) are the second returns to the surface of section from a set of bridging arcs, propagated in forward time from the GTO departure position, and the blue points correspond to the stable manifold trajectories, propagated in reverse time from the Lissajous injection points. An initial guess is produced with a pair of blue and red points that are close in position in the $\hat{x}$-$\hat{z}$ projection and is plotted in Figure 10(b). The initial guess is reformulated into a multiple shooting problem and corrected via a Newton predictor-corrector algorithm.

The solution space for a transfer into a Lissajous orbit with a single maneuver is constructed via a numerical continuation scheme. In Figure 9(a), the values of $\theta_2 = 335.35^\circ$, $152.85^\circ$ are identified as desired injection points with $T_{\theta_1} = 0$ into the selected Lissajous orbit. The initial guess in Figure 10(b), from a fixed $\lambda$ and $\delta$, is divided into a series of discontinuous arcs, consistent with a multiple shooting scheme, and a feasible solution is constructed via a Newton predictor-corrector algorithm. Recall that the trajectories, consistent with scenarios in Table 2, are propagated in reverse time and a single general DSM is performed, as illustrated in Figure 3. The objective is the construction of the two-dimensional solution surface of transfers from the
GTO departure positions along the Sun-Earth ecliptic. The selected initial guesses in Figure 10(b) do not correspond to the ideal injection locations, i.e., $\theta_2 = 335.35^\circ, 152.85^\circ$. The first step is a transfer into one of the two possible ideal injection locations. By reformulating the constraint conditions in scenario D in Table 2 with Equation (7), the fundamental variables for the transfer are: $T_{\theta_1}, \theta_2, T_M, T_{Arc}, \lambda, \delta,$ and $\Delta V_{gen}$, where $\Delta V_{gen} = [\Delta V_x, \Delta V_y, \Delta V_z]^T$. There are nine free variables and two departure constraints corresponding to: apsis and altitude. Note that the departure conditions, via Equation (7), are defined as a four-dimensional vector. The goal is a search for a transfer into a desired injection point, defined by $\theta_2$, therefore, four fundamental variables, $T_{\theta_1}, T_M, \lambda,$ and $\delta,$ remain constant so the solution space representing the transfers is a one-dimensional curve. A pseudo-arclength continuation strategy is leveraged to create a curve of transfers with varying $\theta_2$ and $T_{Arc}$ values. A curve can also be generated by fixing any combination of two free variables from $T_{\theta_1}, T_M,$ and $T_{arc}$ with the fixed $\lambda$ and $\delta$ angles. A curve of transfers is displayed in Figure 11(a). This curve is not a complete representation of the solution space and the only point of interest is a transfer with corresponding $\theta_2 = 335.35^\circ$ or $152.85^\circ$; one such instance is identified in Figure 11(a). The time outside the SEZ reported in Figure 9(a) corresponds to a trajectory with $T_{\theta_1} = 0$, therefore the transfer
from 11(a) is utilized as an initial point to generate a separate curve with varying $T_{\theta_1}$ and $T_M$. To construct the curve in Figure 11(b), $\lambda$, $\delta$, and $\theta_2$ are fixed along with one of the following variables: $T_M$ or $T_{Arc}$. Two points on the solution curve in Figure 11(b) possess the desired $T_{\theta_1} = 0$. Finally, a surface of transfers is generated from the sample transfer with the desired $\theta_2$ and $T_{\theta_1}$ values, i.e., the values that correspond to the desired injection point on the Lissajous orbit. The surface, or family of transfers, is created through pseudo-arclength continuation process by fixing $\theta_2$ and $T_{\theta_1}$, and one of the following variables: $T_M$, $T_{Arc}$, and $\lambda$. Recall that the objective is an exploration of the solution space for all possible GTO orientations and, in this application, only transfers such that the departure is from a GTO periapsis located along the Sun-Earth ecliptic are considered, i.e. $\delta = 0^\circ$. The enclosed transfer solution surface for a region around the Earth, such that $-40^\circ \leq \lambda \leq 20^\circ$ is depicted in Figure 12(a). Note that, in Figure 12, $T_{\text{tot_{mani}}}$ and $T_{\text{tot_{arc}}}$ are the total time along the stable manifold and bridging arc transfer segments, respectively. In the transfer scenarios presented in Table 2, there is no information about the maneuver implemented at the GTO periapsis; a Transfer Injection Maneuver (TIM) is necessary to shift from the GTO departure position to the transfers on the solution curve, e.g., Figure 12(b).

![Figure 12](image)

**Figure 12:** (a) Solution Surface from a region corresponding to $-40^\circ \leq \lambda \leq 20^\circ$ near the Earth. (b) Solution Surface with GTO inclination and Transfer Injection Maneuvers (TIMs).

A summary of the steps to generate the solution surface for this example follows:

1. Identify injection location, $\theta_2$, along the invariant curve of a quasi-periodic orbit.
2. Fix the departure location from Earth, i.e., fix $\lambda$ and $\delta$. Note that the departure location is an apsis with respect to the Earth with a fixed altitude.
3. Generate transfer arcs from the fixed departure location towards a surface of section.
4. Generate stable manifold trajectories from the desired quasi-periodic orbit. These are propagated in reverse time toward a surface of section.
5. Identify an initial guess from a Poincaré map constructed via the crossings onto the surface of section.
6. Search for a solution with the desired $\theta_2$. A continuation strategy is implemented by fixing any combination of two from the variables: $T_{\theta_1}$, $T_M$, $T_{Arc}$ along with $\lambda$ and $\delta$. 

16
7. Search for the solution with desired $T_{\theta_1} = 0$, by fixing $\theta_2$, $\lambda$, $\delta$ and either $T_M$ or $T_{Arc}$.

8. Explore the solution space for a fixed $T_{\theta_1}$, $\theta_2$, $\delta$. The surface is created from a series of curves created from a continuation strategy. The curve is created by fixing one of the variables: $T_M$, $T_{Arc}$, or $\lambda$.

These steps can be generalized to search the solution space for any single DSM transfer to a specific quasi-periodic orbit. A separate solution surface is constructed from the previous steps for a different initial departure position near the Earth, for example, given $\lambda = 90^\circ$ and for $\delta = 0^\circ$, see Figure 13(a), and $\lambda = 180^\circ$ and $\delta = 0^\circ$ in Figure 14(a). Recall that every point on this surface corresponds to a single maneuver transfer.

Figure 13: (a) Solution Surface from a region corresponding to $40^\circ \leq \lambda \leq 130^\circ$ near the Earth. (b) GTO inclination and Transfer Injection Maneuvers (TIMs) information.

Figure 14: (a) Solution Surface for direct transfers. (b) Solution Surface with GTO inclination and Transfer Injection Maneuvers (TIMs).
In Figures 12(b), 13(b), and 14(b), the magnitude of the TIM is within the range of $740 \leq \text{TIM} \leq 780$ m/s, however, there is a larger variation in the magnitude of the DSM. The surface information from Figures 12-14 fill the region of $-180^\circ \leq \lambda \leq 180^\circ$, therefore, it is possible to identify a feasible transfer into the selected injection points on the Sun-Earth L$_1$ Lissajous trajectory from a range of GTO orientations.

**OPTIMIZED TRANSFERS TO L$_1$ LISSAJOUS ORBITS**

Optimal direct transfers into L$_1$ Lissajous orbits are constructed from a range of departure positions near the Earth by leveraging insight from the solution surface. In particular, optimal direct transfers into a specific injection location, $\theta_2$, on a Lissajous orbit are explored here. The solution surfaces from Figures 12(a), 13(a), and 14(a) reveal transfers with a single DSM from a range of GTO departure positions, i.e., the complete set values for $\lambda$, with $\delta = 0^\circ$, i.e., on the Sun-Earth ecliptic. But these transfers are also initial guesses to compute optimal transfers. For demonstration in this investigation, optimized transfers are constructed from a GTO departure position corresponding to a range of $\Omega$ values, as noted in Table 1, at a specific epoch of Jun 2, 2022 12:00:00.000. Recall that the optimization process is performed in the Sun-Earth rotating frame, thus the GTO, as expressed in terms of the rotating frame, possess a range of $i, \lambda$, and $\delta$ values, as plotted in Figure 2. To reduce the total $\Delta V$ necessary for the transfer, the number of maneuvers is increased to four: the TIM, two DSMs, and one Lissajous Injection Maneuver (LIM). Recall that the single DSM transfers from Figures 12(a) and 13(a) each included two maneuvers, the TIM and a single DSM, labeled DSM1, because the transfer incorporated a stable manifold trajectory to approach the desired injection point into the Lissajous orbit. Now, an additional DSM, termed DSM2, and a maneuver at the injection point, denoted as LIM, are introduced to construct the optimal transfers in search of a minimal $\Delta V$ solution. The location of the DSMs along the final transfer trajectory is a free variable in the optimization process. A direct optimization scheme is implemented via MATLAB’s fmincon optimization function to compute a locally optimum transfer that minimizes the following function:

$$J = \min \left\{ \Delta V_{TIM} + \sum_{i=1}^{b} \Delta V_{DSM}^i + \Delta V_{LIM} \right\}, \quad (12)$$

where the trajectory is subdivided into a set of $N$ nodes and $N$ forward propagated arcs, as illustrated in Figure 15, with two deep space maneuvers such that, $b = 2$. As an example, in Figure 15, two DSM are illustrated via red points, the TIM is represented by a purple point at departure, and the LIM is plotted as a cyan point at the injection point. The initial state along the transfer trajectory, $\vec{X}_1$, is a GTO state and the position at the end of the transfer, illustrated as a cyan point in Figure 15, is a state on the Lissajous orbit. Full state continuity, position and velocity, is enforced throughout the transfer except for nodes that correspond to a maneuver, in which only position continuity is enforced. A cone constraint, representing the SEZ, is enforced throughout the transfer with $F_p$ defined from Equation (11). The data represented on the solution surfaces in Figures 12(a) and 13(a) is utilized as an initial guess for the optimization scheme and a summary of the optimal transfer is plotted in Figure 16. For the selected epoch, there is an orientation, i.e., $\Omega = 180^\circ$ in Figure 16(a), that corresponds to a $\Delta V$ minimum. However, the $\Omega$ value corresponding to a minimal $\Delta V$ is only observed at the selected departure date. Additionally, a range of trajectories in Figure 16(b) require an

![Figure 15: Multiple maneuver transfer scenario to Lissajous orbit](image-url)
initial excursion to the Sun-Earth L_2 region prior to arrival near L_1. This behavior is observed for transfers with GTO departure positions at the range of 160° ≤ λ ≤ 320°. The optimal solutions in this example are based on the surface of solutions in Figures 12(a), 13(a), and 14(a) that correspond to a specific Lissajous injection point, identified with θ_2 = 335.35°. However, similar surfaces of solutions can be created for any θ_2 angle identified in Figure 9. The method of exploring transfers from a GTO into a selected Lissajous orbit can also be expanded for any general quasi-periodic orbit, such as a quasi-halo, near any Lagrange point in the Sun-Earth system.

CONCLUDING REMARKS

The construction of feasible transfers from a GTO to a Sun-Earth L_1 Lissajous trajectory leverages the stable manifold information associated with periodic and quasi-periodic orbits as well as a bridging arc. The goal is a departure from periapsis for any GTO orientation to insert into a specific location on a Lissajous path. Two distinct scenarios are explored, one with a tangential DSM and one with a general DSM. Transfers with a DSM are constructed via a multiple shooting algorithm and a surface of transfer solutions is generated with pseudo-arclength continuation. Transfers into an L_1 southern halo orbit are computed by including a tangential DSM along the trajectory and the transfer solutions appear on a one-dimensional curve for GTO departure positions placed in the Sun-Earth Ecliptic plane. General DSM transfers into specific injection points along an L_1 Lissajous orbit are constructed as a two-dimensional surface. An additional communications constraint is introduced to prevent trajectories from crossing into a defined Solar Exclusion Zone. Optimal transfers into the Lissajous orbits are computed by using the transfer information from the surface of transfer solutions as initial guesses and by incorporating additional maneuvers. The method described in this analysis is extendable to periodic and quasi-periodic orbits near any Lagrange point in a CRTBP model.

ACKNOWLEDGMENT

The authors thank Mr. Michael Mesarch for introducing this research topic. Assistance and guidance from colleagues in the Multi-Body Dynamics Research group at Purdue University is much appreciated as is the support of the Purdue University School of Aeronautics and Astronautics including access to the Rune and Barbara Eliasen Visualization Laboratory. This material is based upon work supported by the National Science Foundation Graduate Research Fellowship Program under Grant No. DGE-1333468. Any opinions,
findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.

REFERENCES


