Leveraging Discrete Variational Mechanics to Explore the Effect of an Autonomous Three-Body Interaction Added to the Restricted Problem

Natasha Bosanac, Kathleen C. Howell, and Ephraim Fischbach

Abstract With recent improvements in ground and space-based telescopes, a large number of binary systems have been observed both within the solar system and beyond. These systems can take the form of asteroid pairs or even binary stars, with each component possessing a similar mass. In this investigation, periodic motions near large mass ratio binaries are explored using the circular restricted three-body problem, which is modified to include an additional three-body interaction. Discrete variational mechanics is leveraged to obtain periodic orbits that exhibit interesting shape characteristics, as well as the corresponding natural parameters. Shape characteristics and structural changes are explained using the stability and existence of equilibrium points, enabling exploration of the effect of an additional three-body interaction and conditions for reproducibility in a natural gravitational environment.

1 Introduction

With recent improvements in ground and space-based telescopes, a large number of binary systems have been observed both within the solar system and beyond. These systems can take the form of asteroid pairs or even binary stars (Margot et al. 2002; Raghavan et al. 2006). In each of these examples, however, the system is typically described by larger values of the mass ratio than the Sun-planet or planet-moon combinations commonly examined within the solar system. Furthermore, limitations on the accuracy and capability of various observational techniques may

N. Bosanac (✉) • K.C. Howell
School of Aeronautics and Astronautics, Purdue University, 701 W. Stadium Ave., West Lafayette, IN 47907, USA
e-mail: nbosanac@purdue.edu; howell@purdue.edu

E. Fischbach
Department of Physics and Astronomy, Purdue University, 525 Northwestern Ave., West Lafayette, IN 47907, USA
e-mail: ephraim@purdue.edu

G. Gómez, J.J. Masdemont (eds.), *Astrodynamics Network AstroNet-II*, Astrophysics and Space Science Proceedings 44,
DOI 10.1007/978-3-319-23986-6_1

result in an uncertainty in the mass ratio of the binary, significantly impacting the potential motions of a nearby small body orbiting within a chaotic system.

In this investigation, the motion of a small body, such as an exoplanet or moonlet, is examined using the Circular Restricted Three-Body Problem (CR3BP) as a foundation. For a binary star system, the CR3BP may provide a simple, autonomous approximation to a dynamical environment that is otherwise influenced by additional forces such as gravitational radiation. In addition, near a binary asteroid, the restricted problem can provide an approximation to complex higher-order gravitational effects for preliminary analysis of the motion of a spacecraft or a moonlet (Chappaz 2015). When using the CR3BP to represent the dynamics in either form of binary system, the motion of a comparatively small body is governed by the existence of periodic orbits, which form an underlying dynamical structure. For instance, stable orbits may aid in identifying potential motions of an exoplanet that persist for a long time interval near a binary star system. Alternatively, unstable orbits may be valuable in studying natural transport of matter near a binary asteroid. In either case, periodic orbits offer useful information in identifying the possible motions near large mass ratio binary systems.

An alternative dynamical model for the binary system is also derived based on the CR3BP, but extended to incorporate an additional autonomous term in the potential function, producing a Modified Circular Restricted Three-Body Problem (MCR3BP) (Bosanac 2012). Given the absence of experimental data gathered within the vicinity of a binary star, for example, it is possible that the gravitational field within this system might not be accurately modeled solely using pairwise gravitational forces. Thus, in this investigation, the impact of an additional three-body interaction is examined. Many-body forces are well-established in nuclear physics as critical contributions to the accurate modeling of a force field on the atomic scale (Fischbach 1996). The motion of a small body orbiting a binary object, however, serves as an interesting larger scale application for studying the impact of a three-body interaction in gravitational systems. This three-body interaction is added to the potential function of the CR3BP and is assumed to possess a form that is inversely proportional to the product of the distances between all three bodies in the system: when the three bodies are in a tighter configuration, the three-body interaction is stronger. A constant k is used to scale the strength of this additional contribution relative to the gravitational field (Bosanac 2012). In this modified dynamical environment, families of orbits are inherited from the CR3BP, but may undergo structural changes for a sufficiently disturbing three-body interaction, causing interesting and potentially new periodic motions to emerge.

Interesting behaviors, identified in previous work via dynamical systems theory techniques, are explored using discrete variational mechanics (Bosanac et al. 2015). Although this technique has typically been used in astrodynamics to identify optimal paths under a control force, the underlying formulation is applied to the computation of natural periodic motions that resemble a given reference path, as well as the corresponding values of the system parameters, μ and k (Moore 2011). The constrained optimization problem formulated using discrete variational mechanics potentially overcomes some of the sensitivity associated with using continuous shooting techniques to compute motions in a chaotic system where the natural parameters are not accurately known. This methodology is leveraged to compute periodic orbits at various values of the system natural parameters, μ and k, that exhibit desired shape characteristics (Bosanac et al. 2015). By reproducing a motion of interest in the MCR3BP, discrete variational mechanics enables the exploration of the effect of a three-body interaction, as well as the identification of any unique solutions that may emerge. Although the additional autonomous term examined in this investigation is assumed to take the form of a three-body interaction, a similar analysis can be performed for autonomous forces that are derivable from a potential function, such as a time-averaged quantity or a higher-order gravitational term for a body fixed in a given coordinate frame.

2 Dynamical Models

The dynamical environment in the vicinity of a binary system is modeled using the CR3BP as a foundation. This dynamical model reflects the motion of a massless particle under the influence of the point-mass gravitational attractions of two primaries. To study the effect of a three-body interaction, an autonomous term is added to the pairwise gravitational interactions in the potential function in the CR3BP (Bosanac 2012). The resulting model, denoted the MCR3BP, has been introduced and explored by Bosanac et al. (2013) The resulting autonomous potential in the MCR3BP influences the equations of motion and still possesses an integral of motion. Accordingly, families of periodic orbits still exist throughout various regions of the configuration space (Bosanac et al. 2015).

2.1 Circular Restricted Three-Body Problem

When modeling the dynamics in the vicinity of a binary system using the CR3BP, a body of interest, P_3, is examined as it orbits near the larger and smaller primaries, P_1 and P_2. Each of these bodies, P_i, possesses a mass m_i and remains fixed in a right-handed coordinate frame, $\hat{x}\hat{y}\hat{z}$, that rotates with the primaries and is centered at the system barycenter. This rotating frame is oriented such that the $\hat{z}$ axis is oriented parallel to the angular momentum of the primaries as they orbit their mutual barycenter. Furthermore, nondimensional quantities are used to enable comparisons between different systems of similar mass ratios. In particular, length quantities are normalized such that the distance between the two primaries is equal to unity, while time quantities are nondimensioned to ensure that the mean motion of the primaries is also equal to unity. A characteristic mass quantity is also introduced and set equal to the sum of the masses of the primaries. Accordingly, P_1 and P_2 possess nondimensional mass values equal to $(1-\mu)$ and μ, respectively (Szebehely 1967). Using this normalization, the two primaries, P_1 and P_2, remain fixed along the $\hat{x}$-axis of the rotating frame at the locations $(-\mu, 0, 0)$ and $(1-\mu, 0, 0)$, respectively. When modeled using the CR3BP, the equations of motion of P_3, located at the coordinate (x, y, z) relative to the barycenter, are compactly written in the rotating frame as:

$$\ddot{x} - 2\dot{y} = \frac{\partial U^*}{\partial x}, \qquad \ddot{y} + 2\dot{x} = \frac{\partial U^*}{\partial y}, \qquad \ddot{z} = \frac{\partial U^*}{\partial z} \tag{1}$$

where the pseudo-potential function, $U^* = \frac{1}{2}(x^2+y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}$, and the distances between P_3 and each of the two primaries are $r_1 = \sqrt{(x+\mu)^2 + y^2 + z^2}$ and $r_2 = \sqrt{(x-1+\mu)^2 + y^2 + z^2}$. The autonomous psuedo-potential function, U^*, admits an energy integral, labeled the Jacobi constant, equal to $C_J = 2U^* - \dot{x}^2 - \dot{y}^2 - \dot{z}^2$ (Szebehely 1967).

2.2 Modified Circular Restricted Three-Body Problem

Using the CR3BP as a foundation, the equations of motion of P_3 in the MCR3BP are derived using the derivatives of a potential function. In particular, the scalar pseudo-potential governing the motion of P_3 is assumed to consist of the following terms when formulated in the $\hat{x}\hat{y}\hat{z}$ frame that rotates with the primaries:

$$U_k^* = \underbrace{\frac{1}{2}(x^2+y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}}_{\text{pairwise potential}} + \underbrace{\frac{k}{r_1 r_2}}_{\text{three-body potential}} \tag{2}$$

where k is a constant that scales the potential term corresponding to the three-body interaction. Since the distance between the two primaries is equal to a constant value of unity, only the variables r_1 and r_2 appear in the denominator of the three-body potential term. Furthermore, the magnitude and sign of k can be selected as either positive, negative or zero. A positive value of k corresponds to an attracting three-body interaction, while a negative value of k produces a repulsive interaction. A value of k equal to zero, however, causes the dynamics in the MCR3BP to reduce to those of the CR3BP. The particular values of k are constrained within the range $k = [-0.20, 0.70]$ for numerical and dynamical reasons. The lower limit of $k = -0.20$ produces a repulsive three-body interaction that is approximately half as strong as the natural gravitational environment, while an upper limit of $k = 0.70$ reduces numerical sensitivities during integration. Using derivatives of the potential function as defined in (2), the equations of motion for P_3 are compactly written as:

$$\ddot{x} - 2\dot{y} = \frac{\partial U_k^*}{\partial x}, \qquad \ddot{y} + 2\dot{x} = \frac{\partial U_k^*}{\partial y}, \qquad \ddot{z} = \frac{\partial U_k^*}{\partial z} \tag{3}$$

Since the three-body potential term is formulated to be autonomous, a constant energy integral still exists and is equal to $C_k = 2U_k^* - \dot{x}^2 - \dot{y}^2 - \dot{z}^2$. As the value of k approaches zero, this energy constant reduces to the Jacobi constant in the CR3BP.

2.3 Equilibrium Points

Particular solutions in the form of equilibrium points offer insight into the nonlinear dynamical systems modeled in the CR3BP and the MCR3BP. When the equations of motion of the CR3BP are formulated in the rotating frame, five equilibrium points exist, as depicted via green dots in Fig. 1 (top) for a system with a mass ratio of $\mu = 0.30$, and are labeled L_i, for $i = 1, 2, 3, 4, 5$. The presence of a three-body interaction, indicated by nonzero values of k, impacts the location, stability and even existence of these equilibrium points (Bosanac et al. 2015). At the sample mass ratio of $\mu = 0.30$, the planar equilibrium points are numerically computed for values of k within the range $k = [-0.20, 0.70]$ and plotted in Fig. 1 (top) with blue dots corresponding to positive values of k, and red dots indicating the location of the L_i for negative values of k. For an increasingly attractive three-body interaction, the collinear equilibrium points are displaced farther from the primaries. The converse is true for an increasingly repulsive three-body interaction. In either case, L_4 and L_5 are no longer located at the vertices of equilateral triangles formed with the two primaries. At a critical negative value of k equal to -0.1985, however, Fig. 1 (top)

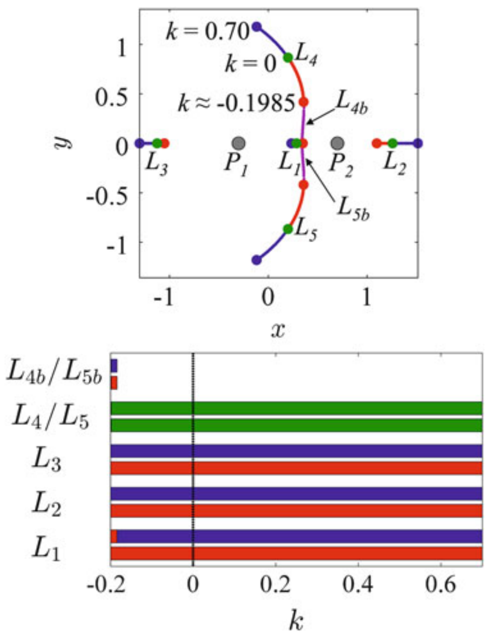

Fig. 1 Equilibrium points in the MCR3BP for $\mu = 0.30$. *Top figure*: Location of planar equilibrium points in the CR3BP (*green*) and the MCR3BP for $\mu = 0.30$: $k > 0$ (*blue dots*), $k < 0$ (*red dots*), and $-0.1985 < k < -0.1839$ (*purple*). *Bottom figure*: Stability of the two planar modes for each equilibrium point as a function of k for $\mu = 0.30$, with purely oscillatory modes (*blue*), real eigenvalues (*red*), or complex conjugate eigenvalues (*green*)

reveals that L_4 and L_5 disappear. Furthermore, purple dots in Fig. 1 (top) depict the existence of two additional equilibrium points, labeled L_{4b} and L_{5b}, that exist for large negative values of k within the range $k = [-0.1985, -0.1839]$ as identified by Douskos (2014).

To explain the formation of L_{4b} and L_{5b}, an overview of the stability of each equilibrium point is useful. In Fig. 1 (bottom), a qualitative measure of the linear stability of each equilibrium point is displayed for a mass ratio equal to $\mu = 0.30$. In this figure, the two horizontal bars associated with each equilibrium point are colored such that blue portions indicate an oscillatory mode, red sections correspond to the presence of a stable and unstable pair of real modes, and green depicts complex conjugate eigenvalues. While the qualitative stability of the planar modes of L_2 and L_3 remain unchanged across the examined range of values of k, L_1 undergoes a stability change at $k = -0.1839$. In particular, the oscillatory mode that corresponds to the existence of the planar Lyapunov family near L_1 becomes real. As explored in previous work by the authors, this stability change in L_1 is accompanied by a pitchfork bifurcation that produces the equilibrium points L_{4b} and L_{5b}. An increasingly repulsive three-body interaction causes these equilibrium points to evolve away from the x-axis in the rotating frame, as depicted in Fig. 1 (top) via purple dots. As k approaches the critical value of $k = -0.1985$, L_4 and L_{4b} meet and disappear.

Prior to examination of the reproducibility of periodic orbits that exhibit a given behavior, a summary of the in-plane stability and existence of each equilibrium point over various large values of μ is useful. Figure 2 portrays the qualitative planar stability of L_1 and L_2 for negative values of k, represented on the horizontal axis, and values of μ in the range $\mu = [0.10, 0.30]$ on the vertical axis. Blue points in this two-dimensional space indicate that the associated equilibrium point possesses one pair of real modes and one pair of oscillatory modes, as inherited from the CR3BP. Red

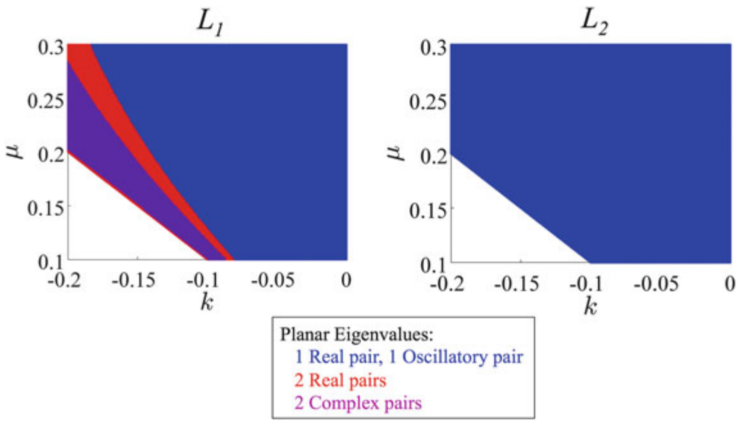

Fig. 2 In-plane stability of L_1 and L_2 for combinations of (k, μ) within the range $\mu = [0.10, 0.30]$ and $k = [-0.20, 0.0]$

points correspond to values of the system natural parameters at which an equilibrium point possesses two real pairs of modes, indicating that motion in its vicinity may not remain nearby indefinitely. Purple regions identify values of μ and k where the planar modes are comprised of two pairs of complex conjugate eigenvalues. Any white regions of Fig. 2 locate combinations of μ and k where the associated equilibrium point does not exist. Analysis of the left plot in Fig. 2 reveals that at each value of μ between 0.10 and 0.30, L_1 undergoes a stability change. As μ is decreased towards 0.10, this stability change occurs with a less negative value of k. Beyond a sufficiently large negative value of k, L_1 disappears, as indicated by the white triangular region. In comparison to the white region on the rightmost plot of Fig. 2, L_2 also apparently disappears at the same critical value of k as L_1. However, the blue region that spans each value of the natural parameters for which L_2 exists reveals that no stability change occurs prior to its disappearance. The qualitative stability of these equilibrium points can influence the underlying structure of the entire system, potentially affecting additional types of steady-state solutions, including periodic and quasi-periodic motions. Such an impact on the dynamical environment in a binary system under the influence of a three-body interaction may be examined using both dynamical systems theory and discrete variational mechanics (Bosanac et al. 2015).

2.4 Periodic Orbits

Since the MCR3BP produces an autonomous dynamical fields when formulated in the rotating frame, families of planar, periodic orbits exist within the configuration space. These continuous families of orbits, characterized by motion that repeats after a minimal period, T, form an underlying structure within the phase space. In particular, stable orbits attract or bound nearby motion, while unstable orbits cause nearby trajectories to depart (Contopoulos 2002). Stable periodic orbits are invaluable in both trajectory design and astronomy applications, as they are typically surrounded by families of quasi-periodic orbits. Unstable orbits, however, are associated with stable and unstable manifold structures that may describe natural transport within the vicinity of a binary system. Regardless of their stability, periodic orbits can also be classified by their direction of motion relative to one or both primaries. In particular, a trajectory that is prograde at some instant of time possesses a state with an angular momentum vector, relative to either primary, that is oriented parallel to the $\hat{z}$-axis (Bosanac 2012). Conversely, a state that is retrograde possesses an angular momentum vector that is oriented anti-parallel to the $\hat{z}$-axis. By examining the periodic orbits within various regions of configuration space, the effect of an additional autonomous force contribution on the underlying dynamical structures near a binary system can be investigated.

3 Constrained Optimization Using Discrete Variational Mechanics

To explore the effect of an additional autonomous force contribution, discrete variational mechanics is used to compute the natural parameters of the system for which periodic orbits exhibit a desired shape characteristic. By determining the range of values of μ and k for which a desired motion is possible, the reproducibility of the effect of a three-body interaction in the natural gravitational environment is examined. Such analysis reveals whether the effect of this additional term is unique, and enables identification of periodic motions that otherwise do not exist in the CR3BP. Unlike collocation or multiple-shooting, which require that the dynamics of a continuous-time system be exactly satisfied at a discrete set of nodes or along multiple trajectory arcs, discrete variational mechanics begins with a discretization of the action integral (Ober-Blobaum et al. 2011). Using the variation of the discrete action at a finite set of nodes, a discrete version of Hamilton's principle is used to constrain the motion along a path (Moore 2011). This methodology does not require sequential integration, and may mitigate the difficulties associated with a poor initial guess. To demonstrate this process, some background on variational mechanics is presented, followed by a discussion of its numerical implementation in computing natural paths in the MCR3BP.

3.1 Variational Principles for Continuous Time Systems

Variational principles in Lagrangian mechanics are first summarized for continuous time systems. Consider an autonomous mechanical system described by the generalized coordinates q and generalized velocities $\dot{q}$. The associated continuous Lagrangian, $L(q(t), \dot{q}(t))$, is then integrated along a path from a time $t_0 = 0$ to a subsequent time t to construct the action integral, defined as follows (Greenwood 1988):

$$A = \int_{t_0}^{t} L(q(t), \dot{q}(t))dt \tag{4}$$

In a holonomic system, a natural path, $q(t)$, results in a stationary action integral with respect to path variations, given fixed endpoints (Lanczos 2012). This statement, known as Hamilton's principle, can be mathematically expressed as:

$$\delta A = \delta \int_{t_0}^{t} L(q(t), \dot{q}(t))dt = \int_{t_0}^{t} \left(\frac{\partial L}{\partial q} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \right) \delta q dt = 0 \tag{5}$$

The solution to this equation is written as follows:

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0 \tag{6}$$

Thus, a natural path in a continuous time autonomous and holonomic system must satisfy the well-known Euler-Lagrange equations (Greenwood 1988).

3.2 Discrete Variational Mechanics

The continuous variational concepts in Lagrangian mechanics are straightforwardly modified to accommodate discrete time systems, as summarized by Marsden and West (2001). Since numerical integration or observation of the motion of an object inherently results in a discretely-sampled path, a discrete time Lagrangian formulation is valuable. Consider, for instance, a continuous path $q(t)$ discretely sampled N times at constant time intervals of length h to produce a discrete path $\tilde{q}$ (Marsden and West 2001). While the generalized coordinates along this discrete path are denoted $q_i = \tilde{q}(ih)$, velocities can be replaced by finite difference approximations such as $\dot{q}_i \approx \frac{(q_{i+1}-q_i)}{h}$. A discrete time Lagrangian can also be defined for autonomous systems as $L_d(q_i, q_{i+1}, h)$, approximating the integral of the continuous time Lagrangian, $L(q(t), \dot{q}(t))$, over the i-th time interval of length h as the system travels from q_i to q_{i+1} (Moore 2011). Once this integral is approximated numerically using a quadrature rule, a discrete action is constructed as the summation of $L_d(q_i, q_{i+1}, h)$ over $N-1$ time intervals, such that:

$$A_d = \sum_{i=0}^{N-2} L_d(q_i, q_{i+1}, h) \approx \int_{t_0}^{t} L(q(t), \dot{q}(t))dt \tag{7}$$

By taking the variation with respect to path variables, assuming fixed points and leveraging summation by parts, a discrete time version of Hamilton's principle can be mathematically written as:

$$\delta A_d = \delta \sum_{i=0}^{N-2} L_d(q_i, q_{i+1}, h) = \sum_{i=1}^{N-2} \left[\left(\frac{\partial L_d(q_i, q_{i+1}, h)}{\partial q_i} + \frac{\partial L_d(q_{i-1}, q_i, h)}{\partial q_i}\right) \delta q_i\right] = 0 \tag{8}$$

The solution to this expression supplies the discrete Euler-Lagrange equations, which must be satisfied across each time interval $[t_i, t_{i+1}]$:

$$\frac{\partial L_d(q_i, q_{i+1}, h)}{\partial q_i} + \frac{\partial L_d(q_{i-1}, q_i, h)}{\partial q_i} = 0 \tag{9}$$

for $i = [1, N-2]$. Although this solution is only true for holonomic autonomous systems with no external forcing, modifications to the discrete Euler-Lagrange equations have been presented and used by numerous authors in previous works to enable extension to forced and nonautonomous systems (Moore 2011; Marsden and West 2001).

3.3 Formulation of Constrained Optimization Problem

The discrete variational principles can be used in a constrained optimization problem to find a natural periodic orbit that resembles a given reference path. In a continuous time system, the objective function to be minimized can be written as the integral of a cost function, C, from time $t_0 = 0$ to t along the path such that $J(q, \dot{q}) = \int_{t_0}^{t} C(q(t), \dot{q}(t))dt$. This functional is transformed into a finite-dimensional objective function for a discretely-sampled path by replacing the integral with a summation of discrete cost functions evaluated along the $N-1$ time intervals:

$$J_d(\tilde{q}) = \sum_{i=0}^{N-2} C_d(q_i, q_{i+1}, h) \approx \int_{t_0}^{t} C(q(t), \dot{q}(t))dt \qquad (10)$$

where the discrete cost function, $C_d(q_i, q_{i+1}, h)$, is constructed using an appropriate quadrature rule to compute the numerical integral of the continuous cost function C over the i-th time interval (Moore 2011). Since the optimization problem is used to compute a periodic orbit that resembles a given reference path, a continuous cost function is defined using the distance between a point located at $q = (x, y)$ and the corresponding point along the reference path, $q_{ref} = (x_{ref}, y_{ref})$, i.e., $C(x, y) = (x - x_{ref})^2 + (y - y_{ref})^2$. The discrete cost function is then calculated via application of the midpoint rule, and summed over each time interval to produce the objective function in Eq. (10).

Prior to implementing the constrained optimization problem, an expression for the discrete Lagrangian must be defined. This approximation, $L_d(q_i, q_{i+1})$, to the integral of the exact Lagrangian over the i-th time interval is calculated using the well-known midpoint rule such that:

$$L_d(q_i, q_{i+1}, h) = hL\left(\frac{q_i + q_{i+1}}{2}, \frac{q_{i+1} - q_i}{h}\right) \qquad (11)$$

The continuous Lagrangian used to evaluate this expression for the MCR3BP is written as:

$$L(x, y, \dot{x}, \dot{y}) = \frac{1}{2}(\dot{x} - y)^2 + \frac{1}{2}(\dot{y} + x)^2 + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \frac{k}{r_1 r_2} \qquad (12)$$

As the value of k approaches zero, this function reduces to the Lagrangian of the CR3BP. Once this Lagrangian is used to compute $L_d(q_i, q_{i+1})$ via the midpoint rule, it can be differentiated and used in Eq. (9) to form the discrete Euler-Lagrange equations, which must be satisfied at each interior node along the discrete path.

A constrained optimization problem is used to compute a discretely-sampled path that minimizes the objective function, while also satisfying the dynamics of the system (Bosanac et al. 2015). Accordingly, the discrete Euler-Lagrange equations must be satisfied at each interior node. These constraints are met by modifying the planar position variables $q_i = (x_i, y_i)$ at each node, as well as the time interval h and the natural parameters, μ and k. Since there are N discrete nodes along each path, the resulting minimization problem can affect $2N + 3$ design variables and is summarized as:

$$\min \qquad J_d(\tilde{q}) = \sum_{i=0}^{N-2} C_d(q_i, q_{i+1}, h)$$

subject to the constraints from Eq. (9), reflecting the system dynamics:

$$\frac{\partial L_d(q_i, q_{i+1}, \mu, k, h)}{\partial q_i} + \frac{\partial L_d(q_{i-1}, q_i, \mu, k, h)}{\partial q_i} = 0 \qquad\qquad i = 1, \ldots, N-2$$

Additional constraints are also applied to enforce periodicity along the discrete path via position and momentum continuity. Furthermore, linear equality constraints can also be applied to the values of h, μ, or k to enforce the orbital period or properties of the dynamical system. This optimization problem, constructed using discrete variational mechanics, is used in the sequential quadratic programming algorithm available in MATLAB's *fmincon* routine to compute a natural path that exhibits a desired motion as well as the corresponding system natural parameters.

4 Exploring the Effect of an Additional Autonomous Force Contribution

The use of discrete variational mechanics enables analysis of the effect of an additional three-body interaction via the computation of orbits that exhibit desired shape characteristics as well as the corresponding values of the natural parameters, μ and k. Previous work by the authors has identified structural changes in the evolution of various families of periodic orbits, resulting in identification of interesting behaviors that exist under the influence of a three-body interaction (Bosanac et al. 2015). These motions of interest have been isolated and are explored in this investigation. Using discrete variational mechanics, such behaviors are reproduced for various values of μ and k. The existence of a desired shape characteristic is compared to the qualitative stability of the equilibrium points, potentially aiding the identification of unique motions under the influence of a three-body interaction. The example explored in this investigation involves interesting shape characteristics that emerge along a family of large retrograde circumbinary orbits.

4.1 Retrograde Circumbinary Orbits

If observational data suggests that an exoplanet, for example, follows a circumbinary orbit in a binary star system, a simply-periodic family of retrograde orbits that exists in the exterior region may be of interest. For this family of orbits, the effect of a three-body interaction has been explored using composite stability representations in previous work by the authors (Bosanac et al. 2015). In the CR3BP, orbits in the retrograde circumbinary family evolve from large and circular paths to those existing closer to the primaries and exhibiting loops in the vicinity of L_4 and L_5 as displayed in Fig. 3a (Bosanac et al. 2014). Several members of this family are plotted in black in the rotating frame, with the arrow indicating direction. Gray filled circles locate the primaries, while red diamonds indicate the equilibrium points. As the orbital period is increased further, additional loops form near these equilibrium points in a fractal manner. For nonzero values of k in the MCR3BP, L_4 and L_5 are shifted in configuration space and no longer exist at the vertices of equilateral triangles. Accordingly, the loops that form along orbits in this family will also be shifted in configuration space. Beyond a critical value of k, however, the loops along the retrograde exterior orbits exhibit 'pointed tips' that evolve towards L_1, as depicted in Fig. 3b. To investigate the significant changes in the physical configurations of the orbits in the retrograde exterior family, retrograde exterior orbits with 'pointed tips' are reproduced at various values of the natural parameters, μ and k, using discrete variational mechanics. These results are then visualized in a (μ, k) two-dimensional space to determine the correlation between the stability of L_1 and the existence of this interesting behavior. By examining the values of μ and

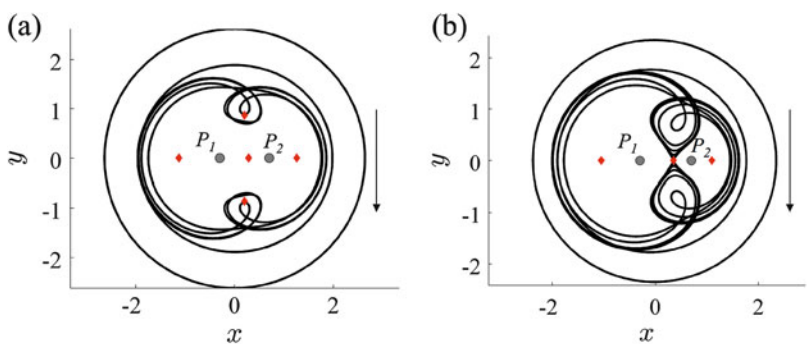

Fig. 3 Retrograde exterior family for (**a**) $\mu = 0.30$ and $k = 0.0$, and (**b**) $\mu = 0.30$ and $k = -0.20$

k at which an orbit resembling a given reference path exists, requirements on the dynamical environment for reproducibility can be determined. These bounds on the values of μ and k are valuable in determining if such motion exists only under the influence of an additional three-body interaction.

Retrograde circumbinary orbits that exhibit ‘pointed tips’ are recreated for various values of the system natural parameters using discrete variational mechanics to explore the effect of a three-body interaction. Consider the retrograde exterior orbits depicted in Fig. 4, existing in the MCR3BP for $\mu = 0.30$ and $k = -0.20$, and an orbital period of $T = 26.26$ nondimensional time units. Each primary is identified via a gray filled circle, with equilibrium points located using red diamonds. This orbit, which exhibits the motion of interest, is discretized using 600 nodes and used as a reference path in the constrained optimization problem formulated using discrete variational mechanics (Bosanac et al. 2015). Periodic orbits exhibiting a similar behavior are then computed for various values of μ sampled within the range $\mu = [0.10, 0.30]$. For each value of the mass ratio, sampled within the given range, the minimization problem is solved using sequential quadratic programming in Matlab’s *fmincon* for a single, unconstrained value of k. Then, the value of k is constrained and successively increased (or decreased) away from this initial value of k until the minimization problem either cannot be solved or produces an orbit that exhibits significantly different behavior, identified via an uncharacteristically large value of the cost function. The resulting process, applied to the reproduction of retrograde circumbinary orbits with the desired characteristics, recovers orbits within a specific region of the two-dimensional space of the natural parameters, μ and k, as displayed in Fig. 5. In this plot, each black point in (k, μ) space indicates the existence of an orbit which exhibits loops with ‘pointed tips’ that approach L_1, with two sample orbits at distinct values of μ and k displayed in the margins to confirm the recovery of the desired motion. Using Fig. 5 as a reference, the desired motion only appears to exist for negative values of k, i.e., when the three-body interaction is repulsive. However, to explore the specific nonzero values of k bounding this region, additional analysis, using concepts from dynamical systems theory, is required.

To explain the existence of ‘pointed tips’ that occur along the retrograde exterior family for a repulsive three-body interaction, a comparison to the stability of L_1 is

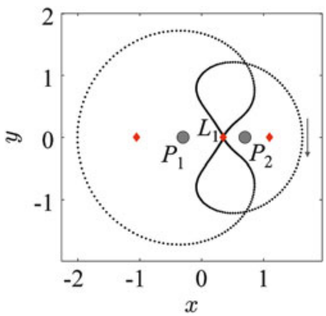

Fig. 4 Reference retrograde exterior orbit for $\mu = 0.30$ and $k = -0.20$, exhibiting ‘pointed tips’

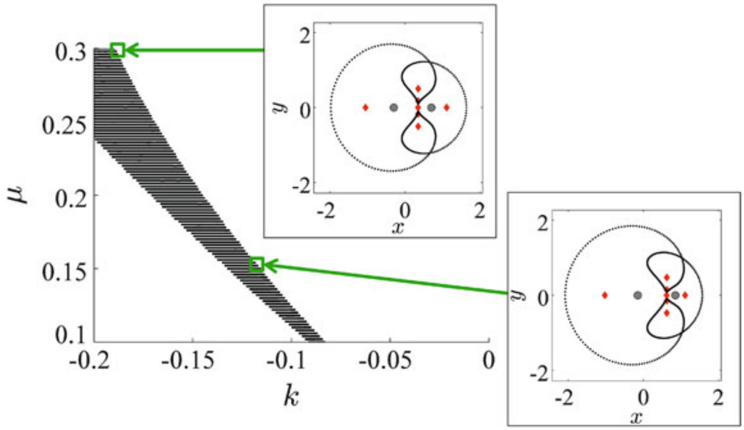

Fig. 5 Existence of retrograde circumbinary orbits with loops that exhibit 'pointed tips' for combinations of k and μ that are indicated via *black points*. Sample orbits are displayed in the margins

valuable. Recall from Fig. 2 that a straightforward stability analysis for L_1 reveals a change in stability for all values of μ within the examined range of $\mu = [0.10, 0.30]$. In fact, for sufficiently negative values of k, the pair of eigenvalues corresponding to the planar oscillatory mode inherited from the CR3BP undergoes a stability change, resulting in L_1 possessing only stable and unstable modes, and causing the disappearance of the well-known L_1 Lyapunov family that exists at all values of μ in the CR3BP. These changes to the equilibrium points, their manifolds and, therefore, the underlying structure of the MCR3BP, influence the foundational dynamical environment near a binary system. For instance, this change in the stability of L_1 may cause the observed structural change along the retrograde exterior family of orbits. The combinations of μ and k at which this family of orbits exhibits 'pointed tips' that evolve towards L_1, as depicted in Fig. 5, are overlaid on the qualitative stability of L_1 using the same color scheme as in Fig. 2: blue points indicate the existence of a planar oscillatory mode, while red regions indicate only real stable/unstable modes, and purple shading corresponds to two pairs of complex conjugate eigenvalues indicating spiraling motion towards and away from the equilibrium point. The resulting plot, displayed in Fig. 6, features black points for each combination of μ and k at which the observed motion exists. Using this figure as a reference, this motion of interest appears to exist only when L_1 possess no oscillatory modes within the plane of motion of the primaries. Such stability properties are not characteristic of L_1 at any value of the mass ratio in the CR3BP. Furthermore, since this black region does not encompass $k = 0$ for any large values of mass ratio, the effect of the three-body interaction on these retrograde exterior orbits at large negative values of k may not be exactly reproduced within the CR3BP, but, rather, mimicked. Accordingly, such motion may be a unique effect of the three-body interaction examined within this investigation.

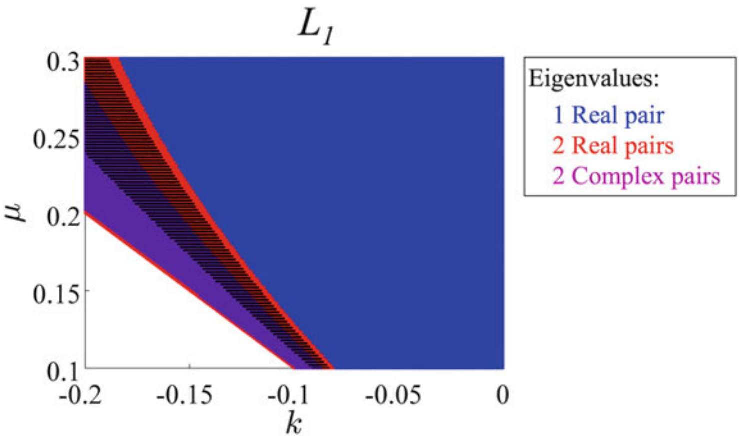

Fig. 6 Existence of retrograde circumbinary orbits with loops, indicated via *black points*, overlaid on a plot of the qualitative stability of L_1

5 Summary

The influence of an additional autonomous force contribution, in the form of a three-body interaction, on the natural gravitational environment of the circular restricted three-body problem is examined using discrete variational mechanics. Previous works by the authors have used stability analysis and dynamical systems theory to study the evolution of families of orbits and identify interesting motions that occur under the influence of a three-body interaction. In this investigation, the existence of these motions of interest is examined in the MCR3BP, as well as their potential for reproducibility in a natural gravitational environment. Discrete variational mechanics is used to overcome the inherent sensitivity in using a shooting method to compute periodic orbits and to enforce the reproduction of desired shape characteristics. The resulting constrained optimization problem enables the computation of periodic orbits exhibiting a particular behavior and the associated bounds on the values of μ and k. A straightforward comparison to the qualitative stability of the equilibrium points in the MCR3BP facilitates explanation of the existence of these interesting orbits and a determination of their uniqueness. Accordingly, dynamical systems theory and discrete variational mechanics are leveraged to explore the effect of a three-body interaction on the dynamics in the restricted problem.

Acknowledgements The authors wish to acknowledge support from the Zonta International Amelia Earhart Fellowship during this work, as well as from the School of Aeronautics and Astronautics at Purdue University.

References

Bosanac, N.: Exploring the influence of a three-body interaction added to the gravitational potential function in the circular restricted three-body problem: a numerical frequency analysis. M.S. Thesis, School of Aeronautics and Astronautics, Purdue University, West Lafayette (2012)

Bosanac, N., Howell, K.C., Fischbach, E.: Exploring the impact of a three-body interaction added to the gravitational potential function in the restricted three-body problem. In: 23rd AAS/AIAA Space Flight Mechanics Meeting, Hawaii (2013)

Bosanac, N., Howell, K.C., Fischbach, E.: Stability of orbits near large mass ratio binary systems. Celest. Mech. Dyn. Astron. **122**(1), 27–52 (2014)

Bosanac, N., Howell, K.C., Fischbach, E.: A natural autonomous force added in the restricted problem and explored via stability analysis and discrete variational mechanics. In: AAS/AIAA Space Flight Mechanics Meeting, Williamsburg (2015)

Chappaz, L.: Exploration of bounded motion near binary systems comprised of small irregular bodies. Celest. Mech. Dyn. Astron. (2015). doi:10.1007/s10569-015-9632-5

Contopoulos, G.: Order and Chaos in Dynamical Astronomy. Springer, Berlin (2002)

Douskos, C.N.: Effect of three-body interaction on the number and location of equilibrium points of the restricted three-body problem. Astrophys. Space Sci. **356**(2), 251–268 (2014)

Fischbach, E.: Long-range forces and neutrino mass. Ann. Phys. **247**, 213–291 (1996)

Greenwood, D.T.: Principles of Dynamics, 2nd edn. Prentice-Hall, Englewood Cliffs (1988)

Lanczos, C.: The Variational Principles of Mechanics. Courier Dover, Toronto (2012)

Marsden, J.E., West, M.: Discrete mechanics and variational integrators. Acta Numer. **10**, 357–514 (2001)

Margot, J.L., Nolan, M.C., Benner, L.A.M., Ostro, S.J., Jurgens, R.F., Giorgini, J.D., Slade, M.A., Campbell, D.B.: Binary asteroids in the Near-Earth object population. Science **296**(5572), 1445–1448 (2002)

Moore, A.: Discrete mechanics and optimal control for space trajectory design. Ph.D Thesis, Control and Dynamical Systems, California Institute of Technology, Pasadena (2011)

Ober-Blobaum, S., Junge, O., Marsden, J.E.: Discrete mechanics and optimal control: an analysis. ESAIM Control Optim. Calc. Var. **17**(2), 322–352 (2011)

Raghavan, D., Henry, T.J., Masion, B.D., Subasavage, J.P., Jao, W., Beaulieu, T.H., Hambly, N.C.: Two suns in the sky: stellar multiplicity in exoplanet systems. Astrophys. J. **646**, 523–542 (2006)

Szebehely, V.: Theory of Orbits: The Restricted Problem of Three Bodies. Academic, London (1967)