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DYNAMICAL EVOLUTION ABOUT ASTEROIDS WITH HIGH FIDELITY GRAVITY FIELD AND PERTURBATIONS MODELING

Andrea Colagrossi[*], Fabio Ferrari[†], Michèle Lavagna[‡] and Kathleen Howell[§]

The paper presents different strategies to model the gravitational field in the vicinity of irregular celestial bodies, such as asteroids and comets. The gravitational attraction of these irregular objects has been modeled, through accurate shape discretization, with a constant density polyhedron or an ensemble of point masses. In the latter case, an optimization algorithm to distribute the mass elements within the volume of the body has been developed. All the different modeling techniques are compared in order to highlight their advantages and drawbacks. In addition, an extensive analysis of the results is performed with the purpose to find the model that has an optimal balance between level of accuracy and required computational effort.

INTRODUCTION

An irregularly-shaped body influences the surrounding space according to the Newton's law as any other celestial body. However, its shape is not spherically symmetric, like the one produced by a centrobaric body, and the irregularities of the surface are reflected in the generated gravitational field. This characteristic is probably the one that influences most the dynamical environment in the vicinity of Solar System's smaller bodies, in fact, the resulting dynamics is extremely different from the Keplerian one.

In this paper, few different techniques to model the gravitational potential generated by an irregular body are presented and compared. The main goal of this work is to determine the level of accuracy of the considered modeling approaches, within the neighborhood of the attractor. The present analysis uses some reference shapes, real and ideal, to perform this task. The computational burden is always monitored in order to be able to find a trade-off between the accuracy and the time required to compute the gravity field.

In the first part of this paper, the models to evaluate the gravitational field of an irregularly shaped bodies are presented and some theoretical aspects are discussed. Then, the accuracy of each model is assessed within the domain of study and the different results are compared.

GRAVITY FIELD MODELS

The study of gravity models for irregularly-shaped bodies has been greatly enhanced in the past decades to allow for a better description of the dynamics around the main bodies of the Solar

[*]Researcher, Department of Aerospace Science and Technology, Politecnico di Milano, Milan, 20156, Italy.
[†]Ph.D. Candidate, Department of Aerospace Science and Technology, Politecnico di Milano, Milan, 20156, Italy.
[‡]Associate Professor, Department of Aerospace Science and Technology, Politecnico di Milano, Milan, 20156, Italy.
[§]Professor, School of Aeronautics and Astronautics, Purdue University, West Lafayette, 47906, Indiana, USA.

System, since none of them is exactly spherical. A common way of modeling non central fields is to use harmonic series expansion of the spherical potential. A fundamental assumption is that the series converge to the actual potential associated to the attractor. This is the reason why harmonic expansions have several drawbacks when dealing with irregularly-shaped bodies. Two alternatives, more suitable in case of small irregular bodies, are used here. They both arise from a simple but effective principle: the unknown potential function of any complex body can be represented with an ensemble of smaller components, whose potential function is known. The elementary units are point masses for the mass concentration approach, referred as *mascons* in the following, and constant-density polyhedra for the *polyhedral* approach. The former one has been introduced by Geissler[1] and the latter by Werner and Scheeres[2,3] .

Mascons approach

The mascons approach is surprisingly simple from a conceptual point of view because it uses several point masses to reproduce the body's mass distribution. The resulting gravity field depends on the number of employed masses, and on how they are distributed. While, the mass of the total body is always preserved, there exist different methods to distribute the point masses in the body's domain. Classical strategies[1] employ an evenly spaced grid and will be referred as *gridded mascons approaches*. However, in this paper, the point masses are also distributed according to an optimization process, which is fundamental for the optimized version of the mascons technique. The obtained results are presented and then compared with those related to the gridded version. The purpose is to increase the efficiency of this method, which is very attractive in term of necessary computational speed.

The gravitational potential of the whole body, making use of N_m mascons, is:

$$U = G \sum_{i=1}^{N_m} \frac{m_i}{r_i}, \tag{1}$$

where m_i and r_i are, respectively, the mass value and the distance from each point mass. The previous equation converges to the true gravitational field for $N_m \to \infty$.

As already said, for a fixed number of masses there is not a unique way to distribute them inside the volume circumscribed by the body. Hence, the *optimized mascons approach* is developed to find the optimal allocation of mascons, trying to recreate the field of an assumed *model of reality*, that will be described in the following. It is worthwhile to point out that for the case of asteroid modeling, the desired optimum distribution of mass is not the real mass distribution of the actual body, which is usually unknown. Certainly, if the exact mass model were available, the optimum solution would be the one closest to the reality, with the purpose to obtain the most accurate gravitational field. Notwithstanding, the optimization algorithm attempts to match the field generated by the best reference model.

Optimized Mascons

The aim of this research work is to find a good representation of the gravity field around an irregularly-shaped body to be used to describe the dynamics of a particle in its vicinity. Therefore, the goal of the optimization problem is to find the solution that minimizes the error in the gravitational contribution inside the equations of motion: the gradient of the potential, ∇U. Furthermore, the error on ∇U is an upper bound for the error on the gravitational potential, and thus, the latter is guaranteed not to diverge. In practice, the optimization algorithm finds the best mascons distribution that minimizes the mean of error on the ∇U computed on a 125000 points 3D grid surrounding the main irregular object, up to a distance 5 times the dimension of the body in each coordinate direction. The error is computed as a percent value with respect to the reference model that will be described in the next section. The algorithm can move the mascons within the volume of the body and the mass is not constrained to be the same for each point mass. As long as the global mass of the body is maintained constant and no point mass is outside the physical boundaries of the celestial object, any position and any mass value for the single mascons are allowed. Consequently, this is a constrained optimization problem and it is implemented using a genetic algorithm. The genetic algorithm is a valid algorithm since both the objective function and the constraints are non-smooth, and there is no analytical and differentiable formulation of the problem.

Polyhedron Shape approach

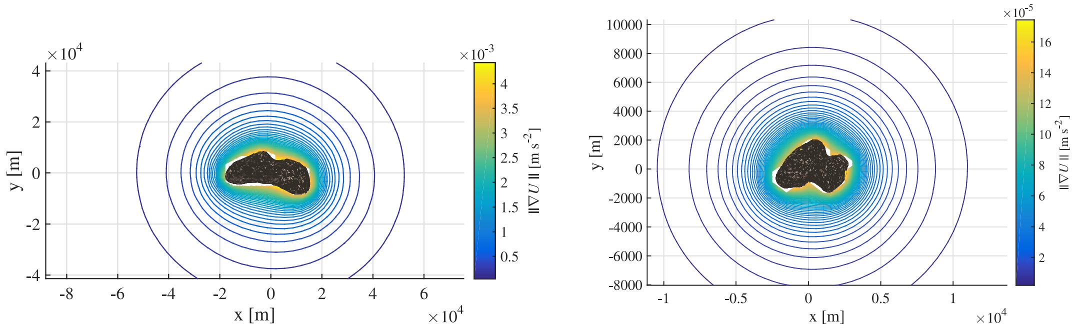
The polyhedron shape approach is based on the concept that any body of arbitrary shape can be approximated with a polyhedron having a variable number of triangular faces. Exploiting the analytic form of the gravitational potential of a homogeneous polyhedron having triangular faces , it is possible to evaluate the field generated by any irregularly-shaped body by collecting all elementary contributions together. The closed-form analytical expression of the exterior gravitational influence of a constant-density polyhedron guarantees that the gravity field is exact in any portion of space for the given shape and density. For this reasons, the field obtained with this technique, together with a shape discretization with an high number of faces, is here assumed as the *model of reality* and used as reference model for the optimized mascons approach. Notwithstanding all these positive features, this method is markedly expensive in terms of the computational cost, as the entire surface must be summed over to achieve one single force value, and certainly this cost increases with the resolution of the shape discretization.

In the work of Werner[2] the gravitational potential of a constant density polyhedron is derived exploiting the Gauss's divergence theorem, and without going into details, the final expression for the potential of the polyhedron is:

$$U = \frac{1}{2} G\sigma \sum_{e \in edges} \mathbf{r}_e \cdot \mathbb{E}_e \cdot \mathbf{r}_e \, L_e - \frac{1}{2} G\sigma \sum_{f \in faces} \mathbf{r}_f \cdot \mathbb{F}_f \cdot \mathbf{r}_f \,, \tag{2}$$

where the first sum extends over all the edges and the second one over all the faces of the polyhedron. In particular, $\mathbf{r}_e$ is a vector from the field point to each edge, $\mathbb{E}_e$ is a dyad defined in terms of the face and edge normal vectors associated with each edge, L_e express the potential of a 1-D line, $\mathbf{r}_f$ is a vector from the field point to each face, $\mathbb{F}_f$ is a dyad defined for each face, ω_f is the solid angle, with sign, subtended by a face when it is seen from the field point and, finally, σ is the average density of the homogeneous polyhedron.

Some celestial bodies, whose shape is known due to spacecraft fly-bys or radar observations, are analyzed in this paper: 67P Churyumov-Gerasimenko, *67P C-G*, comet (ESA's Rosetta Mission, 2014, http://rosetta.esa.int). 216 Kleopatra Main-Belt asteroid, 4179 Toutatis Apollo-Alinda asteroid, 433 Eros Amor asteroid (Planetary Data System Asteroid and Dust Archive of NASA, http://sbn.psi.edu/pds). 1580 Betulia Amor asteroid (Asteroid Radar Research Group of JPL, http://echo.jpl.nasa.gov).



(a) 433 Eros Gravitational acceleration, $\|\nabla U\|$. (b) 67P C-G Gravitational acceleration, $\|\nabla U\|$.

Figure 1: Gravity field with $\sim 10^4$ faces polyhedron model.

Gravitational Influence of Selected Objects

The bodies that are analyzed in this research work are representative of different classes of shape for real celestial bodies. In particular, the comet Churyumov-Gerasimenko is representative for bodies with two relevant bulges, the asteroid Kleopatra for objects that have a dog-bone shape, Toutatis and Eros represents those bodies that have one dimension longer than the other two, and Betulia characterize the asteroids that are more similar to a usual spherical shape.

The field is computed in a 3D grid of $50 \times 50 \times 50$ points along the axes of the body-fixed reference frame. The dimensions of the analyzed space are relative to the selected body, indeed the grid is 5 times the dimension of the body in each coordinate direction. For example, the projections on the xy-plane for the gravitational acceleration of the asteroid 433 Eros and the comet 67P Churyumov-Gerasimenko are shown in figure 1. These results were computed by means of the polyhedron approach, and it is reasonable to wonder what are the differences with the results obtained exploiting the mascons approach.

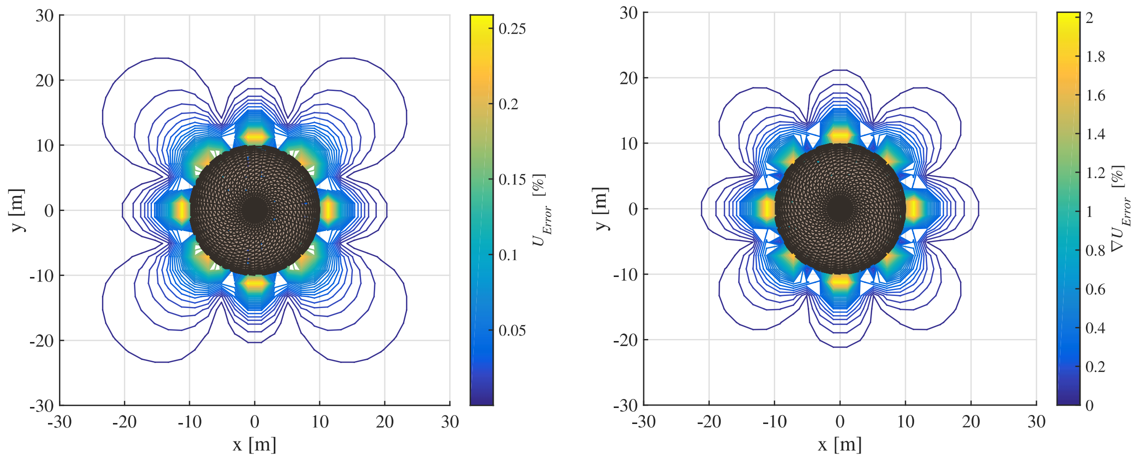
GRAVITY MODELS COMPARISON

First, to have a preliminary and insightful overview on the differences between the polyhedron and the mascons approach, their results can be compared using simple geometries, such as spheres and ellipsoids.

Sphere and Ellipsoid Analysis

The two gravitational modeling techniques under study are here compared on a sphere and on two axially symmetric ellipsoids. The study of these simple geometries allows focusing the analysis on the general differences between the two different approaches, with the goal to define some guidelines that are valid for all the particular real situations.

The sphere has radius equal to $10\,\mathrm{m}$ and its field is computed using a ~ 3000 faces polyhedron and ~ 5000 mascons on an evenly spaced grid. The overall mass of the spherical body is maintained as constant between the two different techniques. The potential and the gravitational acceleration are evaluated in a 3D grid with 125000 points around the central body, and then the relative error between the two techniques is computed. The polyhedron model is assumed to be more accurate than the mascons one, since its fidelity depends only on the shape approximation of the body. In this case,



(a) Percent error on gravitational potential. (b) Percent error on gravitational acceleration.

Figure 2: 5000 mascons and 3000 faces polyhedron approach comparison on a $10\,\mathrm{m}$ sphere.

the sphere is very well reproduced with the chosen number of faces, and therefore the polyhedron model is taken as a reference to estimate the relative error of the mascons approach. As example, the relative error on the gravitational potential is computed, in percentage, as:

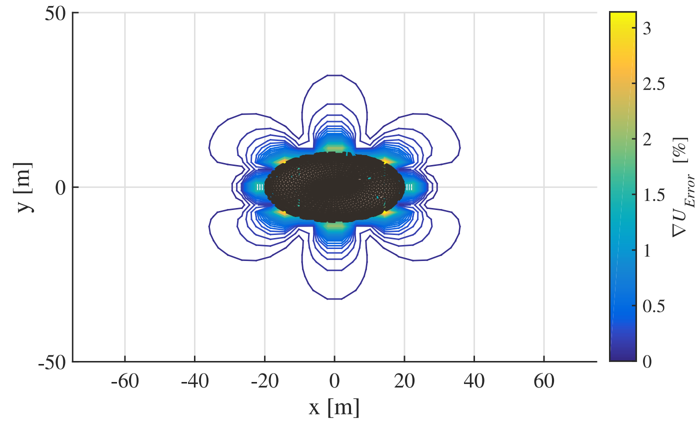
$$U_{Error} = 100 \times \left| \frac{U_{Poly} - U_{Mascons}}{U_{Poly}} \right|, \tag{3}$$

and in a similar manner the error on the acceleration is available.

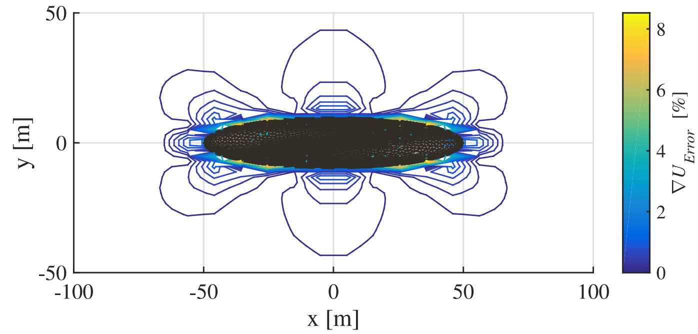
The result obtained, on the $xy-$plane, is reported in figure 2. As a first remark, the error is high close to the surface and decreases at a greater distance from the body. This can be explained considering than the real solid body is replaced with a discrete number of point masses, which are singularities for the field, and close to them this difference is more evident. Then, the lobed shape of the error can be noted, and it can be explained assuming that the point masses are represented as spheres that fill the volume of the body, and that they touch without overlap. In this way, a cubic differential volume of the real continuous body can be replaced with its inscribed sphere, but each sphere occupies only 52% of the original volume of the cube, and its density is twice the original. So, if the gravitational field of a unit mass cube is compared with the field of a unit mass inscribed sphere a lobed error pattern is obtained. Hence, the lobed shape of the error is the typical result for the mascons approach, and the obtained pattern depends on the distribution of the point masses inside the body.

In figure 2a, the maximum error of the mascons approach is around 0.25%, while in figure 2b the error is one order of magnitude higher, and its maximum value is approximately equal to 2%. This is another general feature of the comparison of the mascons approach with the polyhedron model; in fact, the acceleration field is normally less accurate than the potential field. In the mascons approach, the source of the field is a point mass, which is a singularity, and the differentiation process enhance this problem. Thus, the mascons gravitational acceleration has a large error with respect to the potential, and the region in which the error is still relevant extends well away from the body.

Obviously, the modeling error is proportional to the mascons resolution, and with a larger number



(a) Percent error on gravitational acceleration, $2/1/1$ ellipsoid.



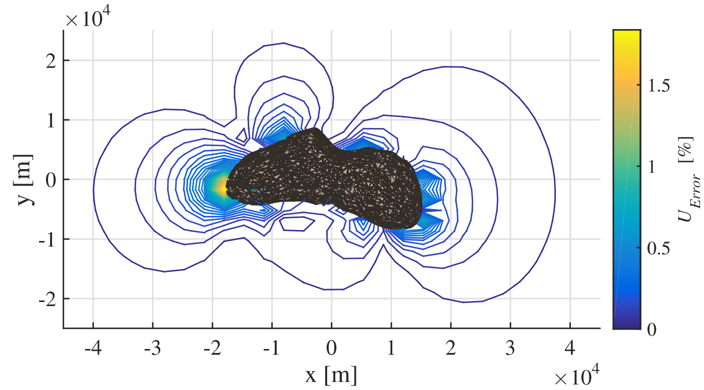
(b) Percent error on gravitational acceleration, $5/1/1$ ellipsoid.

Figure 3: 5000 mascons and 3000 faces polyhedron approach comparison on $20\,\mathrm{m}$ and $50\,\mathrm{m}$ ellipsoids.

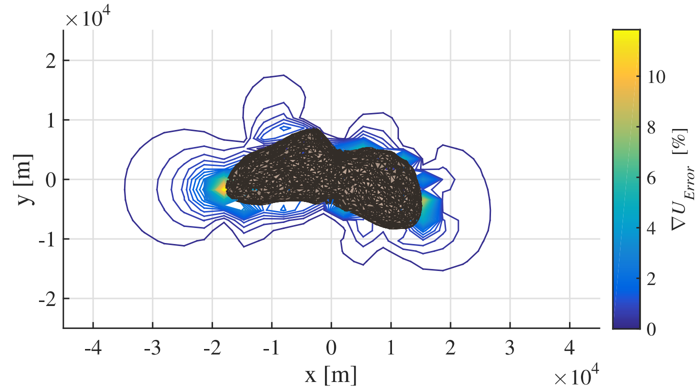
of point masses, the error can be further reduced. However, according to[3] , the number of employed mascons is inversely proportional to the cube of the size of each mascon. As a consequence, to reduce the error of the mascons model of one order of magnitude, the number of point masses has to be increased, in general, of three orders of magnitude.

The same analysis can be repeated also with simple elongated bodies; in this case two axially symmetric ellipsoids. These bodies are assumed to have the same density and transversal dimensions of the sphere presented before. So, the semi-axes along $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ are equal to $10\,\mathrm{m}$. The ellipsoids have been discretized with a ~ 3000 faces polyhedron and ~ 5000 gridded mascons. The $\hat{\mathbf{x}}$ dimension characterizes the two different cases with two different elongation levels: one ellipsoid has the third semi-axis equal to $20\,\mathrm{m}$, while the other has that semi-axis equal to $50\,\mathrm{m}$. They will be referred to as $2/1/1$ and $5/1/1$ ellipsoids, respectively. The relative errors between the mascons and the polyhedron acceleration are reported in figure 3.

Same considerations apply, as discussed before with the spherical case, showing that common features are shown to be related with the mascons approach and are independent from the geometry of the object that is being considered. Nevertheless, looking at the two different ellipsoids, another important characteristic of this modeling technique can be highlighted. The maximum error in figure 3a is in the order of 3%, and in figure 3b is around 8%. For the spherical case, it was $\sim 2\%$.



(a) Percent error on gravitational potential.



(b) Percent error on gravitational acceleration.

Figure 4: ~ 2500 mascons and $\sim 10^4$ faces polyhedron approach comparison on 433 Eros.

This is not casual, and in general, the mascons approach produces less accurate results when working with elongated bodies. The spherical symmetry of the field generated by a point mass works better with bodies that maintain this symmetry also in the mascons distribution. With oblong geometries, several spherical fields must be aligned to produce the elongated shape of the overall contribution; this leads to a higher error on the longer side of the object. In fact, the error close to the shortest side of the ellipsoid is lower than the one along the $x-$axis, as can be seen from the small lobes on the left and the right side of the body. The mascons approach is more suitable for computing the field close to quasi-spherical and not elongated geometries.

The comparison between polyhedron and mascon approach is also performed considering the real geometries of the selected celestial bodies. As an example, the Eros case is reported in figure 4. The error appears to be higher and less intuitive to understand, because of the complexity of the real asteroid shape. However, some features that have been described before are still present and evident; this is in general true for all the real geometries studied within this work.

As a general remark for the gridded mascons approach, the shape of the error is dependent from the number of point masses that are used. This is because the distribution of mascons in a grid changes for a different number of masses, and these different arrangements are not determined by the shape of the body, but just from a simple subdivision of the internal volume. In this way, some

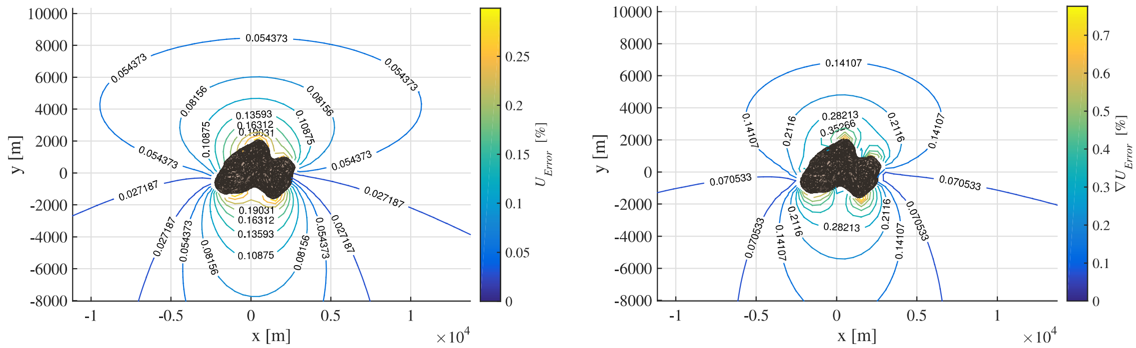


Figure 5: Percent error on gravitational potential and acceleration of 15000 faces polyhedron on 67P with respect to the Hi-Fi polyhedron model.

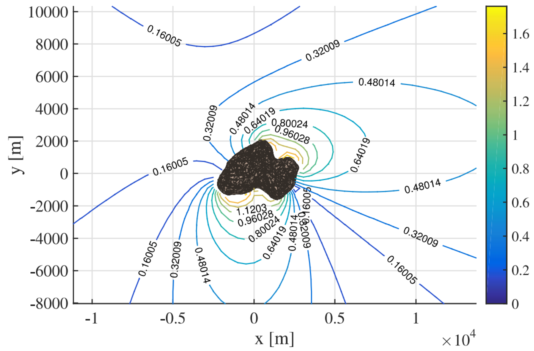
characteristics of the field of the object can be accurately represented with a certain resolution of the grid and not with a different one. So, even if the magnitude of the inaccuracy normally decreases if the number of mascons is increased, the shape of the error does not follow a well defined trend and the error analysis should be performed from case to case.

All these aspects will be discussed later. Now it is useful to understand the influence of the number of faces on the results of the polyhedron shape approach.

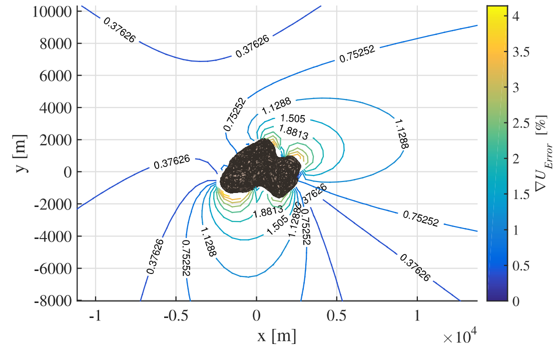
Polyhedron Shape Approach Analysis

The polyhedron shape approach gives results with an accuracy determined by the resolution of the shape discretization. For this reason, different results of this modeling technique have been compared for different number of faces of the polyhedron shape model. The Delaunay triangulation, which is used in this paper, allows rescaling the original shape model and obtaining a new one with an arbitrary number of faces. Obviously, the geometry of the body is slightly modified and the models with a low number of polyhedra lose several surface details. Notwithstanding, the scaling algorithm tries to preserve the overall shape and volume of the body, and if a reasonably low number of faces is employed, the model drops only the finer details on the surface of the object. In fact, the global topology is preserved because the algorithm only applies minor modifications to the positions of the vertices of the polyhedron, in order to complete the surface with a different number of triangular faces. In the following analysis, the Hi-Fi polyhedron model, $\sim 2 \times 10^4$ faces, is the reference to estimate the relative error of the different Lo-Fi models. The strategy to perform the comparison is the same that has been presented before, and the only difference lies in the involved quantities. Now, indeed, is not the mascons approach to be compared with the Hi-Fi polyhedron model, but several Lo-Fi polyhedron models are compared with a higher fidelity field obtained with the same technique.

In figure 5, the Hi-Fi model is compared with a very similar Lo-Fi model composed by $N = 15000$ faces. While, in figure 6, the Lo-Fi model has $N = 1000$ faces and the difference with the reference field is more noticeable. As expected, the error is higher when a lower fidelity model is used, and the gravitational acceleration has a large error if compared with the potential. However, in this case, the difference is not as apparent as in the mascons case, since the polyhedron has not dimensionless elements and the differentiation process operates on smooth quantities. Even in this example, the error is higher close to the surface, mainly because the details on the surface are different or not



(a) Percent error on gravitational potential.



(b) Percent error on gravitational acceleration.

Figure 6: Percent error on gravitational potential and acceleration of 1000 faces polyhedron on 67P with respect to the Hi-Fi polyhedron model.

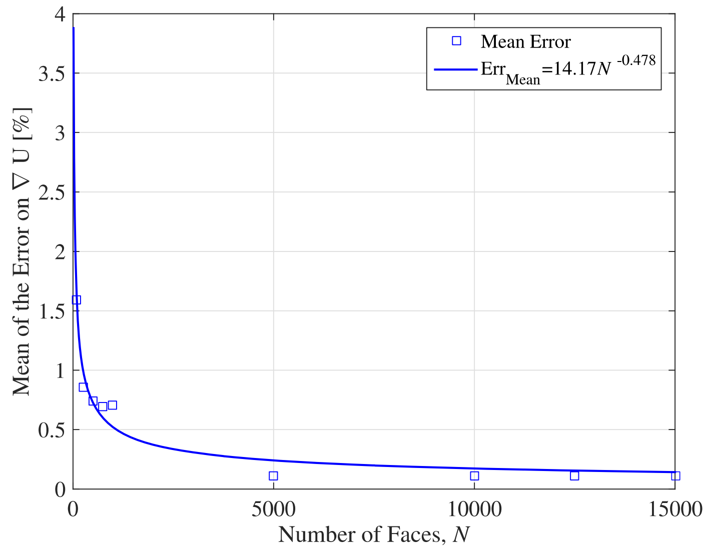
present anymore when a lower number of faces is used.

At this point, in order to have a general trend of the error, as a function of the number of employed faces, several Lo-Fi models are compared with the Hi-Fi polyhedron model. Nevertheless, a more intuitive understanding of the behaviour is possible if the error data are reduced in a handy form.

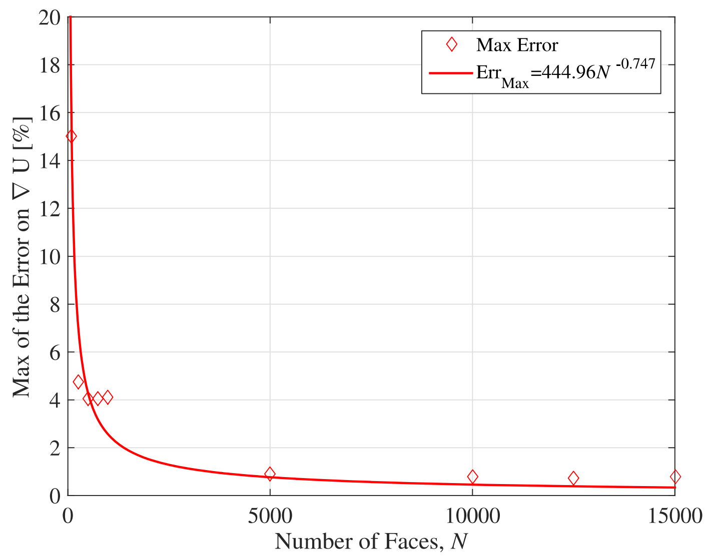
The error data contained in the 3D grid around the body are statistically analysed in order to obtain some quantities that can be easily compared. In the present situation, involving different resolution polyhedron models, the mean and the maximum values of the error in the field are evaluated and compared. They are sufficient to describe the different instances, since the obtained results have approximately the same variance and the error distribution is well defined using these two parameters. Actually, only the mean of the error is necessary to characterize the error field, but the maximum value is useful anyway to know the bounds of the resulting accuracy. Figure 7 shows the evolution of these two quantities for different values of N on the comet 67P Churyumov-Gerasimenko.

The trends in figure 7a and figure 7b are the result of a nonlinear regression on the available data, the obtained best-fitting equations are presented in the legend of the plots. These particular relations have not general validity, since they are evaluated for a particular shape model. However, these trends are obtained also for other shapes, and in general, the mean of the error shows to approximately decrease with the inverse of the square root of N. This result can be used to estimate the increase in accuracy that can be obtained with a different number of faces with respect to an already known error field.

The increase in accuracy is not for free, and the drawback is the increase in the needed computational time. The threshold to define a tolerable time depends on the particular application; therefore, the optimal number of faces that balance a reasonable level of accuracy with an acceptable computational time cannot be univocally defined and the requirements must be stated from case to case. Anyway, the time required to evaluate the different polyhedron fields is shown in figure 8, and these values could be used as a reference for future works. In this analysis, the gravitational potential and acceleration are computed using a MATLAB® code that makes use of parallel computing techniques on a quad-core $2.5\,\mathrm{GHz}$ processor. The code computes the field for a given z value on a 50×50 2D grid, thus the time to compute these quantities on a single field point is $\mathrm{t_C}/2500$. The obtained best-fitting equations show, as expected, a linear relation between the different values of N.



(a) Mean of the error on ∇U.



(b) Maximum error on ∇U.

Figure 7: Percent error on gravitational acceleration of N faces polyhedron on 67P with respect to the Hi-Fi polyhedron model.

Mascons Approach Analysis

A similar analysis is carried out for the gridded mascons approach, with the purpose to see how a different number of mascons affect the result if compared with the Hi-Fi polyhedron model.

The gridded mascons approach fill the volume with point masses arranged in a grid that is obviously dependent from the number of used mascons, as can be seen in figure 9, for the asteroid 216 Kleopatra. In addition, the standard gridded mascons use N_m point masses of equal mass $m = M/N_m$. The resulting field is more accurate if a larger number of mascons is used, but as already said, also the

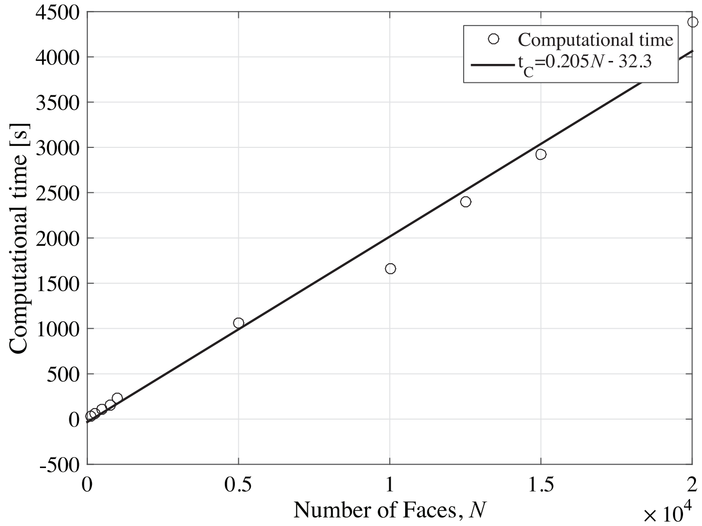
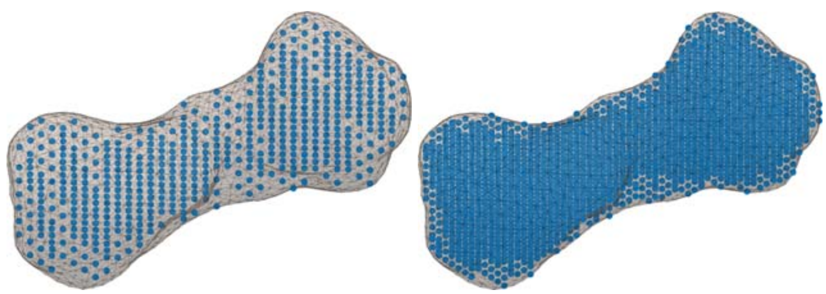


Figure 8: Required computational time with N faces polyhedron model on a 50×50 2D grid, with parallel computing techniques on a quad-core 2.5 GHz processor.

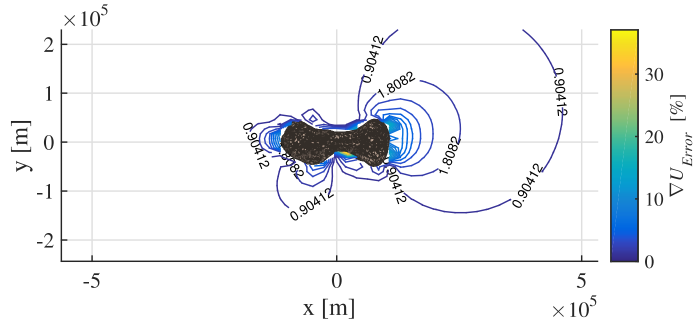


(a) $\sim$ 500 point masses. (b) $\sim$ 2500 point masses.

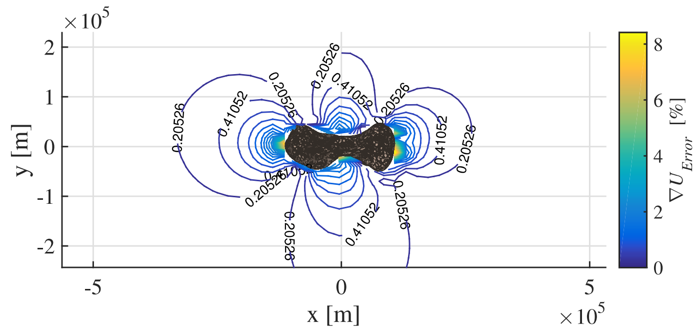
Figure 9: Gridded mascons approach on 216 Kleopatra.

shape of the error is dependent from N_m. A dissimilar distribution of masses can determine a better or worse representation of the particular body, and if the object is filled with a regular grid, the result can be particularly different even if a similar number of mascons is used. That is because the gridded approach creates the evenly spaced grid along $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ regardless of the actual shape of the body. This problem is notably relevant if N_m is low, since, with few masses, adding or subtracting only one element can determine a drastic change in the mass distribution, and as a consequence, in the obtained field.

This concept is more clear looking at figure 10, where the errors of the gravitational acceleration are computed comparing two different mascons models and the Hi-Fi polyhedron. The maximum error for the $\sim$ 2500 mascons, figure 10b, is obviously lower than the error for the model containing $\sim$ 500 mascons, figure 10b, but there is no actual relation between the two shapes and distributions of error. This can be a problem, because there is not a deterministic evolution for the reduction of the error and each case must be analysed to understand the level of accuracy. For sure, the general rule to increase N_m in order to reduce the error is globally valid, but a not negligible level of randomness is still present. Moreover, it is reasonable to wonder if a better way to distribute the N_m mascons



(a) $\sim$ 500 point masses.



(b) $\sim$ 2500 point masses.

Figure 10: Percent error on gravitational acceleration with gridded mascons on 216 Kleopatra.

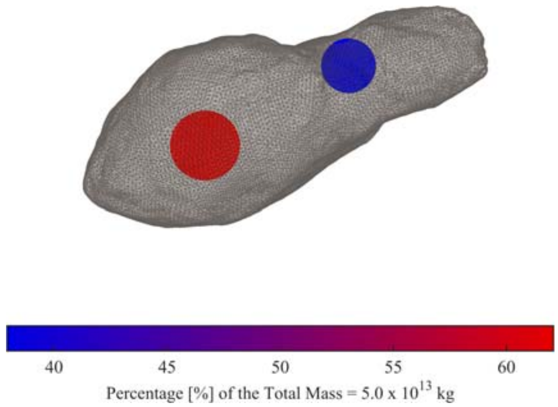
exists. Hence, an optimized version of the mascons approach is developed and presented. Then, both the gridded and the optimized mascons are compared together, as a function of the number of used point masses.

Optimized Mascons Output

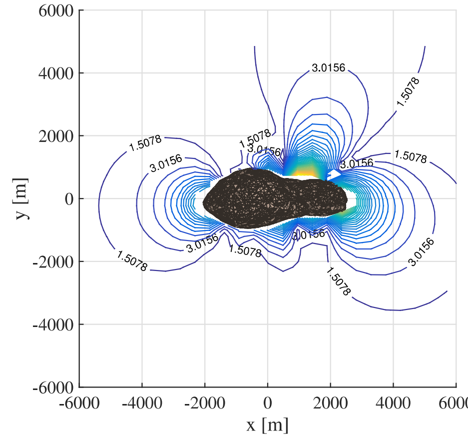
The optimized mascons approach has been applied to all the selected bodies. Some results regarding 4179 Toutatis are shown here, in figure 11.

These plots are representative for the main advantages of the optimized mascons approach. In fact, as a first remark, the reduction of the error follows a reasonable trend, and it is possible to predict the evolution of the error shape. Moreover, the optimized distribution of the point masses inside the body determines low errors with respect to the gridded mascons approach. As a consequence, the optimized mascons model produces acceptable results that can be utilized for a preliminary study of the dynamical environment. The error is high, close to the surface, but this extremely simple model allow the usage of the circular restricted three-body problem to model the dynamical environment far enough from the two point masses. This formulation has several advantages, since a lot of studies on three-body systems exist, which can allow an insightful overview of the environment in the vicinity of the celestial objects to be modeled.[4,5]

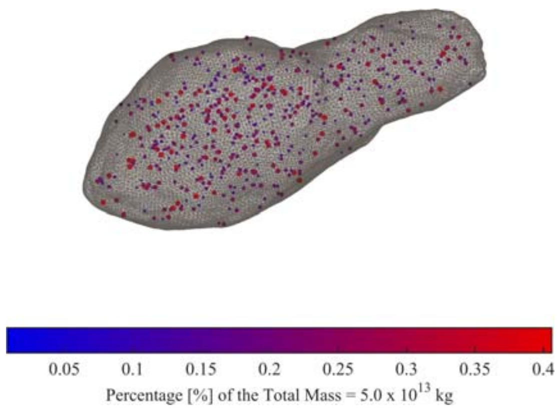
The error clearly decreases for an increasing number of masses, but in order to have a better understanding about the relation between the number of mascons and the error, a statistical analysis, similar to the one presented for the polyhedron approach, is performed.



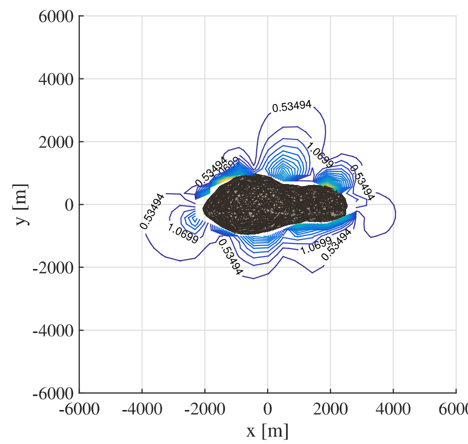
(a) Optimum mascons, $N_m = 2$



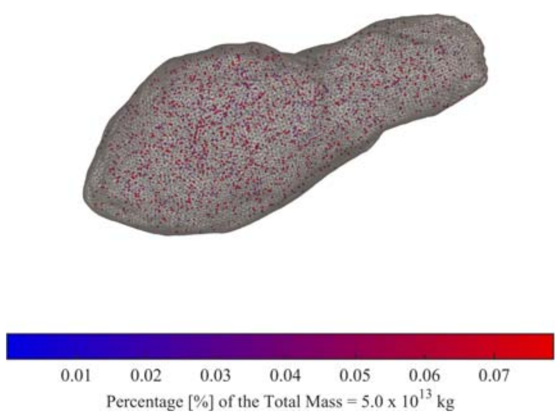
(b) Percent error on gravitational acceleration, $N_m = 2$



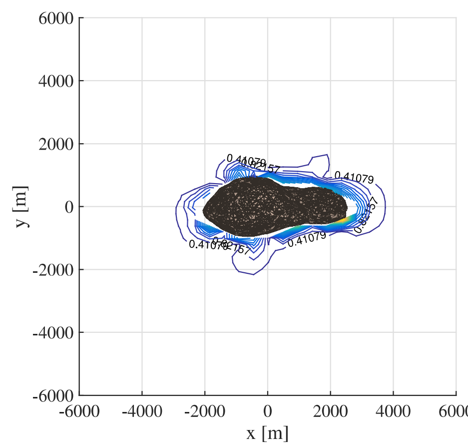
(c) Optimum mascons, $N_m = 500$



(d) Percent error on gravitational acceleration, $N_m = 500$



(e) Optimum mascons, $N_m = 2500$



(f) Percent error on gravitational acceleration, $N_m = 2500$

Figure 11: Optimized mascons approach on 4179 Toutatis.

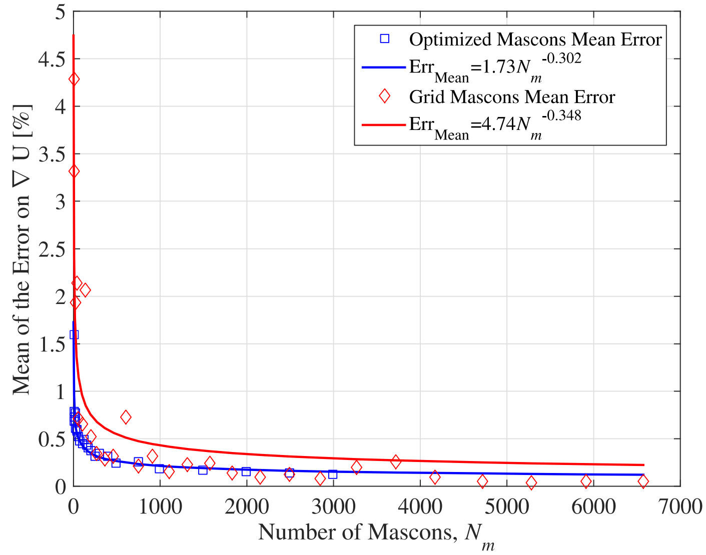


Figure 12: Percent error on gravitational acceleration with N_m mascons on 67P.

Mascons Models Comparison

The gridded and the optimized mascons approaches are here compared through the mean value of the error on the gravitational acceleration around the irregular body. This quantity is used to characterize the error field and as the fitness function to be minimized during the optimization process, to find the best mascons distribution. This section shows how the solution of the optimized approach improves over the standard gridded approach. Figure 12 shows the relation between the mean value of the error and the number of point masses, N_m, on the comet 67P Churyumov-Gerasimenko. It is worth remembering that the relative error is computed with respect to the Hi-Fi polyhedron model.

A nonlinear regression is performed on the available data and the obtained best-fitting equations are presented in the legend of the plot. These relations are quite interesting since they both agree with the 3-to-1 order of magnitude rule, already described in the section about the analysis of sphere and ellipsoids. The optimized mascons results are very well represented by this trend, while the gridded approach has the already mentioned component of randomness. The 3-to-1 trend is generally followed also by the standard approach but to completely know the level of accuracy each single case must be analysed.

The good agreement of the optimized technique with this trend means that the optimization process improves the performances of the mascons approach up to its theoretical limit. Hence, to reduce the error of the optimized mascons model of one order of magnitude, N_m has to be increased by three orders of magnitude. In this way, the accuracy of an optimized mascons model can be precisely estimated knowing the accuracy of a certain reference field. Indeed, the particular fitting equations shown in figure 12 are valid for the analysed case of the comet 67P, but they can be easily scaled for another body once the error for one N_m value is computed.

As already said, with few masses the accuracy of the gridded mascons is extremely dependent from the exact number of employed masses, and furthermore, the improvement due to the optimization process is remarkable. Consequently, for low N_m the optimized approach is extremely valuable and must be preferred to the standard modeling technique. On the other hand, for numerous point masses

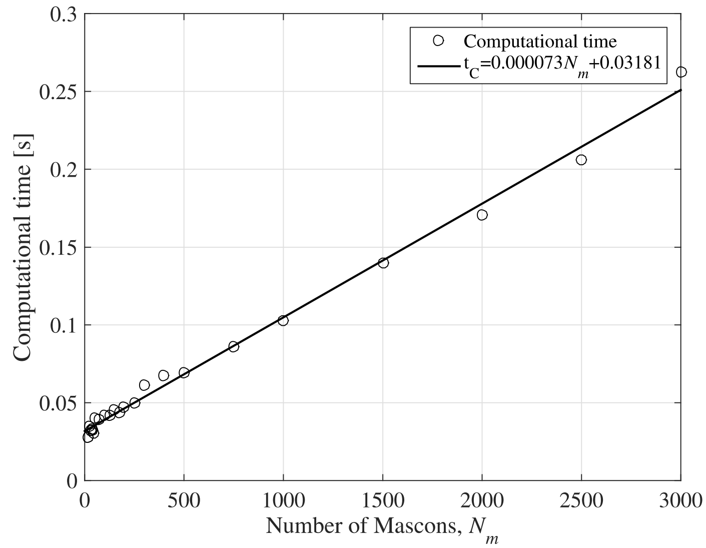


Figure 13: Required computational time with N_m mascons model on a 50×50 2D grid, with parallel computing techniques on a quad-core 2.5 GHz processor.

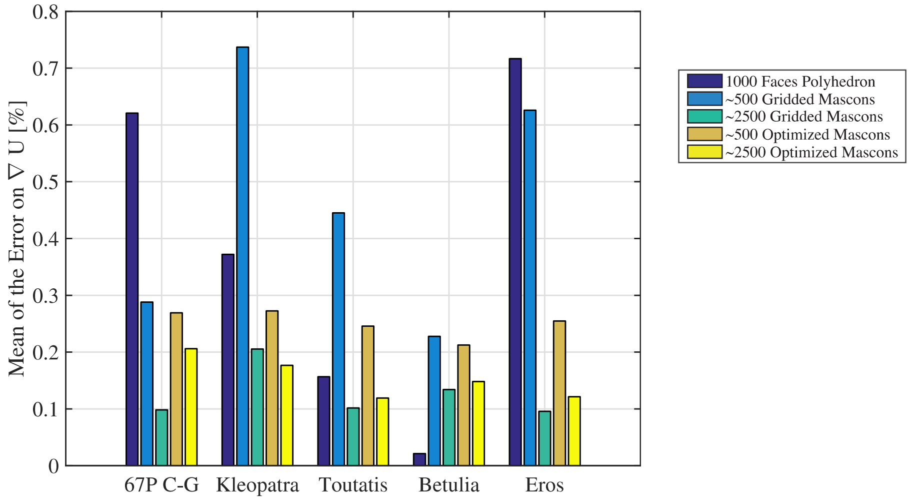
this difference tends to be small, or in practice, unreal. So, if N_m is large enough ($N_m > 2000$), the effort to optimize the mass distribution is not worth. This is reasonable since, when many masses are employed, the differences between distinct mass allocations are very limited.

A large number of mascons requires a longer computational time. Anyway, in this case, the needed time is always tolerable and there is no practical necessity to find a balance between accuracy and computational burden. In the current analysis, the gravitational potential and acceleration are computed using a MATLAB® algorithm that makes use of parallel computing techniques on a quad-core 2.5 GHz processor. The field is computed for a given z value on a 50×50 2D grid, thus the time to obtain these quantities on a single field point is $\mathrm{t_C}/2500$. Similarly to the polyhedron approach, there is an obvious linear relation between N_m and $\mathrm{t_C}$, which is shown in figure 13. These results could be used as a reference for future studies.

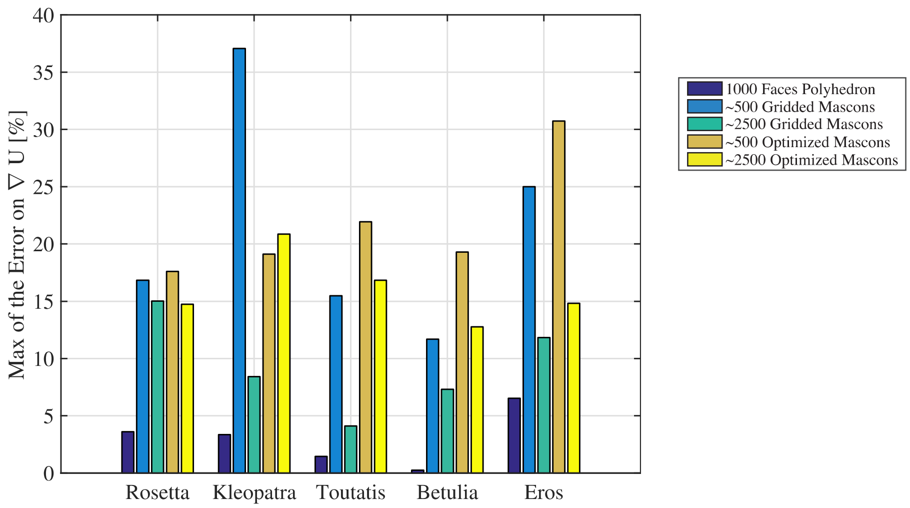
OPTIMUM LO-FI MODEL

The analysis of the different gravity field models can be concluded by comparing the different modeling techniques on the shape of the selected celestial bodies. In this way the optimum Lo-Fi model can be defined; it has a low fidelity with respect to the assumed Hi-Fi model, but it allows saving computational resources. For certain applications, this could be a decisive feature, and therefore, the introduction of this concept is extremely important. In practice, the optimum Lo-Fi model has the best balance between level of accuracy and required computational effort.

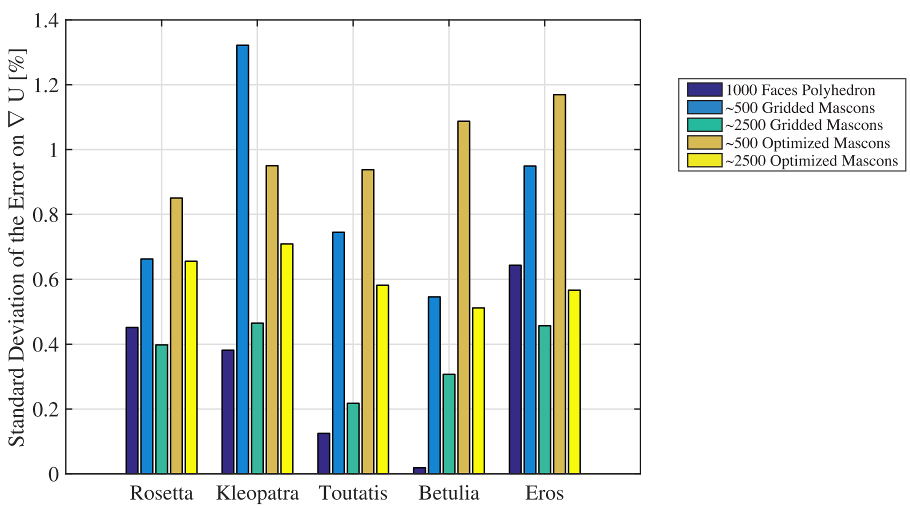
In figure 14, the obtained data are shown by means of two different convenient statistical quantities and a useful bounding parameter. In fact, the mean and the standard deviation define the error distribution, and the maximum establishes the limits of the model accuracy, close to the surface of the body in particular. It is important to note that the error is not guaranteed to be distributed according to a *folded normal distribution*, and the actual probability density function is often unknown. Nevertheless, the mean and the standard deviation can be used to characterize the distribution of the error field independently from this information.



(a) Mean of the error on ∇U.



(b) Maximum of the error on ∇U.



(c) Standard deviation of the error on ∇U.

Figure 14: Optimum Lo-Fi model: statistical analysis.

As a first remark, the optimization process increases the performances of the ~ 500 mascons model, while the ~ 2500 optimized mascons is not better than the equivalent gridded model, and the optimization process just converges to a solution very close to the one already provided by the gridded approach. This was already pointed out before in the text, but is here confirmed for all the different shape models. In addition, looking at the mean and maximum values in figures 14a and 14b, the optimization algorithm works at its best with the mean of the error, in agreement with the definition of the objective function to be minimized. Furthermore, both gridded and optimized mascons approaches produce good results for non-elongated bodies, and are less effective with the oblong ones. Then, focusing on the mean of the error, the mascons approach with a large N_m produces satisfactory results, while the optimized version should be preferred if few masses are employed.

The polyhedron model with 1000 faces has higher mean values of the error, and this can be explained considering the slight shape alteration produced by a coarse discretization of the body. However, the Lo-Fi polyhedron approach performs dramatically better when the maximum value of the error is considered, and this is due to the good results of the polyhedron model in the vicinity of the surface of the body. On the contrary, the mascons approach has several problems in representing the gravitational acceleration close to the singularities introduced by the point masses. Accordingly, it produces high errors in proximity of the body. In general, if the shape of the body is not particularly modified by the scaling algorithm, the polyhedron approach with a reduced number of faces is by far the best Lo-Fi model: the asteroid 1580 Betulia is an example of this situation.

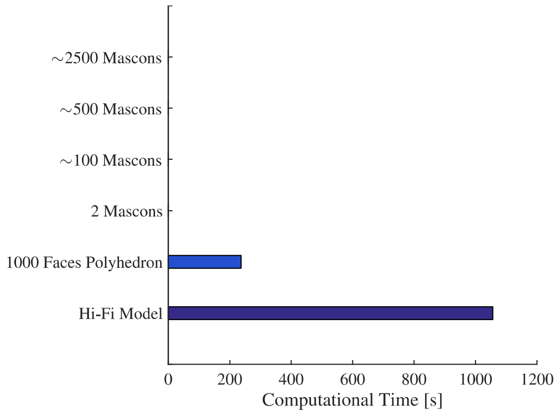
The standard deviation, in figure 14c, is a measure of the dispersion of the error and it is consequently influenced by the maximum value in the field, which can be considered as an outlier in the error distribution. Hence, the behaviour of the standard deviation is similar to the one in figure 14b, and the Lo-Fi polyhedron is the best performer in this regard. The standard deviation is anyway not large; the errors are concentrated close to the mean value and normally 68% of them is below 1%.

At this point, assuming that the shape of the body is fairly modified by the scaling algorithm, the best Lo-Fi model is a combination of a polyhedron model with few faces and an optimized mascons with a small N_m, or in alternative, a gridded mascons with many point masses. The first should be utilized close to the surface of the body and one of the latter when the distance from the centre of mass is large enough.

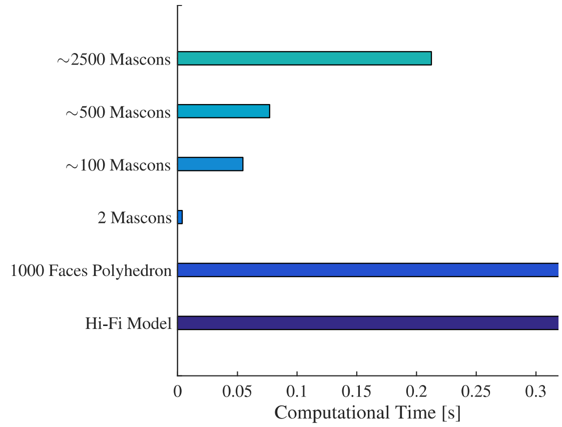
In figure 15, the computational times required by the different enhanced modeling techniques are shown and compared. They are related to MATLAB ® codes running in parallel on a quad-core $2.5\,\mathrm{GHz}$ processor. As usual, the time to obtain the different gravitational quantities on a single field point is $\mathrm{t_C}/2500$. These plots demonstrate the huge difference between the polyhedron and the mascons approach in terms of needed computational resources. However, the Lo-Fi polyhedron model requires a tolerable time, $\mathrm{t_C} \simeq 0.1\,\mathrm{s}$, to be evaluated on single point. So, even if it is slower than the mascons approach, it is anyway preferred when the field point is in proximity of the surface of the body

FINAL REMARKS

The employed enhanced modeling techniques to describe the gravitational field of an irregular celestial object have their own advantages and drawbacks. They do not have convergence problems, thus, they allow having a decent representation of the gravitational field in all the surrounding space, including the surface of the body. The level of achievable accuracy can be easily tuned adjusting the



(a) Global view.



(b) Detail.

Figure 15: Optimum Lo-Fi model: required computational time on a 50×50 2D grid, with parallel computing techniques on a quad-core 2.5 GHz processor.

number of discretizing elements, and consequently, they are extremely flexible.

The polyhedron approach, especially if the representation of the body is particularly accurate, produces superlative results, but it is computationally expensive. The fidelity of the field is only limited by the accuracy of the shape representation; moreover, the Laplacian of the potential is immediately available to determine if a field point is outside or inside the body. The required time to evaluate the field is linearly increasing with the number of polyhedron's faces, while the errors of the model decrease with the square root of the same quantity. Having this information, an acceptable level of accuracy and computational effort can be established according to the current needs.

The mascons approach is markedly faster and produces good results if the field point is far enough from the surface of the body. In general, it produces poor results if it is applied to elongated bodies. The optimized version of this technique improves the performances of the standard version up to its theoretical limit: the error decreases by one order of magnitude for an increase of three orders of magnitude in the number of mascons. However, over a certain number of point masses, the optimization process is not effective anymore and the gridded mascons approach is sufficient to have an acceptable result.

With the analysis, which has been conducted in this paper, the most efficient and effective gravity model is a combination of both approaches. The resulting optimum Lo-Fi technique uses a polyhedron with a moderate number of faces to compute the field close to the surface of the body, and the optimized mascons with a reasonable number of masses when the distance from the centre of mass is large enough. The switch between the two models happens at a distance where the gap between the two techniques is extremely limited, in a way to maintain the continuity of the field.

The present work provides useful information that can be exploited while studying the dynamics in proximity of irregular celestial bodies. The model of the gravitational field is the most important element to simulate trajectories around asteroids or comets, but it must used together with an accurate evaluation of all the perturbing effects that are present in these dynamical environments. However, the whole process is computationally expensive and it is fundamental to optimize the computational speed of the algorithms. Fortunately, the global structure can be improved in several aspects and it is prone to be boosted, using a lighter and simpler programming language and removing some cumbersome sections that were necessary to assess the validity of the code. Additionally, the meshing algorithm can be enhanced including the possibility of a non-uniform resolution of the polyhedron, in order to have a finer discretization only where it is needed. In this way, it is possible to decrease the required computational time, while maintaining the same fidelity of the model. The efficiency of the code can be further increased exploiting not only the parallel computing techniques, but also the Graphic Processing Unit (GPU) computation, in a way to use thousands of processors to complete numerical operations.

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