SEARCH FOR STABLE REGIONS IN THE IRREGULAR
HAUMEA-NAMAKA BINARY SYSTEM

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This work aims to describe the dynamics of small spacecraft around a binary system comprised by irregular bodies Haumea and Namaka. In this model, the dynamics of Haumea and Namaka is assumed as a full two body problem, considering the inclination of Namaka, and the equations of motion of the spacecraft are written incorporating the information from this model, also considering the eccentricity of Namaka’s orbit. We found interest regions in the system, formed by periodic and quasi-periodic orbits, where the integral of the acceleration technique can be applied to find the delta-V required to create delta-V assisted periodic orbits.

INTRODUCTION

Probably Haumea is the most intriguing Kuiper Belt Object discovered up to date. It is a triaxial ellipsoid in fast rotation (3.9155 h), possesses a rocky core and a thin layer of ice, which is carbon depleted. Haumea is also orbited by two moons, Namaka and Hi‘iaka. Namaka is the inner moon, with a considerable eccentricity (0.249) and inclination (13 with respect to the “equator” of Haumea). The outer moon, Hi‘iaka, has low eccentricity (0.051) and inclination (1-2). These characteristics of the system point to a scenario where Haumea was formed by a collision with a same-sized body and, consequently, it is the parent of a collisional family. The other members of this collisional family are found by identifying similar characteristics, in this case, a rocky core surrounded by a thin layer of ice and depletion of carbon. By these features, the Haumea system is a target of interest for a mission to explore distant small bodies in the Solar System. The scientific return of this kind of mission is significant. However, the exploration of this system is very expensive in terms of fuel consumption, if done with the usual large spacecraft that have been employed in past missions throughout the Solar System. The main reason for a change in strategy is the high value of the arrival velocity, which in the case of Haumea could reach 15 km/s. With this arrival velocity, the delta-V for capture is also very high. Then, instead of using large spacecraft for the exploration of the system, we proposed in previous work the use of small probes, like nanosatellites or cubesats for the exploration. These small probes would be delivered in the system by a large spacecraft that could serve as a communications bridge between the small probes and the Earth during the mission. This model of exploration also allows the study of the moons of Haumea,
since the small probes can be inserted in the vicinity of these moons. In this sense, we decide to separate the system in two binaries, the first comprised by Haumea-Hi’iaka and the second comprised by Haumea-Namaka. In a previous work,\textsuperscript{8} we explored the dynamics of small spacecraft around a binary comprised by the irregular bodies Haumea and Hi’iaka. Then, this work aims to describe the dynamics of these small spacecraft around a binary system comprised by the irregular bodies Haumea and Namaka. In this model, the dynamics of Haumea and Namaka (primary system) is considered as a Full Two Body Problem (F2BP),\textsuperscript{9} taking into account the inclination of Namaka with respect to the equator of Haumea (13), and the equations of motion of the spacecraft are written incorporating the information emerging from this model, i. e., the equations of motion are described by the Full Three Body Problem, where we also consider the eccentricity of the Namaka’s orbit. The general mathematical description of the planar circular model appears in References 8 and 10. In this rich dynamical scenario, we are able to define regions of interest for a future mission around Namaka and Haumea, where the small spacecraft can be inserted into periodic or quasi-periodic orbits. With these results, we expect also to find regions where there could exist particles from the formation of the Haumea system. The quasi-periodic orbits are also of great interest for future missions. These orbits can close at some point and their periodicity could be guaranteed by applying an impulsive delta-V at the point of the closure, in order to place the spacecraft back into the previous orbit. If this delta-V is periodically applied in the same point, the orbit become a “forced” periodic orbit. These orbits are called delta-V assisted periodic orbits.\textsuperscript{11,12} Since the magnitude of the perturbation of Hi’iaka on Namaka is smaller than the perturbation coming from the interaction between Haumea-Namaka both with elliptical shapes, for the period of time that we consider in this study (maximum of five years), the perturbation of Hi’iaka on Namaka can be disregarded in the F2BP. Furthermore, we are disregarding the orbits that escape from the Lagrangian point L2, which could eventually reach Hi’iaka. Therefore, the periodic and quasi-periodic orbits found in the model described above will remain in a “full” system comprised by Haumea (and its gravitational potential), Namaka, and Hi’iaka (where Namaka and Hi’iaka can interact with each other). This “full” model in the exact system is important because it allows the application of the integral of the acceleration technique\textsuperscript{13,14} to calculate the minimum delta-V to turn quasi-periodic orbits in this system in delta-V assisted periodic orbits. Due to the fast rotation and size of Haumea, delta-V assisted periodic orbits can likely be feasible for the exploration of the system, instead of landing or even hovering missions, mainly due to the small maneuvering capabilities of small spacecraft and the complexity of these tasks. Also, the periodic orbits can be chosen such that, at the end of the mission, when the delta-Vs cease to be applied, the small spacecraft can collide with a primary, spreading particles from its surface, for analysis.

**FULL TWO-BODY PROBLEM (F2BP)**

In this section, we explain the development of the mathematical model of Haumea and Namaka, considered as a binary system. As explained before, for a period of time of a few years, we can disregard the perturbation of Hi’iaka. Then, let us consider Haumea, with mass \(M_1\), and Namaka, with mass \(M_2\) (throughout the paper, all quantities with subindex 1 and 2 will identify Haumea and Namaka, respectively), both rotating triaxial ellipsoids, with uniform density, and body semi-major axis \(\alpha_i\), \(\beta_i\), and \(\gamma_i\), interacting gravitationally with each other. In previous work,\textsuperscript{8} in the case of Haumea and Hi’iaka, the low inclination of the latter allowed us consider these two bodies with motion restricted to plane. However, in the case of Namaka, this is not possible. The only assumption we made is that both bodies rotate about one of its moment of inertia, which we always assume to be the maximum one. In this case, the motion of the binary is fully described with eight
degrees of freedom, in terms of the distance between Haumea and Namaka \( (r) \), the rotation angle of the body itself \( (\phi_i) \), the inclination of the body with respect to the distance between them \( (\theta_i) \), the precession angle of the “equator” of the bodies \( (\psi_i) \), and the angle of rotation of the system \( (\Theta) \). An inertially fixed frame \((\hat{E}_1, \hat{E}_2, \hat{E}_3)\) was defined such as the direction of \( \hat{E}_1 \) is parallel to the direction of the distance of the two bodies and \( \hat{E}_3 \) is parallel to the rotation rate \( (\hat{\Theta}) \) of the system, and the \( \hat{E}_2 \) direction completes the orthonormal system. We also defined a body fixed frame \((\hat{e}_{i1}, \hat{e}_{i2}, \hat{e}_{i3})\) in which the \( \hat{e}_{i1} \) points to the direction of the larger semi-major axis of the body, \( \hat{e}_{i3} \) is parallel to the rotation rate \( (\hat{\phi}_i) \) of the body, and the \( \hat{e}_{i2} \) completes the orthonormal base.

It is our objective in this section to define the equations of motion of the binary, using the Lagrangian formalism. To do this we need to define the mutual potential of the bodies and the kinetic energy of the system. We will start with the potential. Thus, with all the above considerations, the potential function of the system (developed in a second order expansion in the moments of inertia) is defined as: \(^9^{10}\)

\[
V (r, A_1, A_2) = -\frac{G M_1 M_2}{|r|} - \frac{G}{2|r|^3} \left[ M_2 T_r (I_1) + M_1 T_r (I_2) \right] - \frac{3G}{2|r|^5} r \cdot \left[ M_2 A_1^T \cdot I_1 \cdot A_1 + M_1 A_2^T \cdot I_2 \cdot A_2 \right] \cdot r,
\]

where \( G \) is the universal gravitational constant and \( I_i \) is the inertia tensor associated to the body \( i \), \( r \) is the relative position between the centers of mass of the bodies in an inertial frame, and \( A_i \) is the attitude tensor associated to the body \( i \). This matrix transforms a vector from the inertial fixed frame to the body fixed frame. Since the ellipsoids have free motion, this matrix, in a 3-1-3 classical set of rotations, can be written as: \(^15\)

\[
A_i = \begin{bmatrix}
  c\phi_i c\psi_i - s\phi_i c\theta_i s\psi_i & c\phi_i s\psi_i + s\phi_i c\theta_i c\psi_i & s\phi_i s\theta_i \\
  -s\phi_i c\psi_i - c\phi_i c\theta_i s\psi_i & -s\phi_i s\psi_i + c\phi_i c\theta_i c\psi_i & c\phi_i s\theta_i \\
  s\theta_i s\psi_i & -s\theta_i c\psi_i & c\theta_i 
\end{bmatrix}.
\]

This potential is fully dimensional. However, to facilitate the numerical calculation of the problem, it is convenient to define some quantities to nondimensionalize this potential and, consequently, the equations of motion of the binary. Thus, we define a mass-normalized inertia tensor by dividing the inertia tensor of the body by its mass, \( \mathbf{I}_i = \mathbf{I}_i / M_i \). The unity of length of the system is set as the length of the largest semi-major axis of Haumea, \( \alpha_1 = 960 \) km, and the unit of time defined as the inverse of the mean motion at this point, that is, \( \frac{1}{n_1} \), where \( n_1 = \sqrt{\frac{G (M_1 + M_2)}{\alpha_1^3}} \). Finally, the mass of the primaries are reduced mass, such as \( \nu = \frac{M_1}{M_1 + M_2} \), \( 1 - \nu = \frac{M_2}{M_1 + M_2} \). Due to the size of the potential \( V (r, \phi_1, \psi_1, \theta_1, \phi_2, \psi_2, \theta_2) \), it will not be shown here.

To develop the equation for the kinetic energy, we start again relative to the inertial frame. For the relative motion of the centers of mass of the two bodies, the kinetic energy can be written in the form:

\[
T = \frac{1}{2} \sum_{i=1,2} \left( I_{ix} \omega_{ix}^2 + I_{iy} \omega_{iy}^2 + I_{iz} \omega_{iz}^2 \right) + \frac{1}{2} m v \cdot v,
\]
where \( m = \frac{M_1 M_2}{M_1 + M_2} = \nu (1 - \nu) \). As \( \boldsymbol{v} = \dot{r} \hat{r} + r \dot{\Theta} \hat{\Theta} \), we have:

\[
T = \frac{1}{2} \sum_{i=1,2} (I_{ix} \omega_{ix}^2 + I_{iy} \omega_{iy}^2 + I_{iz} \omega_{iz}^2) + \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\Theta}^2.
\]  

(4)

Since the direction of \( \dot{\Theta} \) is parallel to \( \hat{e}_3 \), we also have:

\[
\begin{bmatrix}
\omega_{ix} \\
\omega_{iy} \\
\omega_{iz}
\end{bmatrix} = \begin{bmatrix}
\dot{\psi}_i \sin \theta_i \sin \phi_i + \hat{\phi}_i \cos \phi_i \\
\dot{\psi}_i \sin \theta_i \cos \phi_i - \hat{\phi}_i \sin \phi_i \\
\hat{\phi}_i + \dot{\psi}_i \cos \theta_i
\end{bmatrix} + A_i \begin{bmatrix} 0 \\ 0 \\ \Theta \end{bmatrix},
\]  

(5)

which results in

\[
\begin{bmatrix}
\omega_{ix} \\
\omega_{iy} \\
\omega_{iz}
\end{bmatrix} = \begin{bmatrix}
(\dot{\psi}_i + \Theta) \sin \theta_i \sin \phi_i + \hat{\phi}_i \cos \phi_i \\
(\dot{\psi}_i + \Theta) \sin \theta_i \cos \phi_i - \hat{\phi}_i \sin \phi_i \\
\hat{\phi}_i + (\dot{\psi}_i + \Theta) \cos \theta_i
\end{bmatrix}.
\]  

(6)

Now, we can write the Lagrangian of the system, \( L = T - V \), and derive the equations of motion for the system:

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) = \frac{\partial L}{\partial q_i},
\]  

(7)

where \( \dot{q}_i = \dot{r}, \dot{\phi}_i, \dot{\psi}_i, \dot{\phi}_i, \dot{\Theta} \) and \( q_i = r, \phi_i, \psi_i, \theta_i, \Theta \). Again, due to the size of the expressions for the equations of motion for \( \phi_i, \psi_i, \theta_i, \Theta \), it is not possible to show them here. However, the equation for \( r \) is decoupled from the others, and it is written as follows:

\[
\ddot{r} = r \dot{\Theta}^2 - \frac{1}{m} V_r,
\]  

(8)

where \( V_r = \partial V / \partial r \). As expected due to the decoupling from the remaining equations, the equation for \( r \) is the same as for the planar case. However, since the behavior of the angles inside \( V_r \) is not the same as for the planar case, the evolution of \( r \) also will not be the same.

In previous work,\(^8\) we compared the initial configurations for the binaries, such as long axis of the body 1 and long axis of body 2 in align, short axis align with the long axis, and so on, always for the bodies synchronous. This was possible since they were considered in planar motion. However, in the present case, we have three degrees of freedom for each body, and we can combine the position of the axis of the bodies with the orbital position, such as the value of the initial \( \psi_i \). Then, instead of compare initial configurations, we show, in Figure 1 and Figure 3, the effect of the inclination in the angles of the bodies, as well as in the behavior of \( r \). Although the numerical simulations were made using a non-dimensional set of unities, we show in Figure 1 and Figure 3 the time evolution of the variables in dimensional unities, to facilitate the visualization of the quantities. We can see that the period of rotation of both bodies match with the real one (18.27 days), and a periodic variation in the inclination of the orbit of Namaka, with 1.2 degrees as amplitude. As we took the bodies synchronous, the periods of rotation of both bodies have 3.95 hours, the period of rotation of Haumea. An important observation is about the variation of \( r \). Its variation is equivalent to an orbit with a small eccentricity \( (e = 3.79 \times 10^{-3}) \).

Now, considering that the orbit of Namaka around Haumea has eccentricity \( e = 0.249 \), we need to add to the previous equations of motion the effect of this eccentricity. This can be made by
Figure 1. Time variation of the distance between the primaries, \( r \) (a), and the rotation of Haumea (b). Precession rate of Haumea (c), and time variation of the orbital inclination of Namaka (d).
Figure 2. Rotation of the system (a), rotation of Namaka (b), precession rate of Namaka (c), and time variation of the orbital inclination of Haumea (as seen from Namaka) (d), as a function of the time.
replacing $\dot{\Theta}$, in Equation (6), by

$$
\dot{\Theta} = \frac{n_o}{(1 - e^2)^{3/2}} (1 + e \cos \Theta)^2,
$$

(9)

where $n_o$ is the mean motion of the system. After, we start again the process of derivation of the equations of motion, obtaining the equations for the eccentric case. We disregard all terms above $e^2$. We will not show these equations here, but there are some points to highlight. First, as expected, the time evolution of $r$ is dominated by the orbital motion, consequently, there is no impact on $r$ due to the initial configuration of the system. The second point is a second frequency associated to the variation of the inclination of Namaka. These two points are shown in Figure ???. Another important point is that, since the variation of $r$ is dominated by the orbital motion in the elliptic orbit, and the eccentricity of Namaka is greater than the one caused by the attitude interaction, mentioned before, any initial configurations has no impact in the evolution of $r$. Thus, to build the Elliptic Restricted Full Three-Body Problem, in the next section, it is not necessary to use the information coming from the F2BP. As in the circular inclined case, there is a great number of combinations of initial configuration to compare, as well as stability studies. As this kind of study is not in the core of this work, we intend to do these studies in a future work.

![Graph](image)

Figure 3. Distance from the binaries (a) and the inclination of Namaka (b), as a function of the time.

ELLIPDIC RESTRICTED FULL THREE-BODY PROBLEM (ERF3BP)

In previous work, we developed the equations for the Restricted Full Three-Body Problem RF3BP. The problem is restricted because the third body (the spacecraft) is consider as massless, and is full, because the two primaries are triaxial ellipsoids. In that case, the information about the distance between the primaries and the angular velocity of the system, necessary to the RF3BP, came from the F2BP. Since in the elliptic case this is not necessary, we use the equations of the RF3BP as a base for the ERF3BP, but the evolution of $r$ comes from the Elliptic Restricted Three-Body Problem (ER3BP). All the development of the RF3BP can be found in Reference 8.

In Reference 16, the author, after some coordinates transformation, found a set of equations for the ER3BP. We use the same set of transformation in the RF3BP to obtain the equations for the elliptic problem, and we also use canonical unities, where the distance between the primaries is set to the unity of distance, and the unity of time is the inverse of the mean motion of the system. Also, due to the coordinate transformations, the independent variable is the true anomaly ($f$). We obtained the following set of equations, in the rotating pulsating system:
\[ \ddot{x} - 2\dot{y} = \frac{r}{p} [x - \nu(x + 1 - \nu)R_{j\alpha_1} - (1 - \nu)(x - \nu)R_{j\alpha_2}], \quad (10) \]

\[ \ddot{y} + 2\dot{x} = \frac{r}{p} [y - \nu y R_{j\beta_1} - (1 - \nu)y R_{j\beta_2}], \quad (11) \]

\[ \ddot{z} = \frac{r}{p} [-\nu z R_{j\gamma_1} - (1 - \nu)z R_{j\gamma_2}], \quad (12) \]

where the “dot” represents a derivative with respect to the true anomaly \( f \). \( p = 1 - e^2 \) and \( R_{j\alpha_i}, R_{j\beta_i}, R_{j\gamma_i} \) are elliptical integrals of the third kind, and are solved using the Carlson’s method.\textsuperscript{17}

Since the independent variable is the true anomaly, we need a differential equation for the time \( t \)

\[ \dot{t} = \frac{r^2}{p^{1/2}}, \quad (13) \]

and, finally, the differential equation for \( r \) as a function of the independent variable \( f \):\textsuperscript{16}

\[ \ddot{r} = 2\dot{r}^2 r + r \left( 1 - \frac{r}{p} \right) = -\frac{2}{p} r^3 + \frac{3}{p} r^2 - r \quad (14) \]

Some considerations need to be made about this model. The first one is that, as in the ER3BP, there is no Jacobi integral in this problem. The second characteristic is that this problem, also like the pure elliptic model, has no natural families of periodic orbits, in the sense of the circular problem.\textsuperscript{16} In this case, any periodic solution must have a period which is an integer multiple of the period of the system \((2\pi \text{ or a multiple of it})\). If this condition is respected, a periodic orbit in this model will be periodic with respect to the inertial axes as well the rotating axes.

In order to show a set of quasi-periodic orbits (surrounding a periodic one), we build a Poincaré section, presented in Figure 4. This type of figure can be used to identify stable regions inside the system, fast and efficiently. With the size and position of the quasi-periodic orbits, we are able to apply the method of the integral of the acceleration,\textsuperscript{13,14} using a “full” model,\textsuperscript{8} in order to calculate the \( \Delta V \) required for the DV APO. We use the integral technique, which takes into account all the perturbations, and is applied to each component of the total acceleration, to find the total velocity added to the spacecraft by the disturbers, whose effect is turning the orbits in non-Keplerian ones. In other words, the method can show how much \( \Delta V \) is required to return the orbit to the same one of the first period. In that sense, this orbit becomes a closed non-Keplerian orbit.

**CONCLUSION**

We developed a model of the Full Two Body problem, with free motion of the binaries (non planar case) and taking into account the eccentricity of the orbit. Our numerical simulations showed that the variation of \( r \) due to the free motion is equivalent to a small eccentricity added to the originally circular orbit. However, when included to the model moderate value of orbital eccentricity, the orbital motion suppress any influence to the evolution of \( r \) coming from the free motion. The model of the ERF3BP presented here is an interesting tool for mission analysis, since all methods used in the R3BP can be used with this model. Furthermore, stable regions found with this model around Haumea can be used for future missions to this system.
Figure 4. Poincaré section generated by the ERF3BP.

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REFERENCES


