Asteroids in the vicinity of the Sun-Jupiter equilateral equilibrium points, commonly termed “Greek” or “Trojan” asteroids, offer insight into the primordial composition of the solar system. An increasing number of studies examine transfers to Near Earth Objects and to the equilateral Lagrange points, but comparatively little investigation of transfers within the neighborhood of the $L_4$ and $L_5$ points has been completed. Low-thrust trajectories that are optimized in terms of propellant consumption potentially offer years or even decades of mission lifetime while simultaneously ensuring a large number of asteroid encounters. This investigation considers the nearly continuous force from a variable specific impulse engine and its incorporation into a primer vector-based, optimal transfer and rendezvous strategy. A shooting procedure yields rapid solutions and generates a framework for an investigation into the trade space between flight time and consumption of propellant mass. Rendezvous between approximated asteroid paths in the planar circular restricted three-body problem is initially investigated, but the methodology is not limited by the dynamical regime and is readily extendable to the quasi-periodic paths representing various asteroids as well as three-dimensional examples in an ephemeris model.

**NOMENCLATURE**

- $x, \chi$: State Variable Vectors
- $t$: Time
- $r, v$: Position and Velocity Vectors, Sun-Jupiter Rotating Frame
- $m$: Spacecraft Mass
- $f_n$: Natural System Acceleration
- $T$: Thrust
- $P$: Engine Power
- $u$: Thrust Direction Unit Vector
- $d_1, d_2$: Sun and Jupiter Distances
- $M_S$: Solar Mass
- $M_J$: Jupiter Mass
- $\mu$: Mass Parameter
- $I_{sp}$: Specific Impulse
- $\gamma$: Asteroid Approximation Angle
- $C$: Approximation Rotation Matrix
- $\tau_0$: Asteroid Time-like Parameter
- $TD$: Thrust Duration
- $J$: Cost Function
- $H$: Problem Hamiltonian
- $\lambda$: Co-states
- $\psi$: Boundary Conditions
- $S$: Switching Function
- $X$: Full State Vector
- $F$: Constraint Vector
- $n$: Number of Rendezvous Arcs
- $\tau_S$: Total Sequence Duration
- $CT$: Intermediate Coast Duration
- $\tau_O$: Observation Time

**I. INTRODUCTION**

Interplanetary objects such as asteroids and comets offer insight into the primordial composition of the solar system and are, therefore, the subject of much scientific interest. Near Earth Objects (NEOs) have recently gained attention as opportunities for manned sample return missions. Likewise, missions to the main belt asteroids and the vicinity of the Sun-Jupiter equilateral points offer opportunities for encounters with multiple objects while retaining the option of robotic sample returns. Such a design concept allows for a broad survey of asteroid types and, therefore, a more complete view of solar system history. This
investigation focuses on potential long-term robotic missions to the vicinity of the Sun-Jupiter $L_4$ libration point, however the design methodology is largely independent of dynamical regime as well as initial and target states; therefore, the low-thrust concept and procedure are readily incorporated into diverse mission design strategies.

High-efficiency, low-thrust propulsion systems are particularly attractive for missions to the Sun-Jupiter equilateral equilibrium points because of the relatively stable natural gravitational dynamics in these regions. Fuel-optimal, low-thrust trajectories, realized by constant specific impulse systems in non-linear dynamical regimes, typically require coasting arcs and the careful balancing of engine capability with transfer time. The inclusion of additional coasting arcs requires engine shut-downs and restarts that may be operationally inefficient and generally infeasible. Therefore, a variable specific impulse (VSI) engine that varies the optimal thrust magnitude is selected to simplify the generation of rendezvous solutions. Accordingly, no coasting arcs are required for rendezvous and the initial generation of optimal trajectories is less restricted in terms of thrust duration. Examples of VSI engines include the Variable Specific Impulse Magnetoplasma Rocket (VASIMR) currently under development by the Ad Astra Rocket Company and the Electron and Ion Cyclotron Resonance (EICR) Plasma Propulsion Systems at Kyushu University in Japan. In general, the computation of locally fuel-optimal trajectories is approached by posing an optimal control problem. The possible formulations to solve the problem include a low-dimension but less flexible indirect approach using optimal control theory or a higher-dimension but more robust direct approach. (A detailed discussion of indirect and direct trajectory optimization methods is addressed in a survey by Betts.) Since the object of this investigation is the analysis of a rather large number of preliminary trajectory designs, the indirect method is employed while the direct method is reserved for later, more detailed trajectory construction. Relatively short times-of-flight (compared to long-duration spiral trajectories) as well as continuation methods aid in achieving robust convergence.

In this paper, independently generated fuel-optimal rendezvous arcs between asteroids are sequenced to create a large number of candidate tours. Similar investigations have been completed by Izzo and Canalias, though these authors implement global search algorithms that produce single tours comprised of brief flyby encounters, optimized end-to-end. In contrast, in this preliminary investigation, asteroid tours are constructed via a three step process:

1. Generate a number of locally optimal thrust segments that link asteroid pairs of scientific interest.

2. Combine these independent solution arcs to yield an ample space of possible tours with an easily approximated propellant cost and time of flight.

3. Select candidate tours with desirable characteristics for a higher-fidelity study.

The resulting trajectories offer tours on the order of months and years in the vicinity of the asteroids, allowing for extensive scientific observation and data collection. For this analysis, the spacecraft is assumed to originate within the $L_4$ region, so transfer costs from Earth to the Trojan asteroid group are not included in the propellant estimates or mission times. The algorithm is illustrated by designing two tours within the asteroid swarm near the Sun-Jupiter $L_4$ point, denoted the “Greek camp”, but the concepts and implementation details are readily extendable to other systems and mission objectives.

II. SYSTEM MODEL

Two key steps are initially necessary to successfully formulate the rendezvous problem, namely the definition of the physical environment to model the dynamics of the system and the construction of the initial and target states. The model for the unpowered spacecraft dynamics is independent of the optimization strategy and can, therefore, be adjusted to introduce various levels of fidelity.

II.1 Equations of Motion

The Sun-Jupiter system is modeled as a Circular Restricted Three Body Problem (CR3BP) with the Sun as one primary and Jupiter as the second. For this preliminary analysis, the perturbing effects of other Solar system bodies are neglected, including the target asteroids. The equations of motion are then formulated within the context of a rotating reference frame where $\hat{x}$ is directed from the Sun to Jupiter, $\hat{z}$ is normal to the orbital plane of the primaries and parallel to orbital angular momentum, and $\hat{y}$ completes the right-handed set. The origin of the coordinate system is the Sun-Jupiter barycenter. Incorporated into the forces that influence the motion in this system are terms that arise from the thrusting of the Variable Specific Impulse (VSI) engine. The system of equations are non-dimensionalized to aid numerical integration efficiency: computed results are converted to dimensional quantities by the proper use of the characteristic quantities and spacecraft parameter values. The characteristic quantities are the Sun-Jupiter distance, the mass of the primaries, the characteristic time, and the initial spacecraft mass. The spacecraft state vector is
The mass parameter \( \mu \) and spacecraft parameters is available in Table 1.

The motion of the spacecraft is described by the following equations:

\[
\dot{x} = \left\{ \begin{array}{c} \dot{r} \\ \dot{v} \\ \dot{m} \end{array} \right\} = \left\{ \begin{array}{c} f_n(r, v) + \frac{T}{m} u \\ -\frac{T^2}{m} \end{array} \right\}
\]

where \( T \) is thrust magnitude, \( P \) is engine power, \( u \) is a unit vector defining the thrust direction, and \( f_n \) represents the natural acceleration of the spacecraft. Furthermore, denote the six-dimensional vector that includes position \( r \) and velocity \( v \) by the vector \( x \), where \( x = [x \ y \ z \ \dot{x} \ \dot{y} \ \dot{z}]^T \). The scalar elements of \( f_n \) are then expressed in terms of the rotating frame as:

\[
f_n = \begin{cases}
2\dot{y} + x - \frac{(1 - \mu)(x + \mu)}{d_1^2} - \mu(x + \mu - 1) \\
-2\dot{z} + y - \frac{(1 - \mu)y}{d_2^2} - \mu y \\
-\frac{(1 - \mu)z}{d_2^2} - \mu z
\end{cases}
\]

where \( d_1 \) and \( d_2 \) are the distances from the Sun and Jupiter to the vehicle, respectively, that is

\[
d_1 = \sqrt{(x + \mu)^2 + y^2 + z^2}
\]

\[
d_2 = \sqrt{(x + \mu - 1)^2 + y^2 + z^2}.
\]

The mass parameter \( \mu \) is

\[
\mu = \frac{M_J}{M_S + M_J}
\]

where \( M_S \) and \( M_J \) are the masses of the Sun and Jupiter, respectively. The power \( P \) is defined as a scalar value between zero and a maximum available power level specified by the engine model, such that

\[
0 \leq P \leq P_{\text{max}}.
\]

Then, the engine thrust \( T \) is evaluated via

\[
T = \frac{2P}{I_{sp}g_0}
\]

where \( I_{sp} \) is the engine specific impulse and \( g_0 = 9.80665 \text{ m/s}^2 \), the gravitational acceleration at the surface of the Earth. Further information on the system and spacecraft parameters is available in Table 1.

II.II Periodic Orbits

Families of stable planar periodic orbits exist in the vicinity of the equilateral libration points, that is \( L_4 \) and \( L_5 \). Near \( L_4 \) and \( L_5 \), analytical approximations are numerically targeted and yield two families of planar periodic orbits associated with each libration point, a short-period and a long-period family. Members of the \( L_4 \) short period family approximate the trajectories of the target Trojan asteroids for the ten-year span from October 3, 2021 to October 3, 2031. The ephemeris trajectories of the asteroids appear in the Sun-Jupiter rotating frame in Fig. 1; note that these tracks are approximately elliptical, as are the short period \( L_4 \) periodic orbits. Furthermore, the asteroid motion is assumed to be confined to the \( xy \)-plane for the selected low-inclination asteroids.

Fig. 1: Projections of ephemeris asteroid trajectories onto \( xy \)-plane, October 3, 2021 to October 3, 2031. Motion is clockwise.

To represent the asteroid tracks in terms of periodic orbits constructed in the CR3BP, select short period orbits of approximately the same major axis, then rotate and translate the position and velocity of a particle moving along such a path to more closely align with the orbits of the actual asteroids. The corresponding rotation matrix is

\[
C = \begin{bmatrix}
\cos \gamma & -\sin \gamma & 0 \\
\sin \gamma & \cos \gamma & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

where the angle \( \gamma \) is selected such that the major axes

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Solar mass ( (M_S) ), kg</td>
<td>1.9891×10^30</td>
</tr>
<tr>
<td>Jupiter mass ( (M_J) ), kg</td>
<td>1.8986×10^27</td>
</tr>
<tr>
<td>Mass parameter ( (\mu) )</td>
<td>9.53816×10^{-4}</td>
</tr>
<tr>
<td>Sun-Jupiter distance ( (l^*) ), km</td>
<td>7.78412×10^8</td>
</tr>
<tr>
<td>Characteristic Time ( (t^*) ), sec</td>
<td>5.95911×10^7</td>
</tr>
<tr>
<td>Characteristic Time ( (t^*_{0}) ), days</td>
<td>6.89712×10^2</td>
</tr>
<tr>
<td>Initial spacecraft mass ( (m_0) ), kg</td>
<td>500</td>
</tr>
<tr>
<td>Maximum engine power ( (P_{\text{max}}) ), kW</td>
<td>1.0</td>
</tr>
</tbody>
</table>

Table 1: System and spacecraft parameter values.
of the periodic orbit and the asteroid track align. The
orbits are transformed as follows:

- Rotate the velocity on the CR3BP orbit by
  \[ \mathbf{v}_{\text{app}} = \mathbf{C} \mathbf{v}_{\text{NP}} \]  
  (10)
  where \( \mathbf{v}_{\text{NP}} \) is the velocity of the new numerically
  produced (NP) periodic orbit with respect to the
  rotating frame.

- Subtract the coordinates of the \( L_4 \) point from the
  position vector, resulting in the vector \( \mathbf{r}_{\text{LC}} \) locat-
  ing the particle with respect to \( L_4 \).

- Rotate the vector \( \mathbf{r}_{\text{LC}} \) such that
  \[ \mathbf{r}_{\text{ANP}} = \mathbf{C} \mathbf{r}_{\text{LC}} \]  
  (11)
  where \( \mathbf{r}_{\text{ANP}} \) is the position on the newly aligned
  numerically produced (ANP) periodic orbit.

- Add the \( L_4 \) coordinates plus an additional set of
  constants such that the altered orbit and the
  asteroid track nearly overlap, resulting in
  \[ \mathbf{r}_{\text{app}} = \mathbf{r}_{\text{ANP}} + \left[ \begin{array}{c} L_{4,x} + \Delta x \\ L_{4,y} + \Delta y \end{array} \right] \]  
  (12)
  with the quantities \( \Delta x \) and \( \Delta y \) determined via
  visual inspection. A more rigorous method of
  selecting the values of \( \Delta x \) and \( \Delta y \) may be imple-
  mented, but the current process is sufficient for
  preliminary investigation.

The result of this procedure is a numerically gener-
ated position \( \mathbf{r}_{\text{app}} \) and velocity \( \mathbf{v}_{\text{app}} \) that approximates
the ephemeris state of the asteroid. In Figure 2, the
ephemeris trajectories are plotted along with their
associated approximations, while the parameters for
translating and rotating the approximate orbits are
summarized in Table 2.

![Fig. 2: Ephemeral asteroid trajectories (dashed)
and approximate periodic orbits (solid) in xy-plane.](image)

Finally, all approximate asteroid states must reflect
the actual asteroid state with respect to time. There-
fore, a non-dimensional, time-like parameter, \( \tau_0 \), is
introduced, one that is defined to be zero at the epoch
October 3, 2021. All future times, and the corre-
sponding asteroid states, are then uniquely determined
by \( \tau_0 \). An approximate asteroid state is determined
by (i) forward propagation of the numerically gener-
ated \( L_4 \) short period orbit; then, (ii) application of the
transformation. The initial states corresponding
to the periodic orbits in the Sun-Jupiter CR3BP that
are used to approximate the asteroid paths are listed
in Table 3.

### III. Trajectory Optimization

The optimal asteroid rendezvous problem is solved in-
directly using the calculus of variations to reformulate
it as a two-point boundary value problem (2PBVP).
The 2PBVP is then solved numerically using a shoot-
ing method; the initial conditions corresponding to
the transfer are determined along with the time evolu-
tion of the subsequent spacecraft position and velocity
states as well as the engine operational states by the
Euler-Lagrange equations. However, the thrust du-
ration \( TD \) must be specified when a VSI engine is
employed. If no limit is placed on either the thrust
duration or the minimum mass consumed, the optimi-
ization process drives \( TD \) and \( I_{sp} \) to infinity while
consuming zero propellant mass.

To fully define the optimization problem, the perfor-
ance index and the boundary conditions must also
be specified. To arrive at the target asteroid with the
maximum final spacecraft mass for a specified thrust
duration, the performance index \( J \) is defined

\[
\max J = m_f. 
\]  
(13)

The boundary conditions and the Hamiltonian are ad-
joined to the performance index, so that Eq. (13) is
expanded to become the Bolza function

\[
\max J' = m_f + \mathbf{v}^T_0 \mathbf{\psi}_0 + \mathbf{v}^T_f \mathbf{\psi}_f + \int_{t_0}^{t_f} [H - \lambda^T \dot{\chi}] dt \]  
(14)

where \( H \) is the problem Hamiltonian, \( \lambda \) is a co-
state vector, the terms \( \mathbf{\psi} \) are vectors comprised of bound-
dary conditions, and the vector terms involving \( \mathbf{\nu} \) are
Lagrange multipliers corresponding to the boundary
conditions. The co-state vector is then

\[
\lambda = \{ \lambda_r, \lambda_v, \lambda_m \} 
\]  
(15)

where \( \lambda_r \) and \( \lambda_v \) are three-dimensional vectors com-
prised of the position and velocity co-states, respec-
tively, and the scalar \( \lambda_m \) is the mass co-state. The
initial and final boundary conditions are

\[
\mathbf{\psi}_0 = \mathbf{x}_f - \mathbf{x}_f(\tau_0) = 0 
\]  
(16)
such that the Hamiltonian with respect to the controls \( T \) becomes. The optimal controls emerge by maximizing the performance index in Eq. (14). With the reformulated Hamiltonian, that is, Eq. (22), the following equations of motion for the co-states emerge

\[
\dot{\lambda} = -\left( \frac{\partial H}{\partial \lambda} \right)^T = \begin{bmatrix}
-\lambda_T^T & -\lambda_T^T \frac{\partial f_T}{\partial \lambda} \\
\lambda_v^T & \lambda_v^T 
\end{bmatrix}
\]

where the initial state for \( \lambda_m \) is set to unity to reduce the number of variables to be determined.

All that remains to define the 2PBVP is construction of the transversality conditions that ensure local optimality. The first differential of the Bolza function, Eq. (14), with respect to the time-like parameter \( \tau_0 \) supplies the condition

\[
\lambda_T^T \frac{\partial x_I(\tau_0)}{\partial \tau_0} - \lambda_T^T \frac{\partial x_T(\tau_0 + TD)}{\partial \tau_0} = 0.
\]

To determine the partial derivatives in Eq. (25), apply the transformation from Section II.II to the velocity and acceleration on the short period orbit that approximates the asteroid trajectory. Thus,

\[
\frac{\partial x_I(\tau_0)}{\partial \tau_0} = \begin{bmatrix} C_I & 0 \\ 0 & C_I \end{bmatrix} \begin{bmatrix} v_I(\tau_0) \\ f_{n,I}(\tau_0) \end{bmatrix},
\]

\[
\frac{\partial x_T(\tau_0 + TD)}{\partial \tau_0} = \begin{bmatrix} C_T & 0 \\ 0 & C_T \end{bmatrix} \begin{bmatrix} v_T(\tau_0 + TD) \\ f_{n,T}(\tau_0 + TD) \end{bmatrix}
\]

where the subscripts I and T are again associated with the initial and target orbits, respectively, and the velocity \( v \) and natural dynamics \( f_n \) are evaluated on the short period orbit prior to transformation. The 2PBVP is then completely formulated to numerically produce the set of design variables

\[
X = \begin{bmatrix} \tau_0 \\ \lambda_{v0} \end{bmatrix}
\]

that satisfy the constraints

\[
F(X) = \begin{bmatrix}
\lambda_T^T \frac{\partial x_I(\tau_0)}{\partial \tau_0} - \lambda_T^T \frac{\partial x_T(\tau_0 + TD)}{\partial \tau_0} \\
0
\end{bmatrix} = 0
\]

and

\[
\psi_f = x_T - x_T(\tau_0 + TD) = 0
\]

where the subscripts I and T indicate the states associated with the current asteroid and target asteroid, respectively. Equation (16) is implicitly satisfied by defining \( x_I \) as the state along the current asteroid trajectory as defined by the parameter \( \tau_0 \). The final, or target, boundary conditions in Eq. (17) are satisfied by solving the boundary value problem.

The calculus of variations is employed to define several properties of the 2PBVP and acquire the derivatives of the co-states. The problem Hamiltonian is

\[
H = \lambda_T^T \dot{\lambda} + \lambda_v^T \left( f_n(r, v) + \frac{T}{m} u \right) - \lambda_m \frac{T^2}{2P}
\]

where the value of \( H \) is constant over the trajectory. The optimal controls emerge by maximizing the Hamiltonian with respect to the controls \( T, P, \) and \( u \) such that

\[
P = P_{\text{max}}
\]

\[
T = \frac{\lambda_v P_{\text{max}}}{\lambda_m m}
\]

\[
u = \frac{\lambda_v}{\lambda_v}
\]

where \( \lambda_v = |\lambda_v| \). From these controls, the Hamiltonian is reformulated and Eq. (18) is rewritten as

\[
H = \lambda_T^T \dot{\lambda} + \lambda_v^T f_n + S \cdot T
\]

where \( S \) is the switching function

\[
S = \frac{\lambda_v}{m} - \frac{\lambda_m T}{2P_{\text{max}}}.
\]

The Euler-Lagrange conditions for optimality modify the performance index in Eq. (14). With the reformulated Hamiltonian, that is, Eq. (22), the following

\[
\lambda_T^T \frac{\partial x_I(\tau_0)}{\partial \tau_0}, \lambda_T^T \frac{\partial x_T(\tau_0 + TD)}{\partial \tau_0} = 0.
\]

Table 2: Orbital characteristics for selected \( L_4 \) asteroids and approximate orbit translations and rotations.

<table>
<thead>
<tr>
<th>Asteroid</th>
<th>Major axis, km</th>
<th>Period, years</th>
<th>( \gamma ), deg</th>
<th>( \Delta x ), km</th>
<th>( \Delta y ), km</th>
</tr>
</thead>
<tbody>
<tr>
<td>659 Nestor</td>
<td>3.631 \times 10^8</td>
<td>11.90238</td>
<td>12</td>
<td>0.5 \times 10^8</td>
<td>-0.3 \times 10^8</td>
</tr>
<tr>
<td>1143 Odysseus</td>
<td>1.829 \times 10^8</td>
<td>11.90307</td>
<td>135</td>
<td>0.8 \times 10^8</td>
<td>-0.55 \times 10^8</td>
</tr>
<tr>
<td>1869 Philoctetes</td>
<td>1.373 \times 10^8</td>
<td>11.90318</td>
<td>17</td>
<td>1.75 \times 10^8</td>
<td>-1.4 \times 10^8</td>
</tr>
<tr>
<td>5012 Eurymedon</td>
<td>2.722 \times 10^8</td>
<td>11.90279</td>
<td>12</td>
<td>0.85 \times 10^8</td>
<td>-0.5 \times 10^8</td>
</tr>
<tr>
<td>5652 Amphimachus</td>
<td>2.024 \times 10^8</td>
<td>11.90302</td>
<td>30</td>
<td>-0.5 \times 10^8</td>
<td>0.3 \times 10^8</td>
</tr>
</tbody>
</table>

Table 3: Periodic orbit states at epoch October 3, 2021, \( \tau_0 = 0 \).
with $\lambda_{rv_0}$ and $\lambda_{rv_1}$ defined as the co-state vectors at the beginning and end of a thrust arc, respectively. The optimization problem can be solved using any boundary value problem or equation solver, for example the `fsolve` function in MATLAB.

IV. MISSION APPLICATIONS

A mission to the vicinity of the Sun-Jupiter “Greek” or “Trojan” asteroid families will almost certainly entail rendezvous with and observation of multiple objects. A strategy is proposed for rapidly generating a large number of candidate asteroid tours and analyzing the resulting design space for trade-offs in terms of fuel consumption, mission time, and scientific opportunity. This trajectory evaluation scheme yields only approximate propellant costs and is, therefore, intended solely for preliminary design analysis. However, overall performance comparisons may still be assessed and specific trajectory concepts are readily transitioned to higher fidelity models that offer more accurate estimates of propellant consumption.

For baseline mission design, the transfer of the spacecraft from Earth to the Sun-Jupiter $L_4$ region must also be addressed. This leg of the mission also provides a wealth of opportunity for optimization and trajectory design. For instance, a Hohmann transfer from Earth to the Sun-Jupiter $L_4$ point is completed in 2.73 years and requires a total 14.44 km/s of $\Delta v$ for Earth departure and matching the $L_4$ point velocity upon arrival. On the other hand, gravity assist maneuvers using solar system bodies such as Earth, Venus, Mars, and Jupiter extend the flight time in return for lowered $\Delta v$ costs. As an example, Callisto constructs an Earth-Venus-Earth-Earth flyby sequence that transfers the spacecraft to the Sun-Jupiter $L_4$ region in 10 years for 9.17 km/s, where this cost includes Earth departure and asteroid swarm arrival. On the other hand, low-thrust propulsion system may be employed to assist in the transfer from Earth to the $L_4$ region, as proposed by Desjean. For this investigation, however, the spacecraft is assumed have already completed this transfer and arrived at the Trojan asteroid swarm with a remaining mass of 500 kg.

IV.I Rendezvous Sequence Generation

The optimization procedure in Section III yields a single rendezvous segment connecting two asteroids and resulting in minimum propellant consumption for a specified thrust duration. The initial and terminal states along these arcs correspond to approximated asteroid positions and velocities such that the spacecraft transfers from the vicinity of one asteroid to that of another. However, once a point solution is generated for a single specified thrust duration $TD$, a simple continuation scheme is applied that produces trajectories over a large range of thrusting times. The continuation process updates the value of $TD$ and uses the previously computed solution as the initial guess for the subsequent 2PBVP. The complete set of thrust arcs that is determined via the continuation scheme, termed a “family”, represents a set of options for a single pre-determined asteroid-to-asteroid link within a design space relating engine operation time and propellant consumed for a spacecraft transfer.

Families of rendezvous segments are independently generated for a variety of initial and target asteroid pairs (e.g., 659 Nestor to 5012 Eurymedon, 1869 Philotetes to 1143 Odysseus, etc.). Note that for every thrust arc segment within these families, the initial spacecraft mass is assumed to be $m_0 = 500kg$, or $m_0 = 1$ non-dimensional unit. The propellant consumed during each interval of engine operation must be incorporated into an equivalent cost for any potential tour scenario comprised of several rendezvous arc segments. Accordingly, for a rendezvous sequence comprised of $n$ thrust intervals, the approximate propellant mass consumed $m_{\text{cons}}$ is obtained via

$$m_{\text{consumed}} = m_0 \left( 1 - \prod_{i=1}^{n} \frac{m_i}{m_0} \right) \tag{30}$$

where $m_i$ is the arrival mass in kilograms at the end of the $i^{th}$ independently generated thrust arc. Aside from propellant consumption, other key design considerations of a proposed tour concept include the time required to complete the overall rendezvous sequence and the time interval within close proximity to the asteroids of interest. The duration of the total sequence, $\tau_S$, is assessed via

$$\tau_S = \sum_{i=1}^{n} TD_i + \sum_{i=1}^{n-1} CT_i \tag{31}$$

where $CT_i$ is the length of time near the target asteroid along the $i^{th}$ thrust arc, that is

$$CT_i = (\tau_0)_{i+1} - (\tau_0 + TD)_i \tag{32}$$

with $(\tau_0)_{i+1}$ representing the time at thrust initiation on the $(i+1)^{th}$ arc and $(\tau_0 + TD)_i$ corresponds to the end of the thrust interval along the $i^{th}$ arc. The time interval in the vicinity of the intermediate asteroids, termed the “observation time” $\tau_O$, is evaluated as

$$\tau_O = \sum_{i=1}^{n-1} CT_i \tag{33}$$

and reflects the time the spacecraft spends loitering near the asteroids and, therefore, available for scientific observation, data collection, and any other pertinent mission activity.

*For the case of impulsive maneuvers, an equivalent total mission cost is $\Delta v_{\text{tot}} = \sum_{i=1}^{n} \Delta v_i$ where $\Delta v_{\text{tot}}$ is the total impulsive $\Delta v$ and $\Delta v_i$ is the equivalent value for one maneuver.*
Equations (30), (31), and (33) are incorporated into an iterative procedure that evaluates all possible combinations of individual segments from the rendezvous families without the necessity to re-optimize each specific sequence. So, for example, combining a family with 10 rendezvous paths from 5012 Eurymedon to 1869 Philoctetes with a 10-member family from 1869 Philoctetes to 1143 Odysseus yields 100 candidate solutions for an asteroid tour originating at Eurymedon, terminating at Odysseus, and including an intermediate coast in the vicinity of Philoctetes. These 100 possible sequences are then filtered for candidate tours exhibiting the desired balance of propellant consumed, observation time, and sequence time. Additionally, the set of sequences can be examined to gain additional intuition concerning the overall design space.

IV.II Sample Trajectory Sequences

The first step in the tour design process is the generation of rendezvous families between the target asteroids. At the initiation of this process, the asteroid order along the tour is not yet available, so locally optimal thrust arc families between asteroid pairs are generated and stored on an individual basis. An organizational framework for constructing an asteroid tour appears in Fig. 3, where the red arrows denote that a family of low-thrust rendezvous segments between the asteroids has been computed.

![Fig. 3: Target asteroids with arrows indicating computed rendezvous trajectory families, i.e., the asteroid-to-asteroid links.]

For a reasonable likelihood of a successful design, it is necessary to ensure that the optimal departure and arrival dates for the thrust intervals are realizable to deliver an end-to-end trajectory. Therefore, the available asteroid rendezvous pairs appear in Table 4 along with the range of departure dates over which families of asteroid-to-asteroid low-thrust segments have been computed. The maximum engine operation time across the family representing a specified link is also noted in the table. To confirm the validity of a selected tour sequence, simply check that the latest departure date plus the interval of engine operation for the $i^{th}$ rendezvous segment does not conflict with the earliest departure date on the ($i + 1)^{th}$ thrust arc. This restriction does omit some valid trajectory options, but streamlines the design process for this preliminary investigation.

<table>
<thead>
<tr>
<th>Pair</th>
<th>Earliest Departure</th>
<th>Latest Departure</th>
<th>Max. TD, yrs</th>
</tr>
</thead>
<tbody>
<tr>
<td>Amph. → Nest.</td>
<td>06/2019</td>
<td>08/2020</td>
<td>4.15</td>
</tr>
<tr>
<td>Amph. → Odys.</td>
<td>07/2025</td>
<td>05/2028</td>
<td>2.27</td>
</tr>
<tr>
<td>Amph. → Odys.</td>
<td>06/2037</td>
<td>04/2040</td>
<td>2.27</td>
</tr>
<tr>
<td>Eury. → Amph.</td>
<td>12/2031</td>
<td>04/2032</td>
<td>3.40</td>
</tr>
<tr>
<td>Eury. → Nest.</td>
<td>09/2025</td>
<td>11/2026</td>
<td>2.08</td>
</tr>
<tr>
<td>Eury. → Phil.</td>
<td>08/2026</td>
<td>06/2027</td>
<td>3.21</td>
</tr>
<tr>
<td>Nest. → Amph.</td>
<td>01/2033</td>
<td>08/2033</td>
<td>3.21</td>
</tr>
<tr>
<td>Nest. → Eury.</td>
<td>12/2024</td>
<td>11/2026</td>
<td>2.08</td>
</tr>
<tr>
<td>Nest. → Phil.</td>
<td>06/2026</td>
<td>01/2027</td>
<td>3.21</td>
</tr>
<tr>
<td>Phil. → Eury.</td>
<td>11/2026</td>
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<td>3.40</td>
</tr>
<tr>
<td>Phil. → Odys.</td>
<td>10/2031</td>
<td>04/2033</td>
<td>2.83</td>
</tr>
</tbody>
</table>

Table 4: Families of low-thrust arcs between asteroid pairs; dates of earliest and latest departure on low-thrust arcs across the family. (Dates are in MM/YYYY format.)

As is apparent in Table 4, some of the departure dates lie outside the ten-year time frame originally specified to create the approximations to the asteroid tracks. For this analysis, the transformation values in Table 2 are assumed to remain valid beyond the original ten-year mission window. Increased fidelity is accomplished by updating the periodic orbit transformation based on $\tau_0$ or by incorporating the ephemeris trajectories directly. Additionally, there are two 5652 Amphimachus to 1143 Odysseus rendezvous families, with a twelve-year interval between them. These two families are nearly the same set of solutions, shifted in phase by one full revolution of the short period orbits. Similarly, all the rendezvous families can be shifted by whole periods corresponding to the approximating orbits subject, of course, to the phase shift in the positions of the asteroids and the similarity of the transformed orbit to the asteroid track. Thus, tours designed within the ten-year window between October 3, 2021 and October 3, 2031 are expected to retain some of their properties if shifted in discrete twelve-year intervals around the nominal dates.

The next step in the trajectory design process is the combination of individual segments into a sequence of thrust arcs between asteroids and coast arcs in the near-vicinity of the asteroids for observations. Two sample tours are examined:

1. Tour ANPO: 5652 Amphimachus to 659 Nestor to 1869 Philoctetes to 1143 Odysseus;
2. Tour NEAO: 659 Nestor to 5012 Eurymedon to 5652 Amphimachus to 1143 Odysseus.

In each tour, the spacecraft is initially assumed to be in the local vicinity of the first asteroid. These
prospective tours are assessed in terms of the trade-off between propellant consumption, engine operation duration, overall end-to-end trajectory length, and the loiter time near the intermediate asteroids.

IV.II.I Sample Sequence 1: Tour ANPO

Sample trajectory sequence ANPO originates with the spacecraft in the vicinity of 5652 Amphimachus and terminates at 1143 Odyssesus, with intermediate coasts near asteroids 659 Nestor and 1869 Philoctetes. The approximated asteroid tracks as well as the paths along the thrust segments in the rendezvous families throughout the tour are displayed in Fig. 4. The colors of the individual segments within the families that link pairs of asteroids as they appear in the figure indicates the engine operation time along that thrust arc; all three families use the color scale indicated in the plot. The dots mark the departure and arrival points on the approximated asteroid tracks. For all potential tours, the spacecraft spends approximately equal intervals near the asteroids 5652 Amphimachus and 1869 Philoctetes.

![Asteroid tracks and families of thrust segments for ANPO tours.](image)

**Fig. 4: Asteroid tracks and families of thrust segments for ANPO tours.**

The propellant mass performance along each family of thrust segments is plotted in Fig. 5. The numbering in the legend corresponds to the families of thrust segments in Fig. 4. Note that in each rendezvous family the final spacecraft mass generally increases with increasing thrust duration $TD$, indicating that less propellant is consumed along longer thrust arcs. The results from the independently generated rendezvous families are combined as described in Section IV.I, with the resulting trade space as illustrated in Fig. 6. The plots in Fig. 6 represent explicit values of the propellant consumption, trajectory duration, and scientific observation time at the intermediate asteroids 5652 Amphimachus and 1869 Philoctetes while indirectly revealing information about the individual thrust arcs comprising the tours. The colors of the points in Fig. 6 reflect the thrust durations within the second rendezvous family, $TD_2$, where black identifies the smallest values of $TD_2$ and white the largest. Likewise, the point size reveals the relative values of $TD_3$, that is, the length of time along the segments in the third thrust arc family, with larger points indicating longer engine operation time, $TD_3$. For a selected trajectory, the duration of the corresponding rendezvous arc from the first family of thrust segments, $TD_1$, is determined from $TD_2$, $TD_3$, $\tau_S$, and $\tau_O$ via Eqs. (31)-(33). Note also that the mass consumed appears in terms of decakilograms (1 dakg = 10 kg).

![Spacecraft mass at the end of the thrust arc in each of the three ANPO thrust arc families.](image)

**Fig. 5: Spacecraft mass at the end of the thrust arc in each of the three ANPO thrust arc families.**

![Trade space for ANPO tours: consumed propellant, overall tour duration, and observation time.](image)

**Fig. 6: Trade space for ANPO tours: consumed propellant, overall tour duration, and observation time.**

Upon examination, the trade space exhibits a clear and relatively linear trade-off between low propellant consumption but higher sequence duration $\tau_S$ computed via Eq. (31) and lower asteroid scientific observation time $\tau_O$, from Eq. (33). For example, the tour with the lowest approximated fuel cost (16.323 kg) also corresponds to the longest sequence interval at 15.113 years and the lowest loiter time, i.e., 4.916 years. This candidate tour is comprised of the longest duration thrust arcs from the three independent families. This trajectory is re-generated from end-to-end to offer a more accurate estimate of the design metrics for comparison to the approximated design parame-
ters. So, when solving the 2PBVP, the spacecraft mass at the initiation of thrust on the \((i + 1)\)th segment is equal to \(m_i\), the spacecraft mass at the termination of the \(i\)th rendezvous thrust arc. Accordingly, the re-generated trajectory is optimized from end-to-end and the thrust segments are no longer independent. This re-converged tour consumes 16.192 kg of propellant, while the overall interval is again 15.113 years and the asteroid science observation duration is 4.916 years. Thus, in comparison to the actual trajectory, the approximated tour estimates a relatively small additional 0.811% in the propellant consumed and differences in the sequence time and observation time, i.e., \(\tau_S\) and \(\tau_O\), on the order of seconds. On the other end of the trade space, the candidate tour with an approximate propellant consumption of 42.077 kg and \(\tau_S = 14.389\) years and \(\tau_O = 8.724\) years actually requires 40.983 kg of propellant when re-computed to optimize the full end-to-end sequence. The mass over-estimate of the combination procedure in Section IV.I is then 2.668%, while the overall tour duration and loiter time decrease by approximately 5 minutes for the end-to-end trajectory.

IV.II.I Sample Sequence 2: TOUR NEAO

In the second example tour, the spacecraft originates alongside 659 Nestor in its orbit, then transfers to the vicinity of 5012 Eurymedon, 5652 Amphimachus, and ultimately 1143 Odysseus. The rendezvous segments and the approximated asteroid tracks appear in Fig. 7. Consistent with the previous sample tour, the color scale indicates the engine operation time for each thrust arc and the dots mark the departure and arrival points along the approximated asteroid tracks. Again, the scientific observation time in the rendezvous sequence is divided nearly evenly between the two intermediate asteroids.

The spacecraft arrival mass corresponding to each of the individual thrust arc families is plotted in Fig. 8, where the numbers match that in Fig. 8. Two of the families exhibit increasing arrival mass with increasing thrust duration \(TD\), but the third transfer family displays a distinct local peak at approximately 550 days of engine operation. Thus, on the final leg of the tour, there is a point when increasing the thrust duration is detrimental in terms of both propellant consumption and mission operation time. The results from the unconstrained rendezvous segments are combined and the trade space is illustrated in Fig. 9. As in Fig. 6, the values of propellant consumption, sequence length, and asteroid observation time are explicitly available in Fig. 9. Indirect information on the individual thrust segments comprising the tour is also available, i.e., the second thrust duration is indicated by the color of the points, with white transitioning to blue then red and eventually yellow as \(TD_2\) increases, while the point size reflects \(TD_3\), the length of the third rendezvous arc, with larger points indicating longer \(TD_3\). In contrast to the trade space representing the ANPO tour, the NEAO tours display a non-linear relationship between \(m_{cons}\) and \(\tau_O\) with a clear minimum at intermediate values of asteroid loiter time. The relationship between propellant consumption and overall tour length is again approximately linear, however the tour with the minimum approximate fuel cost, 5.722 kg, possesses an overall duration \(\tau_S\) of 16.386 years and an observation time of 9.550 years, neither of which are extreme values within the trade space. This candidate tour is composed of the longest duration thrust arcs from the first two rendezvous families and a moderate value for \(TD_3\). The candidate trajectory is re-optimized across the entire tour to provide a more accurate estimate of the design metrics. The reconverged sequence consumes 5.704 kg of propellant, while \(\tau_S\) and \(\tau_O\) are altered only on the order of seconds. So, for this sample tour, the approximated solution over-estimates the required propellant by 0.314%.
V. CONCLUSIONS

A strategy to estimate propellant cost and other performance metrics for a low-thrust trajectory effectuated by a variable specific impulse system has been developed and applied to the generation of asteroid rendezvous tours within the Sun-Jupiter L4 Trojan asteroid swarm. Indirect optimization methods are employed to rapidly yield a large number of rendezvous arcs between selected asteroids. These independently generated asteroid-to-asteroid trajectory arcs are then combined to quickly develop a trade space that offers some insight into the tour design and allows for the selection of candidate trajectories for further analysis. The procedure is not limited by dynamical regime and is readily extended to other mission architectures. In general, the approximation process is expected to be conservative and, thus, slightly over-estimates the required propellant. For the sample tours examined, the combination algorithm yields fuel mass estimates within 3% of identical trajectories optimized from end-to-end while timing results are accurate on the order of minutes.

There are several avenues for further investigation and refinement. In particular, higher fidelity modeling of asteroid motion will increase the accuracy of the resulting designs for both the asteroid-to-asteroid arcs as well as the end-to-end tour trajectory. Additionally, the approximation procedure may be applied to other scenarios requiring multiple low-thrust arcs, whether as part of a baseline or an extended trajectory design. Furthermore, the combination process may be improved by implementing concepts from hybrid systems theory, such as reachability analysis, as well as multi-dimensional and constrained optimization.

ACKNOWLEDGEMENTS

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References

14. Irrgang, L.R., „Investigation of Transfer Trajectories to and from the Equilateral Libration Points L4 and L5 in the
Earth-Moon System,” M.S. Thesis, School of Aeronautics and Astronautics, Purdue University, West Lafayette, Indiana, 2008.
