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MODE ANALYSIS FOR LONG-TERM BEHAVIOR IN A RESONANT EARTH–MOON TRAJECTORY

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Trajectory design in chaotic regimes allows for the exploitation of system dynamics to achieve certain behaviors. For the Transiting Exoplanet Survey Satellite (TESS) mission, the selected science orbit represents a stable option well-suited to meet the mission objectives. Extended, long-term analysis of particular solutions nearby in the phase space reveals transitions into desirable terminal modes induced by natural dynamics. This investigation explores the trajectory behavior and borrows from flow-based analysis strategies to characterize modes of the motion. The goal is to identify mechanisms that drive the spacecraft into a particular mode and supply conditions necessary for such transitions.

INTRODUCTION

Trajectory design in chaotic regimes where complex motion is imparted from various gravitational sources allows for the exploitation of the natural dynamics to achieve desirable orbital behaviors. In the case of the Transiting Exoplanet Survey Satellite (TESS) mission, the science orbit is designed as a 2:1 resonant orbit with respect to the lunar orbital period in the Earth–Moon system.[1] A particular solution resembling the TESS science orbit reflects this resonance configuration and is depicted in both the Earth Mean Equatorial J2000 and the Earth–Moon rotating frames in Figure 1. Ten periods of the spacecraft path (throughout five sidereal lunar cycles) are illustrated, and the associated evolution is indicated by color, ranging from blue to red through a traditional spectrum color mapping. During the nominal 2–4 year mission, the spacecraft will observe distant stars with the goal of identifying extrasolar planets, and the selected science orbit represents a stable option that is well-suited to meet the mission objectives. Extended, long-term analysis of particular solutions that are nearby the science orbit in phase space, reveals transitions into desirable terminal modes induced by the natural dynamics. This investigation explores various components of the trajectory behavior and borrows from flow-based strategies[2–5] to characterize modes of the trajectory. The goal is identification of the mechanism that drives the spacecraft into a particular mode and supplies the conditions necessary for such a transition.

The dynamical regime corresponding to the TESS mission trajectory presents many complex and chaotic interactions. While the science orbit associated with the TESS spacecraft has been numerically demonstrated to be operationally stable,[1] higher-fidelity effects allow for the possibility of unpredictable behavior over longer time intervals. Such a situation is well-suited to analysis of force component contributions for the purpose of potentially fabricating transitions into desirable modes. Well-known analysis strategies for investigating trajectory behavior are augmented by computations of Cauchy–Green eigenmetrics. A combination of approaches may allow for exploitation of the natural dynamics to accomplish mission objectives.

Relevant notions related to orbital resonance in general, as well as the particular resonance associated with the TESS mission orbit, are presented in the ensuing background section. Fundamentals of flow-based strategies and a summary of the dynamical system models are also offered; some material follows directly

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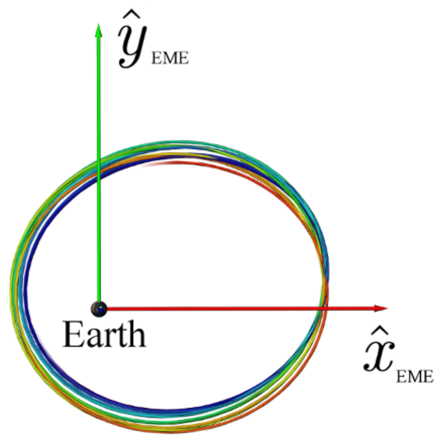
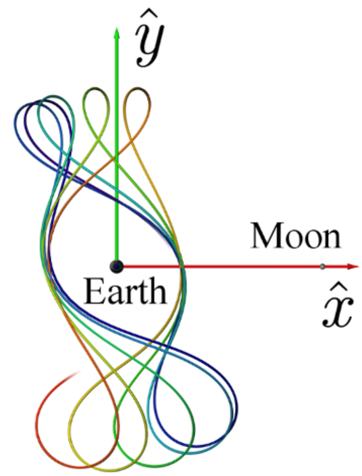


(a) Earth Mean Equatorial of J2000 View



(b) Earth–Moon Rotating Frame View

Figure 1. 10 Periods of a TESS-like Science Orbit Computed in an Ephemeris Model

from Short and Howell[4] as well as Short et al.[5] Several strategies for increasing the understanding of the long-term behavior of the TESS science orbit are highlighted in the analysis. These strategies include a discussion of perturbation theory, particularly with respect to the Kozai parameter, and inspection of model components. The examination of orbital elements and frequency components over time supplies some additional insight. A flow-based approach lends context to particular solutions and the effects of perturbations on these solutions. Finally, conditions for achieving particular results that display desirable behavior modes are offered prior to a few concluding remarks.

BACKGROUND

Resonant orbits and TESS

Many interesting natural solar system behaviors can only be understood within the context of resonances that exist between the motions of the associated bodies. Recently, efforts to exploit resonance for spacecraft trajectory design have become a focus for practical applications. In particular, researchers have explored the possibility of harnessing resonant orbits for transfer trajectories,[6–8] and for stable orbits with desirable repeating motions.[1,9] Such strategies support useful mission options, and reflect the possibilities afforded by carefully navigating a chaotic design space.

Of particular relevance for spacecraft trajectories are orbit-orbit resonances. Such resonances are those where the spacecraft's motion repeats in a near-integer ratio relationship with the motion of a massive system body. For example, in the Earth–Moon system, the Moon orbits the Earth once every sidereal period, q, of $\sim$27.3 days. A spacecraft moving in the Earth–Moon system with a characteristic period, p, that is some multiple of the Moon's period is said to be moving in a p:q resonance with the Moon. If the spacecraft completes two revolutions for each revolution of the Moon, this resonance is designated as a 2:1 resonance. Such exact integer ratio behaviors only appear in simple models of the dynamical system behavior. However, within higher-fidelity-model contexts, motion that oscillates about such resonances frequently persists and is often reflective of stable behavior.

The Transiting Exoplanet Survey Satellite (TESS) mission orbit is selected in such a nearly 2:1 resonance with the Moon in the Earth–Moon system. Thus, the orbit is characteristic of a period roughly half that of the lunar sidereal period, or $\sim$13.67 days. While the orbit does not generally complete exactly two revolutions for each revolution of the Moon, it can be described as "in oscillation" about such a resonance.[1] The orbit is constructed such that its apogee approaches the lunar orbital distance, and remains at about $\pm 90^{\text{o}}$ from the Moon itself, alternately leading and lagging. It is convenient to observe such motion in a rotating frame fixed on the motion of the Moon about the Earth, and such a representation appears in the right panel of Figure 1.

It is desirable for the spacecraft to be operationally stable, requiring no stationkeeping, upon entering and for the 2–4 year duration of the science phase of the trajectory. Throughout the nominal mission and for, at least, 100 years beyond the end of the mission, the spacecraft trajectory is also required to avoid the geosynchronous orbital band at all times. The trajectory is designed to satisfy this requirement throughout all mission phases and employs a "dispose-in-place" strategy after the end of the mission. However, extended, long-term analysis of the trajectory reveals particular modes where the spacecraft behavior can settle into motion consistent with the avoidance of the geosynchronous band. These modes are trajectory phases that are induced by the natural dynamics but, without a more complete understanding of the dynamical environment, it is not certain that a perturbation or disturbance would not shift the modes into some undesirable behavior.

Flow-based analysis

The Cauchy–Green Strain Tensor (CGST or CG tensor) describes the relative stretching of nearby trajectories over a given time interval. As such, the CGST offers a means for identifying relatively larger or smaller directions of expansion. For example, this relative information can be identified from the eigenvalues and eigenvectors of the tensor, such as the Finite-Time Lyapunov Exponent (FTLE) values that are obtained from the Cauchy–Green eigenvalues. The CGST is computed using the familiar State Transition Matrix (STM), a matrix that describes the impact of initial state variations on the behavior along trajectories over time. This STM can be calculated by directly observing the effects of perturbed trajectories, or by integrating the first-order variational equations relative to the system equations of motion.

The flow map, $\phi_{t_0}^t(\boldsymbol{x}_\text{o})$, represents the state of the system that has evolved to a final time t from an initial state $\boldsymbol{x}_\text{o}$ at time t_0. The matrix product, $\mathbf{C} = \Phi_{t_\text{o}}^t(\boldsymbol{x}_\text{o})^\mathsf{T}\Phi_{t_\text{o}}^t(\boldsymbol{x}_\text{o})$, is the Cauchy–Green strain tensor,[10] and the individual matrix, $\Phi_{t_\text{o}}^t(\boldsymbol{x}_\text{o}) = \frac{d\phi_{t_0}^t(\boldsymbol{x}_0)}{d\boldsymbol{x}_0}$, is the STM evaluated along the arc at time t (here, $^\mathsf{T}$ indicates the matrix transpose). Since the state transition matrix supplies the evolution of infinitesimal perturbations to an initial variational state of the system, the Cauchy–Green strain tensor represents a natural object for study that reveals the growth or decay of these perturbations. For example, if several adjacent, initial state vectors are separated by small perturbations and subsequently evolved for a prescribed time, the Jacobian is estimated (here, using only two dimensions for illustration) as described by Haller[11] via finite differencing such as,

$$\Phi_{t_\text{o}}^t(\boldsymbol{x}_\text{o})\bigg|_{(i,j)} = \frac{d\phi_{t_0}^t(\boldsymbol{x}_0)}{d\boldsymbol{x}_0}\bigg|_{(i,j)} = \begin{bmatrix} \frac{x_{i+1,j}(t)-x_{i-1,j}(t)}{x_{i+1,j}(t_0)-x_{i-1,j}(t_0)} & \frac{x_{i,j+1}(t)-x_{i,j-1}(t)}{y_{i,j+1}(t_0)-y_{i,j-1}(t_0)} \\ \frac{y_{i+1,j}(t)-y_{i-1,j}(t)}{x_{i+1,j}(t_0)-x_{i-1,j}(t_0)} & \frac{y_{i,j+1}(t)-y_{i,j-1}(t)}{y_{i,j+1}(t_0)-y_{i,j-1}(t_0)} \end{bmatrix}, \tag{1}$$

where the indices i and j indicate relative initial perturbations in the x and y states, respectively.

Various schemes exist for calculating the Jacobian, including numerical integration of the variational equations, direct calculation from a grid of points, as in Equation (1), or through the use of an auxiliary grid as described by Farazmand and Haller.[12] Direct computation from a grid of points that covers the domain of the simulation allows for the determination of the FTLE in systems where variational equations are not available. Selecting an appropriate grid spacing facilitates the identification of flow features consistent with the order of the grid spacing. A mesh with inadequate resolution may exclude some structures. However, an auxiliary grid that brackets each of the primary grid points increases the accuracy of the resulting Jacobian.

Available from the CGST, the largest finite-time Lyapunov exponent*, is a common metric that quantifies the relative stretching and stability of nearby phase space states as they evolve over time. The FTLE essentially measures the stretching between adjacent trajectories over a prescribed time interval. Mathematically, the calculation of the FTLE is fairly straightforward—it is the largest normalized eigenvalue of

*While there are as many FTLE values as dimensions in the phase space, the largest finite-time Lyapunov exponent is denoted *the FTLE* value of interest in this analysis.

$\sqrt{\Phi_{t_\mathrm{o}}^t(\boldsymbol{x}_\mathrm{o})^\mathsf{T}\Phi_{t_\mathrm{o}}^t(\boldsymbol{x}_\mathrm{o})}$, i.e., the square root of the Cauchy–Green tensor, $\mathbf{C}$. Thus, the FTLE is evaluated from the expression,

$$\lambda = \frac{1}{|T|}\ln\sigma_n\left(\sqrt{\mathbf{C}}\right), \tag{2}$$

where $\sigma_n()$ is the operation that extracts the largest eigenvalue of the operand. The parameter $T = t - t_0$ is both the truncation time for the computation of the FTLE value and a means of normalizing this value.

Also available from the Cauchy–Green tensor are the directions of extremal stretching in phase space. The evolution of a portion of the phase space over time is related to σ_i and $\boldsymbol{\xi}_i$. Specifically, stretching occurs along the eigenvector, $\boldsymbol{\xi}_i$, proportional to $\sqrt{\sigma_i}$,

$$\left|\mathrm{D}\phi_{t_\mathrm{o}}^t(\boldsymbol{x}_\mathrm{o})\boldsymbol{\xi}_i\right| = \sqrt{\sigma_i}\,|\boldsymbol{\xi}_i|\,. \tag{3}$$

Consequently, the greatest degree of stretching occurs along the eigenvector, $\boldsymbol{\xi}_n$, associated with the largest eigenvalue, σ_n. This notion is illustrated in Figure 2, where the double-headed red arrow represents $\boldsymbol{\xi}_n$ and

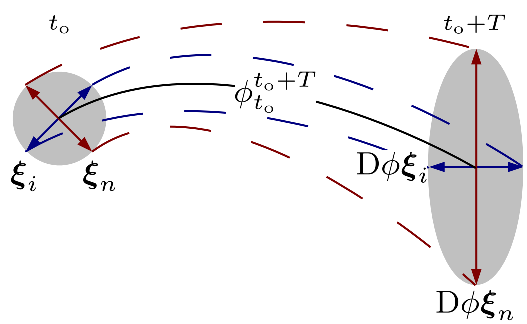


Figure 2. Stretching Associated with Eigenvectors of the Cauchy–Green Tensor

the double-headed blue arrow conveys the remaining directions. As with the largest FTLE value, λ, the eigendirection $\boldsymbol{\xi}_n$ associated with the largest eigenvalue of $\mathbf{C}$ is also a focus in this analysis.

Specific analysis strategies are utilized to explore a TESS-like solution. Observation of the orbital characteristics, augmented with predictive FTLE data, indicates when the trajectory is expected to evolve into different modes. The largest finite-time Lyapunov exponent provides a single measure that describes the magnitude of divergence along a path for a given time horizon. Of special interest, for the purposes of this effort, is the model-agnostic nature of the FTLE—such a metric can be calculated for trajectory arcs regardless of the underlying model complexity or characterization. Values for the finite-time Lyapunov exponent may be computed for simplified models and compared with corresponding values from propagations in higher-fidelity models. Moreover, careful selection of the time horizon and the calculation strategy for the FTLE can yield a useful profile that effectively predicts and characterizes general trajectory trends. In this analysis, FTLE profiles supply a signature of conditions that can "trigger" shifts in a particular solution such that it exhibits various modes of behavior and reveal the transitions between such modes.

Careful inspection of the effects of the model components before, during and after transitions in trajectory behavior raises the possibility for artificially producing such a transition by design. The flow-based FTLE profile suggests a "signature" for trajectory-shifting behavior. Illustrated in Figure 3 is an FTLE profile consistent with a 100-year propagation of the TESS-like orbit. This profile is created by compositing state-transition matrices to form a 1.2-year time horizon for each data point (note where the FTLE profile drops to zero as the time horizon extends beyond the 100-year mark). Thus, the FTLE profile indicates that the trajectory shifts between different types of behavior modes. Generally, there is some correlation with osculating orbital elements, particularly eccentricity, as larger values of the FTLE are consistent with trajectory behavior exhibiting lower perigee radii (r_p) and higher radii of apogee (r_a), while lower FTLE values indicate trajectory phases with higher r_p and lower r_a. However, the FTLE profile also conveys a measure of relative orbital stability under perturbations as trajectory segments characteristic of low FTLE values also reflect a lower propensity for trajectory divergence. Profiles of FTLE values essentially summarize transitional behavior. Knowledge of qualitative transitions, and when they might occur, allows for investigation of these

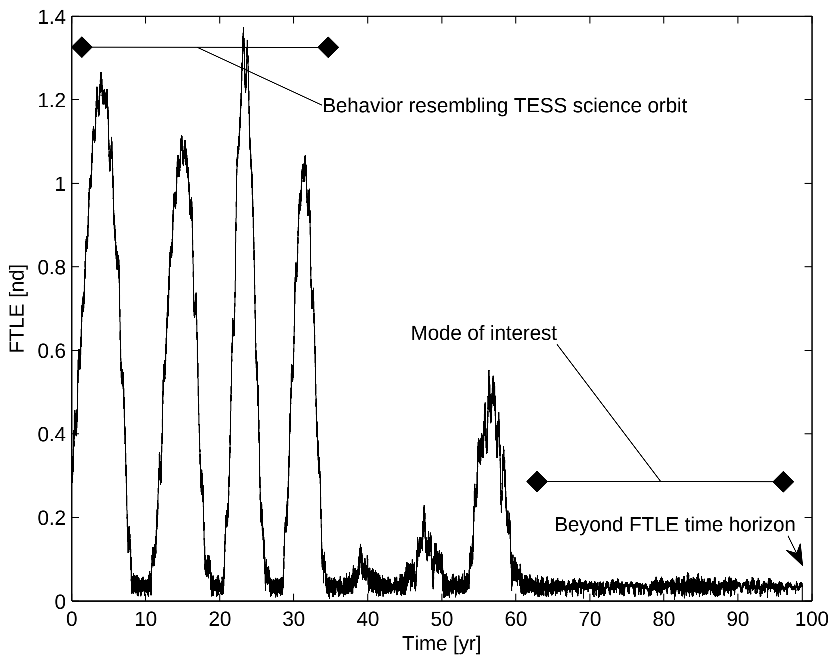


Figure 3. FTLE Profile for 100-year TESS-like Science Orbit Propagation

behaviors through analysis of the system characteristics. Inspection of the gravitational system geometry and the spacecraft trajectory may yield specific conditions that allow for trajectory shifts into desirable modes. Such an ongoing analysis is reflected in this investigation.

System models

Some key space environments involve multiple gravity fields. As such, it is often necessary to incorporate as many of these gravity fields as possible into the governing models to ensure accurate simulation and to capture the essential features of the dynamical interactions. Models directly incorporating more than two gravitating point-mass bodies, however, offer no analytical solutions, and introduce additional complexities, which may be small but significant. Formulating the problem in terms of three bodies produces a model sufficiently complex to reveal many important characteristics while remaining tractable. However, the general three-body problem possesses no closed-form solution.[13] Thus, additional simplifications offer significant insight. Moreover, the computation of the CGST is not contingent upon any assumptions in the derivation of the system differential equations, and can be applied to systems modeled with various levels of fidelity. Analysis based on the Cauchy–Green tensor remains valid regardless of the complexity of the model. In this investigation, various levels of model fidelity, including those consistent with the Circular Restricted three-body Problem (CRP), the Elliptic Restricted three-body Problem (ERP), a BiCircular four-body Problem (BCP), and particular EPHemeris formulations (EPH), are all employed. All of these models represent the inclusion of some natural behavior, and comparisons of the behavior consistent with each model can reveal an associated impact on the trajectory evolution. Some necessary considerations for each model are summarized.

The circular restricted three-body model The circular restricted problem incorporates only the effects of the masses of two larger primaries on a third body of negligible mass, such as a spacecraft. A brief review of the CRP is useful to illustrate the rotating reference frame and summarize the associated equations of motion. The two primary bodies that appear in the model are designated as P_1 and P_2. Position variables, x, y, and z describe the position of the third body, the spacecraft, with respect to the barycenter of the primary system, which also serves as the origin of the rotating and inertial reference frames. The rotating reference frame is defined such that $\hat{x}$ is directed from P_1 to P_2, $\hat{z}$ is perpendicular to the spacecraft plane of motion consistent with orbital angular momentum and $\hat{y} = \hat{z} \times \hat{x}$. The system mass parameter is represented by $\mu = \frac{m_2}{m_1+m_2}$, a function of the masses of the primary bodies. Additionally, distances between the third body and the two primaries are denoted r_{i3}. Specifically, in a coordinate frame that rotates coincident with the circular motion of the primaries, a system of differential equations that describes the motion of the third body incorporates the potential function,

$$U^* = \frac{1-\mu}{r_{13}} + \frac{\mu}{r_{23}} + \frac{1}{2}(x^2 + y^2), \tag{4}$$

and is written,

$$\ddot{x} = \frac{\partial U^*}{\partial x} + 2\dot{y}, \quad \ddot{y} = \frac{\partial U^*}{\partial y} - 2\dot{x}, \quad \ddot{z} = \frac{\partial U^*}{\partial z}, \tag{5}$$

where the first derivatives in x and y appear as a result of the Coriolis acceleration.

The equations of motion in the circular restricted problem are consistent with Szebehely[14] where they admit a single integral of the motion. This integral is termed the Jacobi constant, and is evaluated as,

$$C_{\text{Jacobi}} = 2U^* - v^2, \tag{6}$$

where $v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$, that is, the square of the magnitude of the relative velocity. This integral allows for a reduction of order in the problem, frequently serves in an important role in the definition of surfaces of section, and establishes boundaries on the physical motion of the third body. These boundaries are defined when $v = 0$ in Equation (6), separating regions of real and imaginary velocities. The circular restricted problem represents a model of sufficient complexity to exhibit regions of both chaotic and ordered behavior. Generally, the focus of analysis in this model is conceptual understanding and the exploitation of any dynamical structures that are associated with the chaotic regions to identify useful trajectory arcs. The CRP model is frequently suitable to yield first-order mission design solutions, but useful information is often difficult to isolate amidst the chaos and small variations to initial states can yield large differences in behavior over time.

The elliptic restricted three-body model Also described in Szebehely is an extension of the CRP that incorporates elliptic motion of P_2 with respect to P_1. The reference frame becomes a rotating-pulsating frame such that $\hat{x}$ remains directed from P_1 to P_2 while the characteristic distance invoked to nondimensionalize the system variables is equal to, and varies with, P_2's radial distance from P_1 as it evolves on its elliptic path. Such an increment in model fidelity results in a generally nonautonomous system that is time-periodic and may be rendered autonomous if necessary. The independent variable is defined as the true anomaly f along the orbit of P_2 with equivalences to the nondimensional (nd) time-parameter of the CRP at multiples of π. Thus, introducing these modifications to the system differential equations of motion can be accomplished through a change of variables,

$$w = \frac{U^*}{1 + e\cos f}, \tag{7}$$

where e is the orbital eccentricity of the P_2 orbit. Differentiation is performed with respect to f, $()' := \frac{d()}{df}$, and the equations of motion are structurally equivalent to those in the circular restricted problem, i.e.,

$$x'' = \frac{\partial w}{\partial x} + 2y', \quad y'' = \frac{\partial w}{\partial y} - 2x', \quad z'' = \frac{\partial w}{\partial z}. \tag{8}$$

Unlike the CRP, a constant of the motion is not available in the elliptic restricted problem. This slightly more complex model may aid in isolation of the effects of lunar eccentricity on spacecraft motion.

The bicircular four-body model A simplified four-body model, similar to the model utilized by Koon et al.[15] and further explored by Blazevski and Ocampo,[16] is also employed here. This model incorporates the influence of a third massive body simultaneously with the dynamical effects of the two primary bodies in the circular restricted problem. The additional massive primary is designated P_4. Although the system is not coherent, the Newtonian inverse-square gravity of P_4 acts on the spacecraft in addition to the gravitational effects of the two CRP primaries. The third primary body does not affect the circular Keplerian orbits of the other primaries. If $\mu_4 = \frac{m_4}{m_1+m_2}$, the equations of motion remain the same as Equations (5), but the potential function is now,[17]

$$U^* = \frac{1-\mu}{r_{13}} + \frac{\mu}{r_{23}} + \frac{\mu_4}{r_{43}} + \frac{1}{2}(x^2+y^2). \tag{9}$$

Such a four-body model, while still incorporating significant simplifications, introduces an important transition much like the ERP. The presence of the perturbing fourth body results in a nonautonomous system. As in the elliptic model, the change in the nature of the system decreases the applicability of many of the dynamical systems tools that are available in the CRP, while completely removing others. Again, a constant of the motion, and, consequently, a convenient expression for bounds on the motion, is no longer available. Finally, due to the time-dependent nature of the underlying flow, careful consideration is afforded the initial system geometry.

Ephemeris models Ephemeris models are formulated to be generally consistent with the previous models. Such higher-fidelity systems, similar to the model employed by Pavlak and Howell,[18] are constructed to incorporate position histories for the primary bodies supplied by the JPL developmental ephemerides. The governing equations are derived as the n-body relative equations of motion,

$$\ddot{\boldsymbol{r}}_{ms} = -\frac{\mu_{2b,s}+\mu_{2b,m}}{r_{ms}^3}\boldsymbol{r}_{ms} + \sum_{\substack{j=1\\ j\neq s,m}}^{n} \mu_{2b,j}\left(\frac{\boldsymbol{r}_{sj}}{r_{sj}^3} - \frac{\boldsymbol{r}_{mj}}{r_{mj}^3}\right). \tag{10}$$

Here, μ_{2b} is the familiar mass parameter from the two-body problem, nondimensionalized as appropriate. The position vector, $\boldsymbol{r}_{mj}$, indicates the position of the j^{th} body with respect to the central body, m; the subscript s is associated with the spacecraft. In this model, states defined in the restricted problem are transitioned to body-centered J2000 states, and vice versa, via an instantaneous rotating frame defined by the ephemeris states. Such ephemeris models naturally involve six-dimensional state vectors and trajectory propagation proceeds in all spatial dimensions. Additionally, computation of the CGST employs auxiliary "grid" points about each state variable. Thus, in this model, one CGST computation involves the propagation of 12 perturbations. Given these higher-dimensional considerations, an effective means for parameterizing a search space is required, and in this case the CG eigenvector $\boldsymbol{\xi}_n$ is utilized.

All of these various models ultimately represent contributions to the natural motions in a system. Different impacts associated with specific motions in two- or three-body gravitational fields are reflected in each model. Despite the varying dynamics, motion observed in each model remains subject to the wide applicability of flow-based analysis. Consequently, comparisons can be made between levels of model fidelity and particular model components may be isolated as triggers for specific behaviors through such comparisons.

ANALYSIS

Initial explorations in the phase space nearby the TESS science orbit reveal particular solutions that eventually diverge from the nominal mission orbit but actually exhibit desirable end-of-life behavior. However, the existence of these alternate solutions also implies the potential for undesirable divergence, and a better understanding of the long-term behavior is necessary. Mission requirements constrain the perigee radius for the science phase of the trajectory to remain above 6.7 Earth radii (r_e) to avoid the geosynchronous satellite belt. To minimize communication times for data retrieval, perigee should also remain below 22 r_e for the duration of the nominal science phase (2–4 years). The TESS mission calls for a dispose-in-place end-of-life strategy where the spacecraft will continue in the mission orbit. Avoidance of the GEO belt for at least 100 years is also required. However, after the science phase, the upper bound on r_p for data download is no longer an issue. Thus, behavior that settles into an orbit with higher perigee radii has the potential for greater stability while increasing the distance from the geosynchronous satellites.

To identify solutions that evolve into advantageous behavior modes, and to better understand the local neighborhood of perturbations near the reference solution, several strategies are invoked. Initially, a single-parameter exploration of small variations in the rotating-y component of the state at the Period Adjust Maneuver (PAM), that is, the maneuver to enter the TESS science orbit, reveals various solutions. Among these solutions are those displaying behavior consistent with the FTLE profile depicted in Figure 1. Inspection of the long-term behavior of the orbital elements and observation of individual frequency markers supports the characterization of certain behavioral transitions. From a particular solution, investigations in terms of small perturbations reveal additional insight into the long-term trajectory behavior. Toggling model-fidelity and examining the state components aids in illumination of the complexities of the cumulative effects of the various system influences on the spacecraft path over time. Invoking the flow context provides a clear picture of the truly chaotic nature of the system, and supports identification of potentially useful end-of-life solutions. A strategy for identifying specific conditions for entering these solutions is discussed. The chaotic nature of the spacecraft operating environment supplies many possibilities and all of these varied, but ultimately related, analysis options lead to a better understanding.

Perturbation theory and the Kozai parameter

Perturbation methods offer useful analysis options for understanding the contributing factors in orbital motion. Colombo[19] and Colombo et al.[20] examine the long-term perturbative effects on highly-elliptical and libration point orbits, and explore the possibility of engineering desired end-of-life options for various mission scenarios, including graveyard orbit and re-entry/impact options. The present analysis focuses on understanding the context associated with the effective graveyard orbit represented by the TESS dispose-in-place strategy. Rather than employing analytical or semi-analytical, general perturbation theory, the focus is on special perturbation analysis through numerical simulation. As a stepping-off point from general perturbations, consider the evolution of the Kozai parameter plotted in Figure 4. This parameter reflects a constant of the motion under the Kozai formulation of the three-body problem where short-term oscillations due to the third-body perturbation are essentially averaged out.[1,21,22] The Kozai parameter is defined,

$$K = \cos i \cdot \sqrt{1 - e^2}, \tag{11}$$

and, similar to the Jacobi constant, is not expected to be conserved in higher-fidelity models. However, its evolution can be observed when calculated instantaneously at each time step along the trajectory using osculating elements. In Equation (11), i and e are the inclination and eccentricity of the spacecraft orbit with respect to the lunar orbital plane. As the Kozai mechanism features heavily in the design strategy for the TESS mission and supports some general trends consistent with the science orbit,[1,23] it seems reasonable and appropriate to initiate the present analysis with a discussion of the behavior of the Kozai parameter over time. The TESS orbit in the Design Reference Mission (DRM) has been selected for its transitioning behavior. Thus, the evolution of an initial state slightly perturbed from the TESS DRM trajectory reflects the potential differences in responses. Gangestad et al.[23] identify a suitable Kozai parameter to be 0.65 to maintain the upper and lower constraints on $\mathrm{r_p}$. Apparent in Figure 4 is an initial value of K that oscillates about values lower than 0.65, and then shifts into two later phases oscillating above 0.65. These differences, aside from being expected as associated with a different solution, illustrate two relevant points. First, much of the previous analysis for the TESS DRM focuses on the first few decades of the mission, and is generally consistent with the results showcased in the orbit selected for this analysis. Second, these behavior-shifting trends, characteristic in many of the orbital measures associated with this solution, are reflective of the different modes through which the trajectory evolves. Various metrics reflect these modes and their associated transitions more or less clearly. This brief inspection of the Kozai parameter serves as an introduction to a series of observations that help establish the perturbed orbit that is investigated in this analysis.

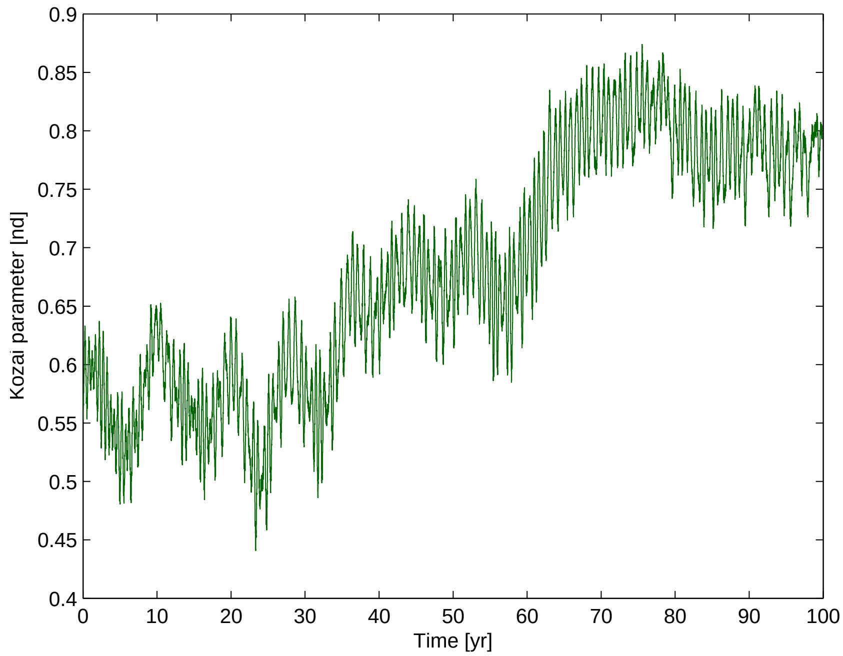


Figure 4. Evolving Kozai Parameter for TESS-like Science Orbit (Ephemeris Model)

Orbital elements

Inspection of the evolution of the osculating orbital elements serves to further establish the long-term behavior exhibited by a given particular solution. Given the TESS DRM orbit, perturbations that shift the initial state vector of the period adjust maneuver are explored. A significant percentage of perturbed trajectories display qualitatively distinct long-term behaviors and are discussed in later sections. As an example, a single "perturbed baseline" trajectory is inspected initially. Specifically, a modest perturbation to increase the rotating-y position coupled with a corresponding decrease in the rotating-z position and a velocity change results in a particular solution characteristic of interesting behavior modes. These modes are apparent in the various panels included in Figure 5 as well as in the evolution of the Kozai parameter depicted in Figure 4. These panels, all associated with the perturbed example trajectory, depict the evolution of each of the osculating Keplerian orbital elements. As necessary, the particular reference for each of the orbital elements is indicated. Also included in the figure is the FTLE profile introduced in Figure 3, repeated here for convenience. There are several notable observations concerning the trajectory evolution. From panel (a) in Figure 5, it is apparent that this perturbed, particular solution leads to the spacecraft r_p initially evolving above 22 r_e after about 10 years. Each of the other panels reveal, to greater or lesser degree, two additional, major transitions in behavior as the evolution approaches 40 years and just after 60 years. As the radius of perigee increases above 22 r_e after 60 years, r_a decreases into the lower-to-mid 50 r_e range consistent with the drop in eccentricity observed in panel (b). This coupled increase in distance from the Earth and decrease in distance from the lunar orbit represents additional stabilization of the orbit. Each of these effects is reflected in the FTLE profile of panel (f). This FTLE profile, consistent with look-ahead times of $\sim$1.2 years for each point, clearly indicates and summarizes the mode shifts along the trajectory while supplying a measure for stability with respect to perturbations relative to the orbit. Larger values of FTLE indicate more sensitive regions of the solution.

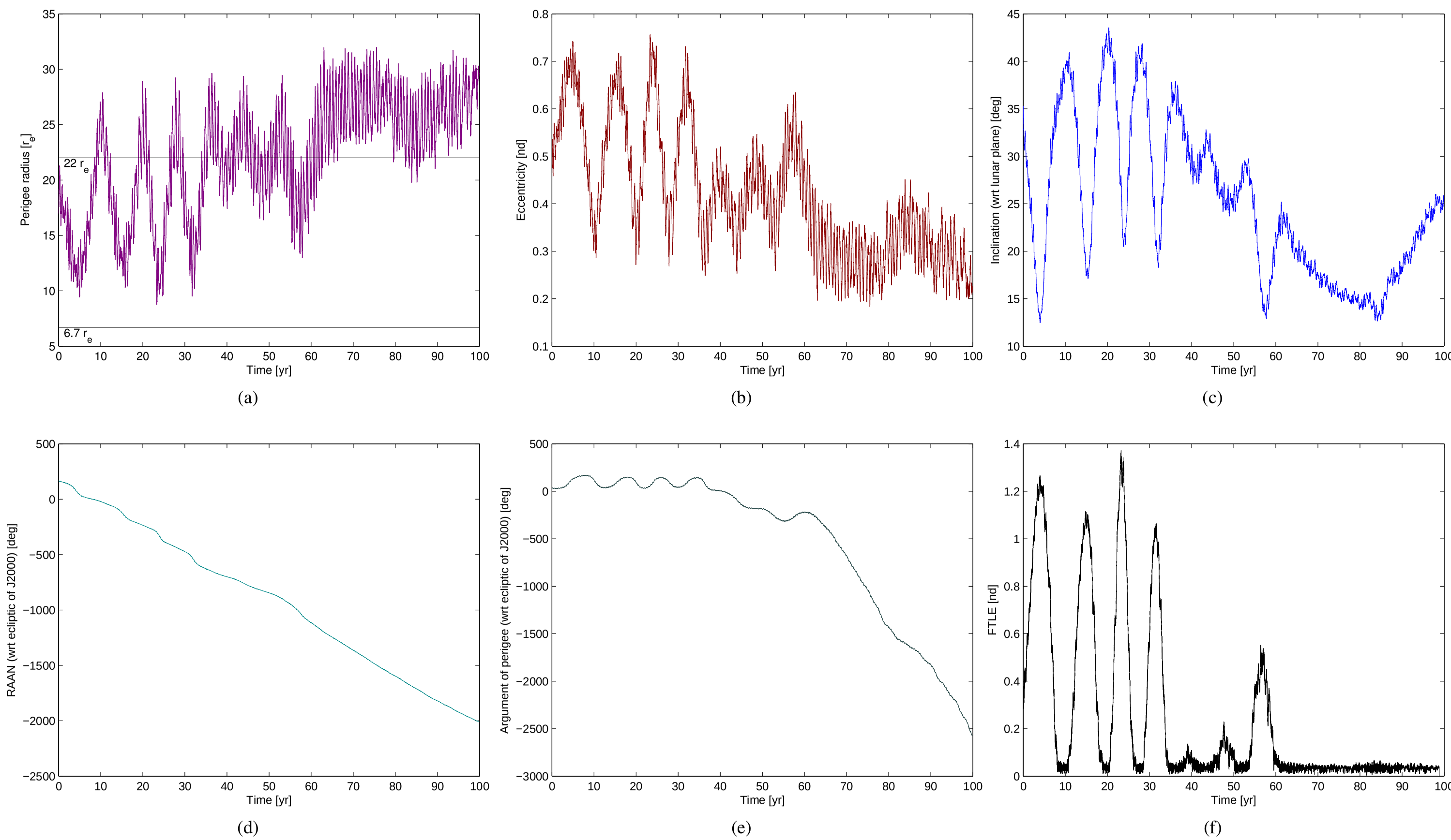


Figure 5. Orbital Elements and FTLE Profile

Frequency analysis

Inspection of the frequencies in the "signal" associated with a spacecraft trajectory can be a useful strategy to identify particular contributions to the motion of the spacecraft. For example, Bosanac et al.[24] employ frequency analysis strategies to characterize the impact of including three-body interactions along with the inverse square pairwise contributions in a three-body system. In Figure 6, the amplitude density associated with the fundamental frequencies and some subharmonics embedded in the time-history of one acceleration component are plotted. These frequencies are associated with the time-history of the y component of the

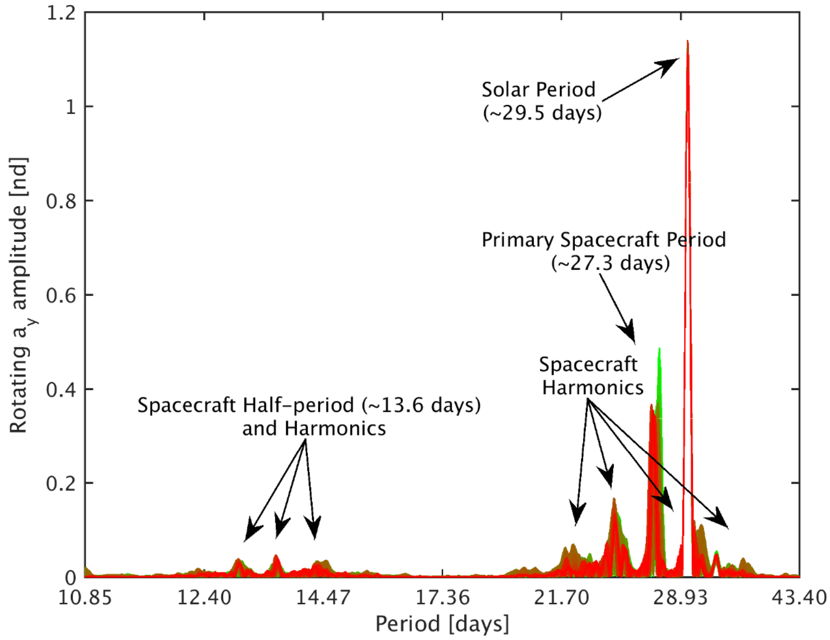
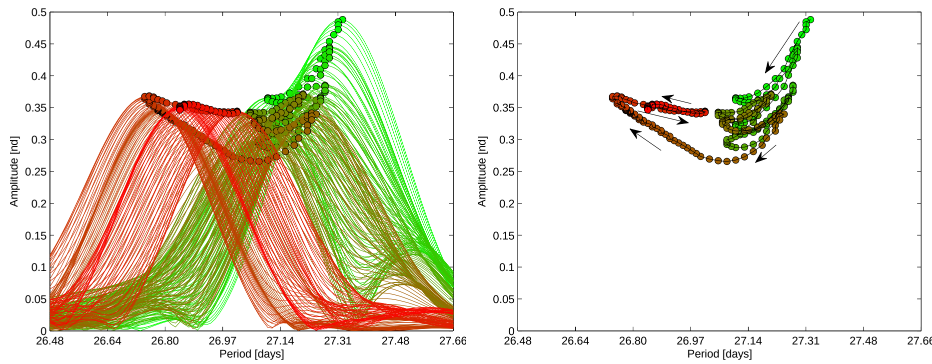


Figure 6. Rotating Frame Frequencies in Spacecraft a_y from 10-year Data Windows

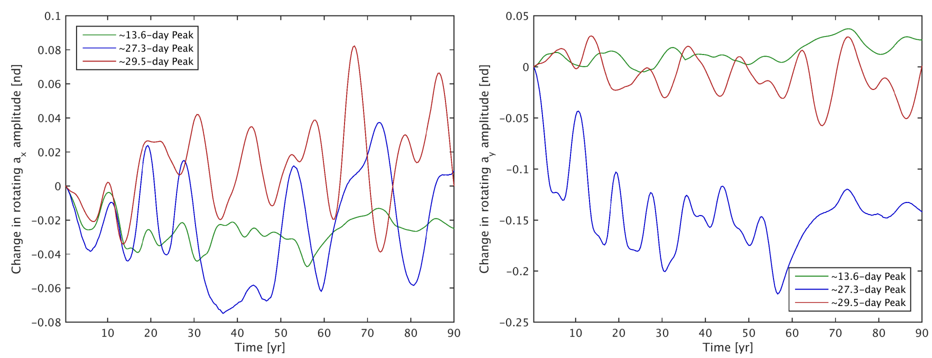
spacecraft acceleration as viewed in the rotating-pulsating-librating frame that instantaneously fixes the Earth and Moon consistent with their ephemerides. Further, the frequency data are associated with the perturbed baseline trajectory evolved under the Sun–Earth–Moon ephemeris dynamics (characterized in Figures 4 and 5). These frequency responses result from Fast Fourier Transforms (FFT) of overlapping data intervals initiated every four months and spanning 10 years each. The curves are colored by their initial time, and range from green to red (see Figure 7 for a closer inspection). The data intervals are pre-processed using a Hanning function that yields more distinct features but reduces the observed amplitudes. A few of the major accelerations are associated with labeled peaks. For example, the 'Primary Spacecraft Period' peak is associated with the y component of the Earth–Moon accelerations from the equations of motion. The solar effects on the spacecraft orbit are associated with the largest peak as a consequence of the Sun's very regular, sinusoidal motion in the rotating frame. However, the peaks associated with the accelerations from the Earth and Moon are distributed into multiple harmonics. This dispersion is a consequence of the evolution in trajectory behavior over time. Allowing the FFT window to slide forward in time leads to an evolving frequency response, and careful peak tracking is employed to inspect the evolving frequency and amplitude of particular signal components. In Figure 7, a trace of the signal contribution associated with the peak corresponding to the principal spacecraft period, as it is reflected in the trajectory's a_y signal, is depicted. The underlying frequency response (plotted in terms of period) is colored according to time with earlier peaks colored in green and later peaks colored in red. As this particular peak evolves, a distinctive difference is observable both in terms of the amplitude and period of the signal component. Consistent with previously explored metrics,



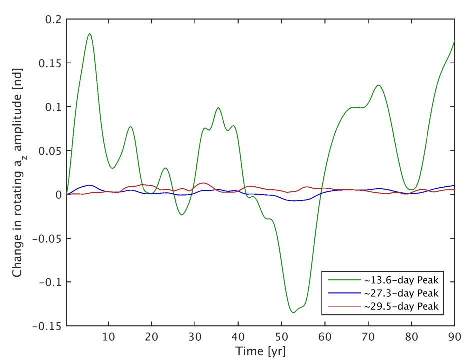
(a) Frequency Peaks Plotted as Periods (b) Without Underlying Curves

Figure 7. Trace of Lunar Frequency Impact on Spacecraft a_y (Over 90 Years)

distinctive modes emerge in the frequency/period trace over time. Specifically, earlier peaks colored in green are dissociated from the later ones colored red. In Figure 8, the signal component amplitudes are depicted in



(a) From s/c a_x Signal (b) From s/c a_y Signal



(c) From s/c a_z Signal

Figure 8. Change in Amplitude for Three Tracked Peaks Over Time

terms of their differences over time from their initial values. Rotating x, y, and z components are plotted and each of the three peaks with indicated periods from Figure 6 is tracked over time. At each point, the subsequent 10-year evolution is invoked to construct the discrete Fourier transform. Thus, the time ranges through the first ninety years of trajectory evolution. Since the frequency evolution can be interpreted not only as the evolution of spacecraft orbital characteristics but also as a consequence of the perturbations embedded in the system, it is challenging to establish any particular behavior as driven by any single factor. However, decomposing the frequency data and comparing the associated time-histories does reveal the change in the net impact in any particular component. It is notable that there are periods of time when both the Earth–Moon and solar impacts in the a_x signal trade off considerably. For example, from about 30–50 years, the Earth–Moon impact seems to decrease leading into the transitioning phases seen in Figures 3 and 5 around 50–60 years. There is some correlation between the a_x signal component associated with the "half-period peak" (green line in Figure 8(a)) and the primary a_y contribution (blue line in Figure 8(b)), which indicates a more apparent coupling of these components. While there is little change in the longer-period response of the z signal, the associated "half-period" trace is largely reflective of the spacecraft's orbital inclination seen in Figure 5(c). Inspection of various frequency components lends some insight, and additional exploration of the model components and nearby solutions expands this insight.

Model component analysis

Comparison of the different components that influence the motion along the path yields some insight into the behavior. However, the goal is to identify particular aspects of the model that drive the trajectory into a given mode, thus, this analysis must be performed relative to a solution displaying the behavior of interest. Since the CG tensor is model-agnostic, evolution of a particular initial state under the influence of different systems allows for the comparison of flow metrics that are derived from the Cauchy–Green tensor. In particular, the FTLE values computed along the trajectory may be compared to indicate a sense of the flow behavior. For these calculations, it is necessary to reset the initial state periodically to remain along the trajectory of interest. Consider the following strategy:

1. Calculate the baseline trajectory (generated from an initial condition of interest perturbed from the TESS DRM state) by evolving under the higher-fidelity ephemeris model, and retaining intermediate states at specified intervals, regularly spaced in time.
2. For each of the intermediate points along the baseline path, calculate the CG tensor for the length of the given time interval.
3. In addition, transform the baseline state as appropriate to conform to lower-fidelity models.
4. Compute the state transition matrix under the lower-fidelity dynamics for each state.
5. Transform the STM as necessary to match the reference frame associated with the baseline, and compute the CGST and FTLE values.

By computing FTLE values for the baseline trajectory and then removing model components and computing comparative values, the impacts of some effects are captured. Selecting an appropriate time scale for discretizing the baseline is a critical consideration so that the influence of the reduced models is allowed sufficient time to be observable without completely diverging. It is possible to impose smaller discretization steps and multiply the resulting state transition matrices to form a composite STM over a longer time interval; but, this method must also be approached with caution since a time slice that is too small may not allow the lower-fidelity effects to be discernible. Traces of FTLE values associated with five separate model formulations are included in Figure 9. For these calculations, time slices are collected at each 0.025 nd time steps and composited to create an STM equivalent to one produced from $\sim$1.2 years (100 nd) of evolution. Experimentation with other time scales for slicing and compositing suggests that the present set of parameters represents an acceptable trade-off for meaningful comparisons. From Figure 9, it is immediately apparent that two general groupings of behavior emerge, that is, one group associated with lower-fidelity models and the other with

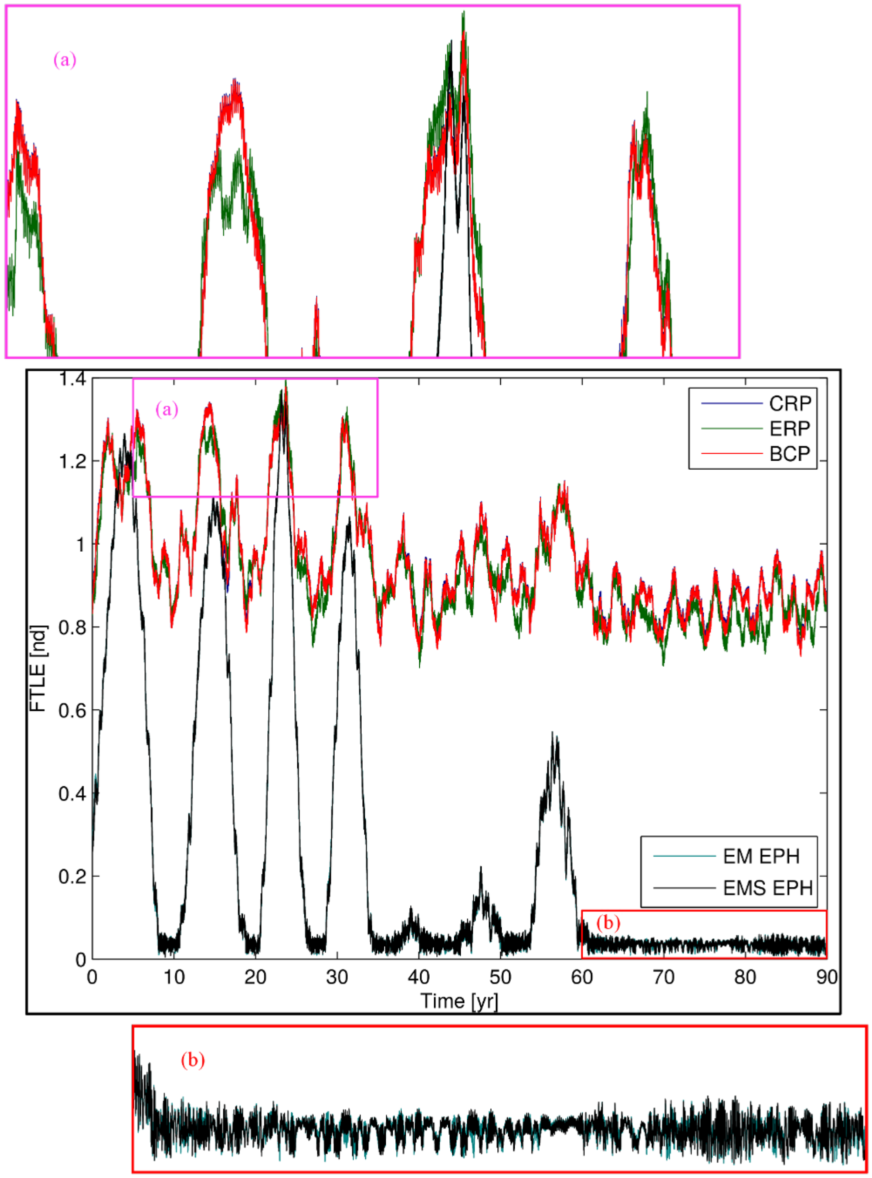


Figure 9. FTLE Profiles for Ephemeris Baseline (Black) and Reduced-Fidelity Propagations

the more realistic ephemeris options. Insets associated with the regions isolated with magenta and red boxes are included in Figure 9 to aid these comparisons. Embedded in the ephemeris options are many higher-order perturbative effects that are not captured by the circular restricted, elliptic restricted, and bicircular restricted problems. However, general consistencies are apparent in the long-term behavior as the several large peaks and troughs appear correlated between lower- and higher-fidelity models. It is also apparent that the addition of the fourth body perturbation appears to produce a lesser effect than the introduction of other components of the lunar orbit. For example, compare the blue and red traces of inset (a) as well as the teal and black lines in inset (b) in Figure 9 (both sets generally overlap) to note the relatively small difference associated with introducing the fourth body (the Sun) into the model. Alternatively, compare the green and red data in inset (a) where the elliptic effects of the lunar orbit are apparent. Given the close association between the Earth–Moon ephemeris and the Earth–Moon–Sun ephemeris traces (teal and black), a reasonable conclusion is that the Earth–Moon dynamics are generally more significant, and individual lunar perturbations may play a larger role in this particular scenario. Further examination of the higher-order perturbations and model components suggests that the problem sensitivity over longer times is a critical component in the trajectory behavior as illustrated by inspection of nearby solutions.

Nearby solutions and FTLE profiles

Further exploration in the phase space nearby the initial conditions associated with the perturbed baseline solution characterized in Figures 4 and 5 adds compelling insight into the trajectory behavior. Various strategies for such an inspection are available. Rather than employing random nearby initial states or using a grid-based strategy, a flow-based approach is adopted. In general, the natural flow in the system will rapidly align with the largest stretching direction in the phase space. Thus, seeding initial states along the eigendirection associated with the largest eigenvalue from the Cauchy–Green tensor allows for a one-dimensional search direction that will most efficiently spread through the phase space. The CGST is evaluated from the initial state for increasing time horizons until the largest eigenvector, $\boldsymbol{\xi}_n$, stabilizes and before the computation numerically overflows. A time horizon of 100 nd time steps is sufficient to accomplish this numerical balance. The search, then, is *designed to maximize the divergence* from the initial state, and the subsequent evolution is associated with the long-term behavior of the set of seeded trajectories. Recall the FTLE profile depicted initially in Figure 3; such a profile may be viewed as a one-dimensional track colored by relative FTLE values and the profile from Figure 3 corresponds to the “row” marked as zero along the vertical axis in Figure 10 where lower-to-higher FTLE values are colored using a blue-to-red spectrum color mapping. Five

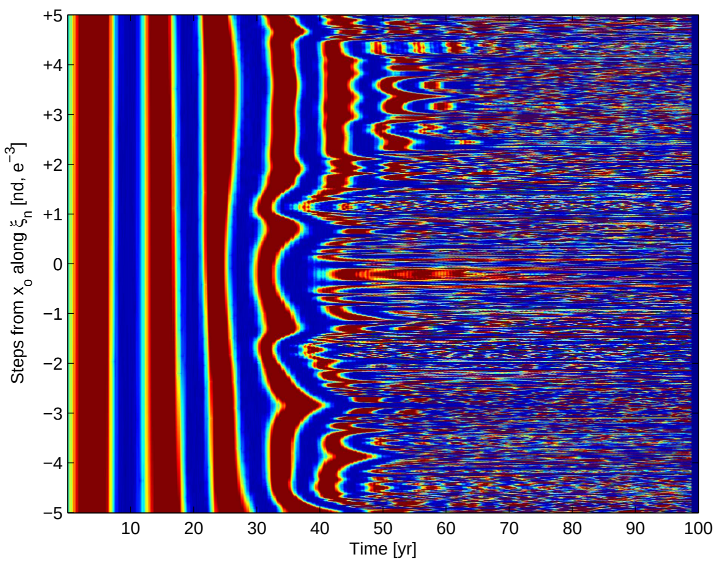


Figure 10. FTLE Profiles from the Baseline (Vertical Axis: 0) and Perturbations Along $\pm\boldsymbol{\xi}_n$

hundred steps along both the $\pm$ directions in $\boldsymbol{\xi}_n$ are tagged as initial state vectors to form the search set. Each step is characteristic of a nondimensional step size of 1×10^{-5}. For reference, in position, 1×10^{-5} nd ≈ 3.84 km while 1×10^{-5} nd $\approx 1.02\times10^{-2}$ $\frac{\text{m}}{\text{s}}$ in terms of velocity, but a step along the eigendirection is distributed, likely unequally, in both position and velocity. The four initial peaks observed in the previous plots (e.g., Figure 3) generally persist under the specified perturbations and are observed as thick red bands in Figure 10. It is apparent that sometime between 40–60 years, the solution space becomes generally unpredictable, as the map bleeds into noise. This time frame correlates to notable shifts in behavior along the original baseline path. The transition into noisy behavior is a consequence of the chaotic regime, and is a reflection of the sensitive dependence on the initial conditions.

Conditions for entering more stable modes

At the conclusion of the science phase of the TESS mission, it may be desirable to evolve the trajectory into a more constant "disposal mode" described by lower eccentricity and higher periapse altitude. Such a trajectory is consistent with the later phases of the perturbed baseline solution investigated in this analysis (Figures 3 and 5). With some insight into the dynamical context and the associated complexity, it is possible to isolate conditions for a solution that mimics the terminal phase of the perturbed baseline. Beginning near the peak at $\sim$57 years in Figure 3, the subsequent behavior in the nearby phase space is plotted as an FTLE profile map in Figure 11. In this case, larger (but fewer) steps along $\boldsymbol{\xi}_n$ are tagged ($\pm$100 steps of size 1×10^{-4} are traversed). The plot in Figure 11 illustrates a few key observations. First, relatively small perturbations

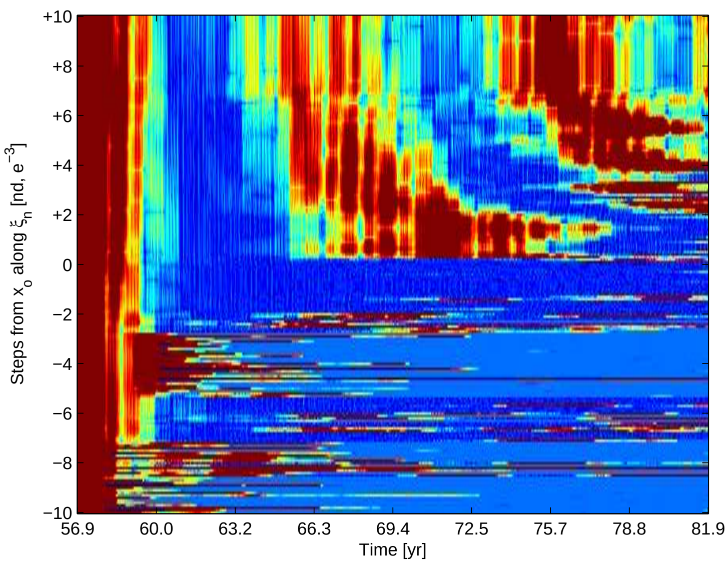


Figure 11. FTLE Profiles, Baseline and Perturbations Just Prior to Transition Around 60 Years

drastically shift the behavior—this fact is more pronounced since the perturbations are induced near the top of a modest FTLE peak (a more sensitive placement). A strip of relatively tame FTLE values (dark blue) is observed near the baseline and persists for a few steps along the negative $\boldsymbol{\xi}_n$ extension. However, perturbing slightly further along the negative $\boldsymbol{\xi}_n$ direction produces profiles with "flat", cerulean FTLE values. These values, despite their calming coloring, are consistent with trajectories that have been ejected from the local neighborhood leading to FTLE stabilizations as the trajectory essentially orbits the Earth–Moon system (in a rotating perspective). This particular plot helps to characterize the nearby behavior after transition.

As a consequence of the observations from Figure 11, solutions reflecting transition into alternate modes are now sought earlier along the evolving trajectory path. To begin this process, an initial state is sought that minimizes the difference between the relative system orientation of the spacecraft, the Earth, the Moon and the Sun with respect to some later configuration that demonstrates desirable behavior. The inertial positions and velocities of the system components may not actually be equal to their respective values at the later time, but the overall system configuration should match as closely as possible. For example, conditions are sought that minimize the difference in the magnitude of the relative state vectors between the spacecraft and the

Moon at two times, i.e.,

$$\Delta_{\text{sM}} = \left| \left(\boldsymbol{x}_{t^*}^{\text{s/c}} - \boldsymbol{x}_{t^*}^{\text{Moon}} \right) - \left(\boldsymbol{x}_{t}^{\text{s/c}} - \boldsymbol{x}_{t}^{\text{Moon}} \right) \right|,$$

where values at time t^* are characteristic of desirable trajectory behavior. This process is repeated with respect to the Sun's relative configuration and, then, a weighted averaging that most satisfies all constraints is isolated at a specific time. One such averaged state is identified at 7.77 years as the trajectory evolves into the first FTLE trough in Figure 5(f). The FTLE profile map associated with the *unaveraged*, baseline state and perturbations is included in Figure 12. Without adjusting the state to meet the necessary geometric constraints,

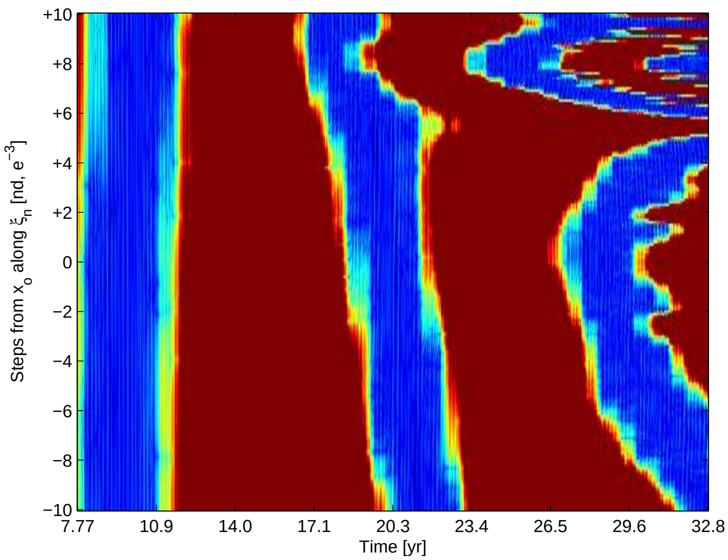


Figure 12. Unaveraged, Baseline and Perturbation FTLE Traces

the FTLE is reflective of the peak-and-trough behavior along the baseline, however some differences between Figures 10 and 12 are expected because $\boldsymbol{\xi}_n$ is computed at the later epoch. Alternatively, employing the *averaged* state, the landscape appears differently, as is apparent in Figure 13. Several features are notable in Figure 13. First, as was observed in Figure 10, transitioning into the later modes is characterized by an apparent increase in chaoticity of the system likely associated with an energy change at the transition. In this case, *the transitioning behavior is induced much earlier* and the resulting FTLE field is populated with several profiles that flatten out consistent with escapes (again, characteristic of cerulean coloring). However, a more contiguous region of dark blue traces that run through the perturbations ranging –[5..25] $\times 10^{-3}$ $\boldsymbol{\xi}_n$ is also noted. These particular traces are reflective of the mode behavior observed in the terminal phase of the original perturbed baseline solution, and they persist for several decades in many cases. Moreover, interspersed throughout the map, roughly 4% of the trajectories maintain the ideal FTLE signatures with slight or no deviations. For example, consider the FTLE trace included as Figure 14(a) that results from a solution that has been propagated from initial conditions defined from a step equal to $-0.0121 \times \boldsymbol{\xi}_n$. The plot's vertical axis is held to scale with Figure 5(f) for comparison, and the FTLE data for the first 7.77 years from Figure 5(f) is also included prior to the red dashed line, where the perturbed state is invoked. The associated radius history is plotted in Figure 14(b). In general, the spacecraft perigee and apogee radii are further from the Earth and Moon, respectively, lessening the impact of perturbations from these bodies.

By inducing transitioning behavior earlier in the mission time-frame, potential long-term, increased stability options are available. A specific example associated with the FTLE trace from Figure 14(a) initiates 7.77 years after the beginning of the TESS mission, and the 100-year evolution of this solution is included in Figure 15. This orbit generally fills out a bounded shell in the rotating frame and serves as an example result-

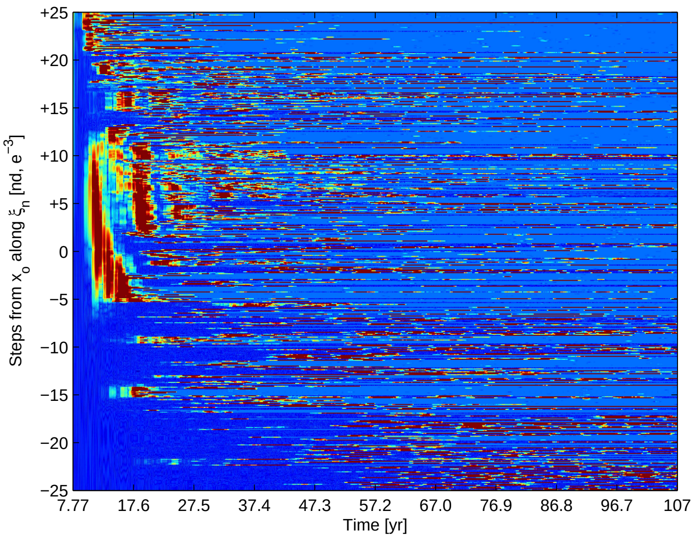


Figure 13. Averaged State and Perturbation FTLE Traces

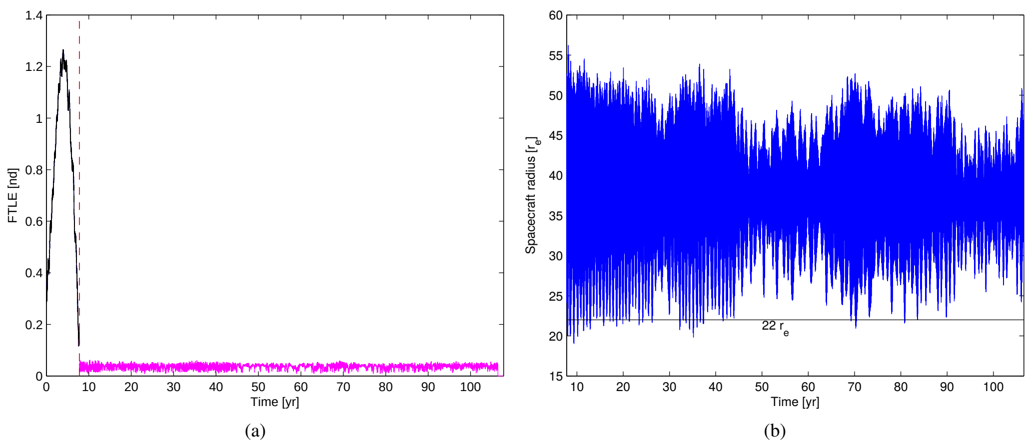
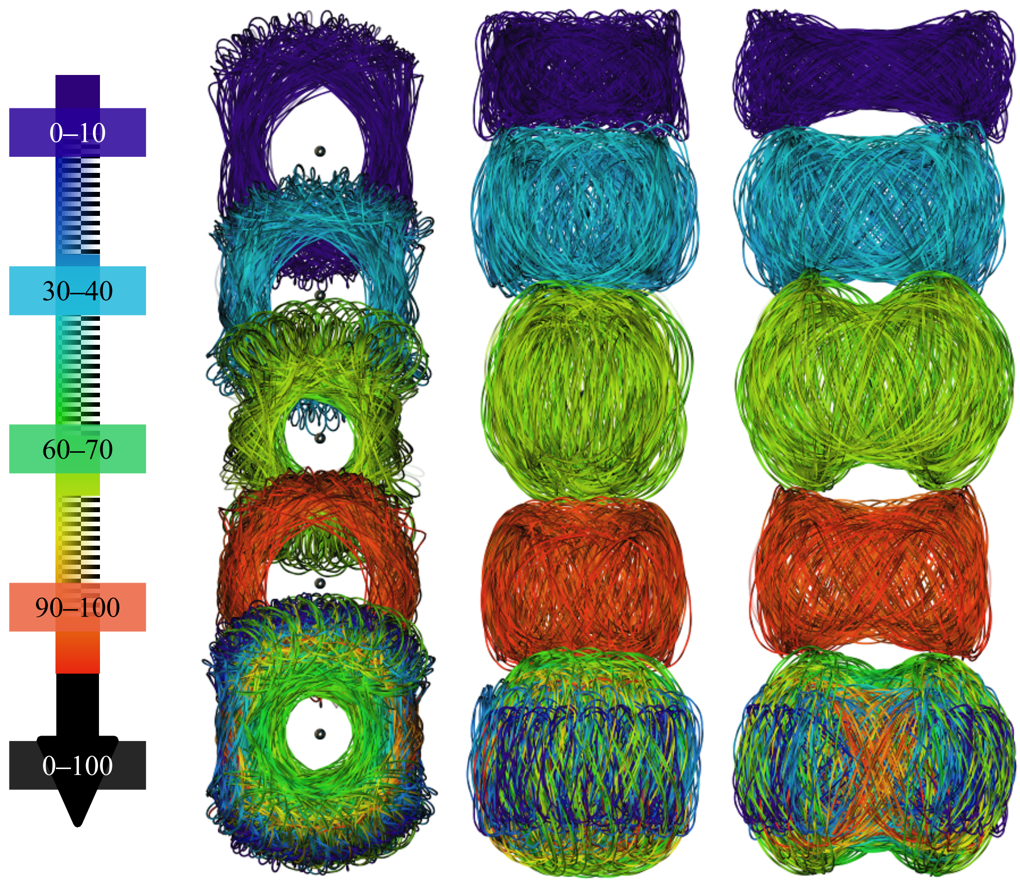


Figure 14. FTLE Trace and Radius Profile from Perturbation to Averaged State

ing from the process detailed here. The orbit is observed to evolve through various modes apparent through the coloring in Figure 15. As the orbit continues through its 100-year propagation, the coloring ranges from initial phases represented in blue through intermediate stages with mid-spectrum colors of yellow and green and ultimately concludes with red coloring interior to the shell traced out by earlier revolutions. The overall evolution is characterized by increased r_p with respect to the Earth while maintaining a modest upper bound on r_a as the orbit maintains an adequate distance from the Moon. Such a solution represents many possible options that may be isolated through numerical exploration of perturbations strategically placed in time. These options are isolated by (1) identifying specific modal behavior exhibited by perturbed solutions, (2) seeking similar system configurations at other times throughout the path evolution, and (3) searching in the flow behavior of the nearby phase space to identify potential trajectory responses.



(a) Time [yrs–7.77] (b) x–y View (c) x–z View (d) y–z View

Figure 15. 100-year Ephemeris Evolution of Orbit from Figure 14 (Rotating Frame View)

The selected science orbit for the TESS DRM supplies an effective end-of-life option with a dispose-in-place strategy. However, the sensitive operating regime allows for the possibility of transitions in the long-term spacecraft trajectory behavior that may or may not be acceptable. Greater knowledge of the complex environment in the vicinity of such a solution can lead to other, useful options. In the example highlighted in this analysis, particular solutions characterized by notable long-term behavior may be realized by identifying a particularly advantageous epoch for shifting into alternate modes and perturbing the spacecraft state to accomplish evolution into the new option. Such an engineered transition would likely require planning to execute one or more maneuvers at an earlier epoch of the trajectory evolution and, while likely feasible, would warrant additional analysis and assessment.

CONCLUSION

The dynamical regime associated with the Earth–Moon–Sun system is rich with complexity and includes intricate perturbations that lead to chaotic behavior. After characterizing an orbit, one that is slightly perturbed from and generally resembling the TESS DRM, this analysis supports a strategy for inducing useful modes of evolving trajectory behavior. However, it is also observed that the chaoticity of the system can potentially perturb results in an unfavorable way, and further analysis is warranted.

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