LOW-THRUST TRANSFERS IN THE EARTH-MOON SYSTEM INCLUDING APPLICATIONS TO LIBRATION POINT ORBITS

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Preliminary designs of low-thrust transfer trajectories are developed in the Earth-moon three-body problem. The solution for a complete time history of the thrust magnitude and direction is initially approached as a calculus of variations problem to locally maximize the final spacecraft mass. The problem is then solved directly by sequential quadratic programming using either single or multiple shooting. The coasting phase along the transfer exploits invariant manifolds and considers locations along the entire manifold surface for insertion. This investigation includes transfer trajectories from an Earth parking orbit to some sample libration point trajectories including $L_1$ halo orbits, $L_1$ and $L_2$ vertical orbits, and $L_2$ “butterfly” orbits.

INTRODUCTION

Within the last 25 years, libration point orbits have emerged as a valuable trajectory alternative for spacecraft mission design. Beginning with the launch of ISEE-3 on August 12, 1978 and its successful direct transfer into a Sun-Earth $L_1$ halo orbit, 1 libration point orbits are now viable options for suitable scientific applications. More recently, an interest in a return to the moon has prompted trajectory design studies for several new mission objectives, including the development of continuous communications links. 2,3,4,5,6 Satellites in orbits that ensure lunar south pole coverage are one such example that has received attention as a possible new application for libration point orbits. 3,4,5,6

Accessing these orbits ultimately requires a means of transfer from an Earth parking orbit. One approach is through highly efficient low-thrust propulsion. This technology has been demonstrated as a primary source of propulsion in missions such as Deep Space 1 7 and SMART-1. 8 The difficulty in constructing low-thrust trajectories is, of course, the computation of a long-duration thrust magnitude and direction history. Typically, these unknown parameters are specified as part of a trajectory optimization problem to locally maximize or minimize a performance index such as final spacecraft mass or total burn time. Investigations to solve these optimization problems have become an area of intense study, and they are typically classified in terms of either direct or indirect methods. (A useful survey of each method is discussed by Betts; 9 with four methods also discussed by Hull. 10) In addition to being classified as direct/indirect, another important distinction between the existing solution methods is whether or not explicit or implicit numerical integration is employed.

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Indirect methods involve the two-point boundary value problem (TPBVP) that arises from the calculus of variations formulation and traditionally involve explicit numerical integration. The TPBVP introduces unknown costates, necessary to parameterize the controls, and additional optimality constraint equations. Compared to most direct methods, solving the TPBVP indirectly (often via indirect shooting\cite{11,12,13}) involves relatively few unknown parameters that are each highly sensitive, especially given the long propagation times and nonlinear dynamics. Despite this difficulty, indirect methods often exhibit rapid convergence compared to a direct method and generally require significantly less function evaluations when implemented with a numerical root-solving procedure. Many early research efforts focused on indirect solutions.\cite{14,15,16} Although many current efforts use direct approaches, several recent studies\cite{11,12,13} have returned to indirect approaches due to the benefits of adjoint control transformations\cite{17} and the low dimensions required to set up the problem. Because guesses for the initial costates are so often problematic in solving a TPBVP, multiple shooting\cite{18,19} was introduced as a means to decompose a trajectory into a series of segments and partition the sensitivities over many “nodes”. This method has been successfully demonstrated in several low-thrust problems.\cite{20,21} Multiple shooting also allows the addition of kinematic constraints to the nodes joining each trajectory segment.

Direct methods typically convert the calculus of variations problem into a parameter optimization problem where a scheme such as nonlinear programming (NLP) is employed to minimize the performance index. Single and multiple shooting methods may also serve in so-called “hybrid” formulations in which components of the TPBVP are established to continuously parameterize the control via explicit integration.\cite{21,22,23} Rather than satisfy all optimality constraints, the cost function is instead minimized. Many direct methods transcribe the states and sometimes the controls through implicit integration. These methods include direct transcription/collocation\cite{25,26,27} and differential inclusion.\cite{28,29} In these direct approaches, the entire trajectory is represented in terms of nodes, and a large number of design variables are used to increase the convergence radius, sometimes removing the costate variables completely. Static/dynamic control is another direct method that has been developed for high fidelity simulation.\cite{30} Yet again, many design variables are required to parameterize a long-duration transfer, and many function evaluations are required.

In general, few low-thrust investigations have focused on transfers to libration point orbits. Gómez, et al.\cite{31} and Howell et. al\cite{32} develop transfer trajectories via the stable manifold with impulsive maneuvers in the Sun-Earth/moon system. Starchville and Melton\cite{33,34} use a dynamical systems approach and combine a thrust arc with a naturally convergent coast to a libration point orbit. They employ a tangent steering law to determine the fuel optimal insertion on individual manifolds in the Earth-moon circular restricted and elliptic restricted problem. Sukhanov and Eiss-mont\cite{35} develop a transfer to a Sun-Earth $L_2$ halo orbit by calculating a spiral trajectory within a two-body model, and then determine the finite burns necessary to replicate an optimal two-burn impulsive transfer in the three-body system. Many researchers, for example, Kluever\cite{22,23} and Conway,\cite{26} incorporate the restricted three-body problem as a precursor to higher fidelity gravitational models, but do not focus on transfers to libration point orbits. Senent and Ocampo\cite{36} consider the benefits of exposing the stable invariant manifold behavior associated with fixed points around entire periodic orbits in the Sun-Earth/moon system. Their low-thrust transfer trajectories also employ a variable specific impulse engine model and a detailed calculus of variations approach that uses a primer vector control law. Similarly, Mingotta et al.\cite{37} consider transfers to halo orbits in the Earth-moon problem with a direct transcription and collocation approach, using constant specific impulse engines. Their results also include subsequent transfer trajectories departing the halo orbits.

This investigation considers the benefits of using the stable invariant manifolds to produce preliminary low-thrust transfer designs within the Earth-moon system. Similar to Senent and Ocampo,\cite{36} an indirect formulation is initially established within the context of the calculus of variations to transfer a spacecraft from an Earth parking orbit to the stable manifold, with the spacecraft asymp-
totically coasting to the desired orbit upon insertion. Instead of solving the complete TPBVP, a “hybrid” approach is utilized and some of the optimality conditions are eliminated in favor of obtaining a direct solution. This method is selected to maintain a relatively low number of design variables, yet widen the convergence radius. Both direct single shooting and direct multiple shooting ⁹ are employed to minimize a performance index of final spacecraft mass using a variable specific impulse engine model. As an extension of a previous study by Grebow et al., ⁵ several target trajectories are considered, including L₁ halo orbits, L₁ and L₂ vertical orbits, and L₂ “butterfly” orbits.

UNCONTROLLED SYSTEM MODELS

The uncontrolled spacecraft motion is assumed to be subject to the dynamics in the Earth-moon Circular Restricted Three-Body Problem (CR3BP). The primary gravitational bodies (Earth and moon) move about their barycenter on circular paths. The third body, i.e., the spacecraft, is assumed to possess negligible mass in comparison to the primaries. The rotating moon move about their barycenter on circular paths. The third body, i.e., the spacecraft, is assumed to possess negligible mass in comparison to the primaries. The rotating -axis is defined along the vector between the primaries; the -axis is parallel to the angular velocity vector associated with the Keplerian primary orbits. Then, the usual barycentric system of equations describing the motion of the third body are written compactly in terms of position, r, and velocity, v, as,

$$\ddot{r} = \left( \frac{\partial U(\vec{r})}{\partial \vec{r}} \right)^T + \vec{h}(\vec{v}),$$

where the dots indicate the nondimensional time derivatives relative to an observer in the rotating frame, the acceleration vectors include the “pseudo-potential” gradient, \(\frac{\partial U(\vec{r})}{\partial \vec{r}}\), and the kinematic velocity function, \(\vec{h}(\vec{v})\), i.e.,

$$\frac{\partial U(\vec{r})}{\partial \vec{r}} = \begin{bmatrix} x - (1 - \mu)(x + \mu)/d_1^3 - \mu(x - 1 + \mu)/d_2^3 \\ y - (1 - \mu)y/d_1^3 - \mu y/d_2^3 \\ -(1 - \mu)z/d_1^3 - \mu z/d_2^3 \end{bmatrix}^T, \quad \vec{h}(\vec{v}) = \begin{bmatrix} 2y \\ -2z \\ 0 \end{bmatrix}.$$ (2)

The values of x, y, and z describe the spacecraft’s position relative to the rotating, barycentric frame. The mass parameter is \(\mu\); then, \(d_1\) and \(d_2\) are the relative distances between the third body and the first and second primary, respectively. The equations are nondimensional, where the characteristic quantities are the total mass, the distance between the primaries, and the magnitude of the system angular velocity. The gradients of the quantities \(\frac{\partial U(\vec{r})}{\partial \vec{r}}\) and \(\vec{h}(\vec{v})\) are also important components in obtaining periodic orbits and later, in obtaining the costate differential equations. These symmetric, nondimensional matrices are determined to include the following,

$$\frac{\partial^2 U(\vec{r})}{\partial x^2} = 1 - \frac{(1 - \mu)}{d_1^4} + \frac{\mu}{d_2^4} + \frac{3(1 - \mu)(x - \mu)^2}{d_1^4} + \frac{3\mu(x - 1 + \mu)^2}{d_2^4},$$
$$\frac{\partial^2 U(\vec{r})}{\partial y^2} = 1 - \frac{(1 - \mu)}{d_1^4} + \frac{\mu}{d_2^4} + \frac{3(1 - \mu)y^2}{d_1^4} + \frac{3\mu y^2}{d_2^4},$$
$$\frac{\partial^2 U(\vec{r})}{\partial z^2} = 1 - \frac{(1 - \mu)}{d_1^4} + \frac{\mu}{d_2^4} + \frac{3(1 - \mu)z^2}{d_1^4} + \frac{3\mu z^2}{d_2^4},$$
$$\frac{\partial^2 U(\vec{r})}{\partial x \partial y} = \frac{3(1 - \mu)(x + \mu)}{d_1^4} + \frac{3\mu(x - 1 + \mu)}{d_2^4},$$
$$\frac{\partial^2 U(\vec{r})}{\partial x \partial z} = \frac{3(1 - \mu)(x + \mu)}{d_1^4} + \frac{3\mu(x - 1 + \mu)}{d_2^4},$$
$$\frac{\partial^2 U(\vec{r})}{\partial y \partial z} = \frac{3(1 - \mu)yz}{d_1^4} + \frac{3\mu yz}{d_2^4},$$
The uncontrolled dynamical equations are used to determine periodic orbits and their invariant manifolds. The state-transition matrix, \( \Phi(t_f, t_0) \) associated with eq. (1) is also available. End-point variations, and the values of \( \Phi(t_f, t_0) \) may be used in a time-fixed, gradient-based Newton-Raphson differential corrections scheme to determine naturally periodic orbits. Such a scheme is used to obtain the initial conditions for orbits of a pre-defined period, i.e.,

\[
\delta \vec{X}_0 = \Phi^{-1}(t_f, t_0) \delta \vec{X}_f,
\]

where \( \vec{X} = [x \ y \ z \ \dot{x} \ \dot{y} \ \dot{z}]^T \), and the subscripts “0” and “f” refer to the initial and final state, respectively. When a specific type of orbital symmetry is targeted, \( j \) “control” parameters are selected from \( \vec{X}_0 \), and \( l \) constraint parameters are selected from \( \vec{X}_f \), \( (j \geq l) \) and, thus, \( \Phi(t_f, t_0) \) reduces to an \( j \times l \) matrix. Then, eq. (5) is used to obtain updates in the initial state until the target states are achieved to within a suitable tolerance. The method is extrapolated to determine neighboring solutions and families of periodic orbits.

### Invariant Manifolds

Henri Poincaré determined that any dynamical system can be analyzed from a geometrical perspective. Knowledge of the phase-space allows the decomposition of the flow into subspaces, thus characterizing the behavior of a system. In obtaining transfer paths in the CR3BP, it is very useful to exploit any knowledge of the flow in the vicinity of a reference solution. Furthermore, it can be demonstrated that the local flow in the vicinity of a periodic solution in this problem may be globalized. Globalizing the stable and unstable invariant manifolds in the vicinity of fixed points along an unstable periodic orbit (of period \( IP \)) yields naturally occurring trajectories that may be exploited for low-energy orbital transfers.

Computing these global manifold trajectories requires initial conditions in the local stable subspace, \( E_s \) and the local unstable subspace, \( E_u \). These conditions are estimated given the state and phase-space information corresponding to any fixed point, \( \vec{X}_{fp}(t_i) \), along the periodic orbit. Of course, any state along the orbit, \( \vec{X}_{fp}(t_i) \), will remain on the periodic orbit when propagated. To globalize the manifold trajectory and compute the flow toward and away from the periodic orbit, conditions on the manifold must be approximated near the fixed point. Thus, given \( \vec{X}_{fp}(t_i) \), a perturbation is added to shift the state into the desired stable or unstable subspace. Computation of this new state is accomplished by defining a perturbation in the direction of the stable or unstable eigenvectors by some small distance \( d_M \). If the eigenvectors associated with the stable and unstable eigenvalues are defined as \( \vec{\gamma}_{Ws}(t_i) = [x_s \ y_s \ z_s \ \dot{x}_s \ \dot{y}_s \ \dot{z}_s]^T \) and \( \vec{\gamma}_{Wu}(t_i) = [x_u \ y_u \ z_u \ \dot{x}_u \ \dot{y}_u \ \dot{z}_u]^T \), respectively, then a normalization results in the definitions,

\[
\vec{V}_{Ws}(t_i) = \frac{\vec{\gamma}_{Ws}(t_i)}{\sqrt{x_s^2 + y_s^2 + z_s^2}}
\]

\[
\vec{V}_{Wu}(t_i) = \frac{\vec{\gamma}_{Wu}(t_i)}{\sqrt{x_u^2 + y_u^2 + z_u^2}}.
\]

Note that the eigenvectors, \( \vec{\gamma}_{Ws}(t_i) \) and \( \vec{\gamma}_{Wu}(t_i) \), are available from the monodromy matrix at the fixed point, \( \Phi(IP + t_i, t_i) \). The initial state vector that shifts the state into \( E_s \) or \( E_u \) is then represented by the full expressions

\[
\vec{X}_s(t_i) = \vec{X}_{fp}(t_i) \pm d_M \cdot \vec{V}_{Ws},
\]

\[
\vec{X}_u(t_i) = \vec{X}_{fp}(t_i) \pm d_M \cdot \vec{V}_{Wu}.
\]

The alternating signs on the displacement from \( \vec{X}_{fp} \) in eq. (7) represent the fact that the trajectory may be perturbed in either direction along the stable or unstable subspace. Propagating the states,
\(\vec{X}_s(t_i)\), in negative time at fixed locations along the entire orbit results in globalization of the stable manifold. Repeating the process on the states, \(\vec{X}_u(t_i)\), in positive time results in globalization of the unstable manifold. Given a series of fixed points along a periodic halo orbit, the six-dimensional eigenvector directions are indicated in Figure 1, where the vectors are projected onto configuration space. Typically, in the Earth-moon system, a value of 50 km for \(d_M\) (converted into nondimensional units) is sufficient to justify the linear approximation, yet still yield reasonable integration times. Globalization of the stable manifolds in the direction of the Earth for an \(L_1\) vertical orbit appears in Figure 2. Note that a representative trajectory winds around the surface as it approaches the periodic orbit.

![Figure 1: Three-Dimensional Eigenvector Projections for Fixed Points Along a Periodic Orbit](image1)

![Figure 2: Stable Manifold Tube for an \(L_1\) Vertical Orbit](image2)
CALCULUS OF VARIATIONS SETUP

A low-thrust transfer with the greatest economy of fuel, and hence a minimum expenditure of mass, is sought. To achieve this goal, a performance index is specified in terms of a Mayer problem in the calculus of variations, i.e.,

$$\max J = k \cdot m_f,$$

where $$m$$ is the total spacecraft mass (subscript "f" denotes a condition at the final time), and $$k$$ is an arbitrary constant that serves a useful purpose when establishing the optimality constraints.\(^{11,13,36}\)

The first-order differential of the performance index (i.e., the noncontemporaneous variation), $$dJ$$, is defined to be zero via a calculus of variations approach, that then requires a suitable test to ensure that the extremal is, in fact, a local maximum. The performance index is subject to the dynamical constraints of the CR3BP with additional thrust acceleration from the VSI engines, i.e.,

$$\dot{X}_p = \begin{cases} \dot{\vec{r}}, \\ \dot{\vec{v}}, \\ \dot{m} \end{cases} = \vec{f}(X_p, \vec{u}_c, t) = \begin{bmatrix} (\partial U(\vec{r})/\partial \vec{r})^T + \vec{h}(\vec{v}) + (T/m) \hat{u}_T \\ -T^2/2P \end{bmatrix}.$$

where the subscript "p" indicates that the state vector for the powered phase includes the additional mass variable, $$m$$. Consistent with the convention of Hull,\(^{10}\) five parameters, $$\Omega$$, $$i$$, $$\theta$$, $$\theta_M$$, and $$\tau_M$$ that specify the kinematic boundary conditions are also incorporated as additional state variables,

$$\dot{z} = \begin{cases} \hat{\Omega}, \\ \hat{i}, \\ \hat{\theta}, \\ \hat{\theta}_M, \\ \hat{\tau}_M \end{cases} = \vec{0}.$$

The three orbital elements, $$\Omega$$, $$i$$, and $$\theta$$ are associated with the instantaneous spacecraft orbit about the Earth at the initial time; $$\theta_M$$ and $$\tau_M$$ are arrival states on the invariant manifold to be described later. Six total controls, $$\vec{u}_c$$, are defined in the problem to ensure a stationary value of the performance index: the three-dimensional thrust direction unit vector, $$\hat{u}_T$$, the thrust magnitude, $$T$$, the engine power, $$P$$, and a slack variable, $$\sigma$$, associated with maintaining an engine power value within the prescribed bounds,

$$\vec{u}_c = \begin{bmatrix} \hat{u}_T \\ T \\ P \\ \sigma \end{bmatrix}.$$

The control equality constraints require the thrust direction, $$\hat{u}_T$$, to be fixed on the unit sphere, and the engine thrust power to be bounded within a maximum value, i.e.,

$$\varphi = \begin{bmatrix} \hat{u}_T^T \hat{u}_T - 1 \\ P - P_{\text{max}} \sin^2 \sigma \end{bmatrix} = \vec{0}.$$

where $$\sigma$$ is employed in converting $$0 \leq P \leq P_{\text{max}}$$ into an equality constraint. Note that this model excludes the use of any direct thrust and $$I_{sp}$$ constraints, and also any inequality bounds on the thrust direction.

**Kinematic Boundary Conditions**

Let the Earth-centered, rotating reference frame be defined by the cartesian coordinates $$\hat{i}$$-$$\hat{j}$$-$$\hat{k}$$, and aligned with the barycentric coordinates defined previously, such that the moon’s orbit plane is the fundamental plane $$\hat{i}$$-$$\hat{j}$$. An Earth-centered, orbit-based rotating frame is also defined by the cartesian coordinates $$\hat{s}_1$$-$$\hat{s}_2$$-$$\hat{s}_3$$. The $$\hat{s}_1$$-vector is parallel along the node line and positive in
the direction of the ascending node. The $\hat{s}_3$-vector is aligned along the instantaneously defined angular momentum vector corresponding to the initial orbit state, and the $\hat{s}_2$-vector completes the right-handed set. The right ascension angle is then defined as the angle from the $\hat{i}$-vector to the $\hat{s}_1$-vector. Inclination is the angle between the spacecraft orbit plane and the Earth-moon orbit plane. The orbit angle is defined as the angle from $\hat{s}_1$ to the initial parking orbit radius vector, $\vec{r}_0$. (All angles are measured positive counterclockwise.) The spacecraft is assumed to depart a circular orbit of unspecified right ascension, $\Omega$, inclination, $i$, and orbit angle, $\theta$. (See Figure 3.) Each time the spacecraft orientation in space is specified, its Earth-orbit state is instantaneously calculated assuming a two-body approximation in the inertial frame, and, then, transformed and shifted into the Earth-moon, barycentric, rotating reference frame. The spacecraft initial mass, departure radius magnitude, and epoch, $t_0$, are all fixed. Thus, the initial kinematic boundary conditions are summarized via the initial state constraint, $\vec{\psi}_0$, and the initial time constraint for the Earth departure orbit,

$$\vec{\psi}_0 \left( \vec{X}_0, \vec{z}_0 \right) = \begin{bmatrix} \vec{r}(t_0) - \vec{r}_0(\Omega, i, \theta) \\ \vec{v}(t_0) - \vec{r}_0(\Omega, i, \theta) \\ m(t_0) - m_0 \end{bmatrix} = \vec{0},$$

$$t_0 = 0,$$

where,

$$\vec{r}_0(\Omega, i, \theta) = \begin{bmatrix} r_0(\cos \Omega \cdot i - \cos \theta \cdot \cos \Omega \cdot c_i) - \mu \\ r_0(\cos \theta \cdot \sin \Omega + \sin \theta \cdot \cos \Omega \cdot c_i) \\ r_0 \sin \theta \end{bmatrix},$$

$$\vec{v}_0(\Omega, i, \theta) = \begin{bmatrix} -\sqrt{\mu/\vec{r}_0(\sin \theta \cdot \cos \Omega + \cos \theta \cdot \cos \Omega \cdot c_i)} + \ell \omega^S r_0(\cos \theta \cdot \sin \Omega + \sin \theta \cdot \cos \Omega \cdot c_i) \\ -\sqrt{\mu/\vec{r}_0(\cos \theta \cdot \sin \Omega - \sin \theta \cdot \cos \Omega \cdot c_i)} - \ell \omega^S r_0(\cos \theta \cdot \sin \Omega - \sin \theta \cdot \cos \Omega \cdot c_i) \\ \sqrt{\mu/\vec{r}_0 \theta} \end{bmatrix},$$

and $\ell \omega^S$ represents the magnitude of the vector denoting the angular velocity of the rotating frame relative to the inertial frame. For the powered phase, the spacecraft arrival point is at a state along the stable manifold associated with the periodic orbit. The stable manifold associated with the arrival orbit is generated a priori and representative trajectories along the manifold are stored in a table. Then, two-dimensional cubic spline interpolation is used to approximate the states along the surface. Let the variables, $\theta_M$ and $\tau_M$ represent free parameters that are used to define a state along the stable manifold within the pregenerated data table. Recall Figure 2 where one

![Figure 3: Departure Orbit Diagram](image)

the stable manifold associated with the periodic orbit. The stable manifold associated with the arrival orbit is generated a priori and representative trajectories along the manifold are stored in a table. Then, two-dimensional cubic spline interpolation is used to approximate the states along the surface. Let the variables, $\theta_M$ and $\tau_M$ represent free parameters that are used to define a state along the stable manifold within the pregenerated data table. Recall Figure 2 where one
trajectory along the manifold appears. The value of $\theta_M$ is an angle-like index associated with the fixed-point, $\vec{X}(t_i)$, that determines where the manifold departs the periodic orbit; essentially this variable identifies one representative trajectory along the surface. The maximum integer value of the $\theta_M$ counter corresponds to the total number of manifold trajectories that are generated around the entire periodic orbit. The value of $\tau_M$ is a time-like variable that is indexed to the backward propagation time along the stable manifold. Each manifold trajectory defined by $\theta_M$ is composed of a preselected total number of time element indices that are tagged by $\tau_M$. (See Figure 4.) The problem is formulated with free final time, thus the only final kinematic boundary conditions, $\psi_f$, are those associated with position and velocity upon insertion at the “match point”, i.e.,

$$
\psi_f(\vec{X}_{p_f}, \vec{z}_f) = \begin{bmatrix} \vec{r}(t_f) - \vec{r}_f(\theta_M, \tau_M) \\ \vec{v}(t_f) - \vec{v}_f(\theta_M, \tau_M) \end{bmatrix} = \vec{0}.
$$

(17)

Figure 4: Sample Arrival States Defined by $\theta_M$ and $\tau_M$ Along the Stable Manifold

**Necessary Conditions for Local Maxima**

The optimization problem is converted to an unconstrained maximization formulation by adjoining all of the constraints (dynamical, control, and kinematic) to the performance index via Lagrange multipliers. The multipliers $\vec{\varphi}$, and $\vec{\nu}$ are adjoined to the kinematic boundary conditions, the multiplier $\vec{\eta}$ is adjoined to the control constraints, and the costate vector, $\vec{\lambda}$, is adjoined to the dynamic constraints, resulting in the augmented performance index, $J'$,

$$
J' = G \left( t_f, \vec{X}_{p_0}, \vec{X}_{p_f}, \vec{z}_0, \vec{z}_f, \vec{\varphi}, \vec{\nu} \right) + \int_{t_0}^{t_f} L' \left( t, \vec{X}_p, \vec{z}, \vec{u}_c, \vec{\lambda}, \vec{\eta} \right) dt,
$$

(18)

where,

$$
G \left( t_f, \vec{X}_{p_0}, \vec{X}_{p_f}, \vec{z}_0, \vec{z}_f, \vec{\varphi}, \vec{\nu} \right) = k \cdot m_f + \vec{\varphi}^T \psi_0 \left( \vec{X}_{p_0}, \vec{z}_0 \right) + \vec{\nu}^T \psi_f \left( \vec{X}_{p_f}, \vec{z}_f \right),
$$

(19)

$$
L' \left( t, \vec{X}_p, \vec{z}, \vec{u}_c, \vec{\lambda}, \vec{\eta} \right) = \dot{\vec{H}} \left( t, \vec{X}_p, \vec{z}, \vec{u}_c, \vec{\lambda}, \vec{\eta} \right) - \vec{\lambda}^T \dot{\vec{X}}_p \vec{z}
$$

(20)

$$
\dot{\vec{H}} \left( t, \vec{X}_p, \vec{z}, \vec{u}_c, \vec{\lambda}, \vec{\eta} \right) = \dot{H} \left( t, \vec{X}_p, \vec{z}, \vec{u}_c, \vec{\lambda} \right) + \vec{\eta}^T \dot{\phi} \left( t, \vec{X}_p, \vec{u}_c \right).
$$

(21)
and transversality conditions, which are summarized as follows: 

If all the constraints are satisfied, maximizing the augmented performance index is identical to maximizing the actual performance index. To satisfy the first-order necessary conditions, a stationary

costate equations, and are of identical dimension \((7 \times 1)\) to the state equations. Equation (26) is trivial to the problem, but is included due to the utilization of the parameters in the calculus of variations problem. Equations (27)-(30) are control optimality conditions, and are used to partially formulate a control law. Finally, eq. (27) implies that \(\lambda_{\xi}\) is parallel to \(\hat{u}_T\), thus,

where the normalization implicitly satisfies the first control constraint in eq. (12). Equation (28) is solved directly to yield a parameterization for the thrust magnitude,
Equation (12) and eqs. (29)-(30) offer the following correlations for the engine power magnitude: (i) when \( \cos \sigma = 0 \), then \( P = P_{\text{max}} \); (ii) if \( \sin \sigma = 0 \), then \( P = 0 \); and, (iii) if \( \eta_2 = 0 \), then \( 0 \leq P \leq P_{\text{max}} \).

Equations (31)-(35) are the transversality conditions, and represent the equality constraints for the free optimization parameters. Only trivial values result from eq. (31), but eq. (32) demonstrates that the final value of the mass costate, \( \lambda_{m_f} \), must be the constant, \( k \), specified in eq. (8). Because \( k \) is an arbitrary positive constant, and because \( \lambda_m \) always increases due to eq. (25), a degree of freedom is eliminated by scaling \( \lambda_{m_0} = 1 \).

The first-order necessary conditions only guarantee a local extremal for eq. (8), and a test for a local maximum is necessary to resolve the potential candidates for a control parameterization. Applying Pontryagin’s Maximum Principle results in the following inequality condition,

\[
\lambda^T \left( (T^*/m) \dot{u}^* - \lambda_m (T^*/2P^*) \right) \geq \lambda^T \left( (T/m) \dot{u}_T \right) - \lambda_m (T^2/2P) .
\]

(38)

Of the two possibilities in eq. (36), clearly eq. (38) requires that,

\[
\dot{u}_T = \lambda_0^*/\lambda_0^s ,
\]

(39)

where the velocity costate vector, \( \lambda_v \), now implies additional meaning as Lawden’s primer vector. The observations (i)-(iii) on the engine power, \( P \), resulting from eqs. (29)-(30) are further reduced by substituting eq. (37) and eq. (39) into eq. (38), that is,

\[
\frac{\lambda^2_0 P^*}{2\lambda_m m^2} \geq \frac{\lambda^2_0 P}{2\lambda_m m^2} .
\]

(40)

Since the first-order necessary conditions identify a value of \( \lambda_m \) that originates at one and always increases, eq. (40) is always satisfied by defining \( P = P_{\text{max}} \). This choice automatically satisfies eqs. (29)-(30) and the second constraint in eq. (12) for the case when \( \cos \sigma \) is always zero. Thus, eq. (37) becomes,

\[
T = \frac{\lambda_0^* P_{\text{max}}}{\lambda_m m} .
\]

(41)

The controls are now uniquely parameterized, and the two-point boundary value problem is completely defined.

**Adjoint Control Transformation**

One of the most difficult aspects in obtaining a solution to optimization problems that use an Euler-Lagrange formulation is generating an accurate initial guess for the costate variables. Low-thrust problems are also typically characterized by long propagation times, rendering the problem even more sensitive to initial conditions. Typical strategies to address the initial guess dilemma often involve solutions to several smaller sub-problems or use analytical results. An alternate approach, the adjoint control transformation (ACT), as investigated by Dixon and Biggs introduces physical control variables and their derivatives as an estimate for the initial costates. These control variables allow exploitation of physical intuition to produce a guess for the values of the initial costate variables. The ACT has also been demonstrated successfully in several low-thrust applications.

Consider a reference frame centered at the spacecraft, defined by the unit vectors \( \hat{v} - \hat{w} - \hat{h} \), i.e., the \( vwh \)-frame. The \( \hat{v} \)-axis is aligned with the velocity vector, \( \vec{v} \). The \( \hat{h} \)-axis is parallel to the instantaneous angular momentum vector, \( \vec{r} \times \vec{v} \). Finally, the \( \hat{w} \)-axis is defined to complete a right-handed system. These unit vectors, and the associated time derivatives, are defined as,

\[
\dot{\hat{v}} = \hat{\vec{v}}, \quad \dot{\hat{h}} = \frac{\vec{r} \times \vec{v}}{||\vec{r} \times \vec{v}||}, \quad \dot{\hat{v}} = \hat{\vec{v}} \times \hat{\vec{v}},
\]

(42)
\[ \dot{\hat{v}} = \hat{\hat{v}} / v - \hat{\hat{v}} / v^2, \]  
\[ \hat{\dot{h}} = \hat{\dot{h}} / h - \hat{\dot{h}} / h^2, \]  
\[ \hat{\dot{w}} = \hat{\dot{h}} \times \hat{\dot{v}} + \hat{\dot{h}} \times \hat{\dot{v}}. \]  
(43)
(44)
(45)

Given a vector and its time derivative, the following relationships are also exploited to fully determine eqs. (43)-(45),

\[ \dot{\hat{v}} = \hat{\dot{v}} / v, \]  
\[ \hat{\dot{h}} = \hat{\dot{h}} / h. \]  
(46)
(47)

Two spherical angles, \( \alpha \) and \( \beta \), as well as their time derivatives, \( \dot{\alpha} \) and \( \dot{\beta} \), specify the orientation of the thrust direction relative to this frame, \( \hat{u}_{Tvw} \), and also the time derivative of the thrust direction, \( \dot{\hat{u}}_{Tvw} \), that is,

\[ \dot{\hat{u}}_{Tvw} = \begin{bmatrix} \cos \alpha \cos \beta & \sin \alpha \cos \beta & \sin \beta \end{bmatrix}^T, \]  
\[ \dot{\hat{u}}_{Tvw} = \begin{bmatrix} -\dot{\alpha} \sin \alpha \cos \beta & \dot{\beta} \cos \alpha \sin \beta \\ \dot{\alpha} \cos \alpha \cos \beta & -\dot{\beta} \sin \alpha \sin \beta \end{bmatrix} \]  
(48)
(49)

Since the equations of motion are integrated in the cartesian, barycentric rotating frame (with unit vectors \( \hat{i} - \hat{j} - \hat{k} \)), a rotation matrix, \( R \), is required to transform the thrust direction, \( \hat{u}_{Tvw} \) (and \( \dot{\hat{u}}_{Tvw} \)),

\[ R = \begin{bmatrix} \hat{i} \cdot \hat{v} & \hat{i} \cdot \hat{w} & \hat{i} \cdot \hat{h} \\ \hat{j} \cdot \hat{v} & \hat{j} \cdot \hat{w} & \hat{j} \cdot \hat{h} \\ \hat{k} \cdot \hat{v} & \hat{k} \cdot \hat{w} & \hat{k} \cdot \hat{h} \end{bmatrix}, \]  
\[ \dot{R} = \begin{bmatrix} \hat{i} \cdot \hat{\dot{v}} & \hat{i} \cdot \hat{\dot{w}} & \hat{i} \cdot \hat{\dot{h}} \\ \hat{j} \cdot \hat{\dot{v}} & \hat{j} \cdot \hat{\dot{w}} & \hat{j} \cdot \hat{\dot{h}} \\ \hat{k} \cdot \hat{\dot{v}} & \hat{k} \cdot \hat{\dot{w}} & \hat{k} \cdot \hat{\dot{h}} \end{bmatrix}, \]  
(50)
(51)
(52)

The subscript “ijk”, denotes that the thrust direction is expressed in terms of unit vectors in the barycentric rotating frame. The definition of the thrust direction from eq. (39), is employed to parameterize the primer vector,

\[ \hat{\lambda}_{v} = \lambda_{v} \hat{u}_{Tijk}, \]  
(53)

where \( \lambda_{v} \) is the magnitude of the primer vector, \( \lambda_{v} = \| \hat{\lambda}_{v} \| \). Then, the equation of motion for the primer vector, eq. (24), is directly involved in parameterizing the position costate vector,

\[ \hat{\lambda}_{v} = -\dot{\lambda}_{v} - \hat{\lambda}_{v} \frac{\partial h(v)}{\partial v}. \]  
(54)

The derivative of the primer vector, \( \dot{\hat{\lambda}}_{v} \), is also available by differentiating eq. (53), and substituting the result into equation (54),

\[ \dot{\hat{\lambda}}_{v} = -\dot{\lambda}_{v} \hat{u}_{Tijk} - \lambda_{v} \hat{u}_{Tijk}. \]  
(55)

Finally, since the Hamiltonian (eq. (22)) is not an explicit function of time, it is a constant integral of motion during powered flight. The boundary condition from eq. (35) renders the constant value of the Hamiltonian, thus, \( \dot{H} = 0 \). Rather than attempt to target this condition at the final time, its value is immediately enforced at the initial time, removing another degree of freedom from the
problem. Substituting the full expression for $\vec{\lambda}$ (obtained via eq. (55) into eq. (54)) as well as eq. (41) into eq. (22), with $H = 0$, and rearranging yields,

$$\dot{\vec{\lambda}} = -\frac{1}{u_{T,ijk}^T \vec{v}} \left( \vec{\lambda} \dot{u}_{T,ijk}^T \vec{v} + \vec{\lambda}^T \left( \frac{\partial \vec{h}}{\partial \vec{v}} \vec{v} - \left( \frac{\partial U}{\partial \vec{r}} \right)^T - \vec{h}(\vec{v}) - \frac{T^2}{2P} \right) \right).$$

(56)

Additionally, eq. (41) is used to parameterize $\vec{\lambda}$ in terms of the thrust, $T$. Thus, the mapping sequence allows $\vec{\lambda}$ to be calculated from $\alpha$, $\dot{\alpha}$, $\beta$, $\dot{\beta}$, $T$, and the Hamiltonian constraint.

**HYBRID DIRECT/INDIRECT SOLUTION APPROACH**

Both direct single shooting and direct multiple shooting are used to determine the low-thrust trajectories. These methods convert the functional optimization problem into a parameter optimization problem by a direct maximization on the performance index, rather than a root-solving process on all of the optimality constraint equations via an indirect approach. Thus, the “hybrid” term, as described by Kluever, reflects the fact that the calculus of variations, TPBVP approach is used for the majority of the development, (e.g., control parameterization via costates) but direct maximization replaces the indirect solving of (the often sensitive) transversality conditions.

Matlab’s® optimization toolbox function, *fmincon*, is utilized for the maximization. The gradient-based method used by *fmincon* for this study is medium-scale sequential quadratic programming (SQP) with finite difference gradients. The function solves a quadratic programming (QP) subproblem at each iteration. An estimate of the Hessian of the Lagrangian is updated at each iteration with the Broyden-Fletcher-Goldfarb-Shanno (BFGS) formula. A line search is performed using a merit function and the QP subproblem is solved using an active set strategy. All integration is explicit, and uses a Runge-Kutta-Verner 8(9) propagator originally written in C, and compiled in Matlab via the MEX interface for speed.

**Direct Single Shooting**

Direct single shooting is useful in describing the problem with a relatively small number of design variables. For the direct single shooting approach, all of the unknown initial conditions and parameters (at each boundary) are selected as design variables. Thus, the design variable vector, $\vec{S}_{ss}$, is defined as,

$$\vec{S}_{ss} = \{ \Omega \ i \ \theta \ \dot{\alpha}_0 \ \dot{\alpha}_0 \ \beta_0 \ \dot{\beta}_0 \ T_0 \ t_f \ \theta_M \ \tau_M \}^T. \quad (57)$$

The first nine design variables in $\vec{S}_{ss}$ establish the state in the departure orbit, as well as the necessary information to predefine the entire state, costate, and control histories. Thus, the ACT is inherently incorporated, and the state and costates are explicitly propagated forward to $t_f$, noting that the initial mass, $m_0$, is scaled such that $m_0 = 1$. The last two design variables specify the state at the “match point” on the invariant manifold. For implementation purposes, the parameter optimization objective function, $F$, is set equal to negative final mass (the constant $k$ is dropped here as it is trivial to the actual maximization), i.e.,

$$F(\vec{S}_{ss}) = -m_f. \quad (58)$$

The constraint vector, $\vec{c}_{ss}(\vec{S}_{ss})$ is equivalent to the terminal kinematic boundary conditions for the powered phase (eq.(17)). To aid in the convergence process, the cost function is not initially maximized, but a feasible solution satisfying $\vec{c}_{ss}(\vec{S}_{ss})$ is obtained. In this initial situation, the cost is set to minimize the least-squares error associated with the terminal constraints,

$$F(\vec{S}_{ss}) = \vec{c}_{ss}^T(\vec{S}_{ss}) \vec{c}_{ss}(\vec{S}_{ss}). \quad (59)$$
Then, the complete NLP maximization process restarts with the feasible solution as an initial condition and eq. (58) as the objective function. The function generator integrated into the NLP package for the process is summarized as follows:

### Direct Single Shooting

**Input:** $\vec{S}_{ss}$

- **Initialize:** $\vec{X}_p$ via eqs. (15)-(16) (scale $m_0 = 1$), and $\vec{z}$
- **Initialize:** $\vec{\lambda}_0$ via eq. (41), eqs. (42)-(56) ($\lambda_{m_0} = 1$)
- **Propagate:** States (eq. (9)), costates (eqs. (23-25)) using control laws of eq. (39), eq. (41), and $P = P_{\text{max}}$ from $t = t_0$ to $t = t_f$

**Terminate trajectory**

**Compute Objective and Constraint:** $F(\vec{S}_{ss})$ via eq. (58) (or eq. (59))), and $\vec{c}_{ss}(\vec{S}_{ss})$ via eq. (17)

### Direct Multiple Shooting

Direct multiple shooting is also used to compute low-thrust trajectories. While the low number of design variables in single shooting methods are advantageous for simplicity, the same feature also means that the variables are susceptible to extreme sensitivity due to small changes propagating into very large nonlinear deviations downstream. This fact is especially true in the low-thrust problem, which is characterized by very long propagation times. Multiple shooting techniques are introduced to alleviate these precise difficulties in solving the TPBVP.\(^9,18,19\) Let the time interval be decomposed into $N$ segments, $t_0 < t_1 < \cdots < t_N = t_f$. (60)

Note that the times for all interior nodes, $t_1, \ldots, t_{N-1}$ are assumed to be fixed here. (The initial time is still fixed, and final time is still free, as defined previously.) Correspondingly, each interior node includes an unknown initial state and costate denoted by a “$+”$ superscript that may be explicitly integrated forward,

$$
\vec{X}^+_p, \ldots \vec{X}^+_{p_{N-1}}, \vec{\lambda}^+_1, \ldots \vec{\lambda}^+_{N-1}.
$$

(61)

All of these unknown parameters become part of the new design variable vector, $\vec{S}_{ms} = \{ \vec{S}_{ss}^T, \vec{X}^T_p, \ldots \vec{X}^T_{p_{N-1}}, \vec{\lambda}^T_1, \ldots \vec{\lambda}^T_{N-1} \}^T$, (62)

and the new constraint vector is

$$
\vec{c}_{ms} = \left\{ \begin{array}{c} \vec{c}_s \\ \vec{c}_{cs} \end{array} \right\},
$$

(63)

where $\vec{c}_s$ and $\vec{c}_{cs}$ are the state and costate constraints evaluated at the interior nodes, and $\vec{c}_{ss}$ is the terminal constraint error that is also used in the direct single shooting approach. At an arbitrary interior node during time $t_i$, the interior constraints are defined,

$$
\begin{align*}
\vec{c}_s(t_i) &= \vec{X}^-_{p_i} - \vec{X}^+_p, \\
\vec{c}_{cs}(t_i) &= \vec{\lambda}^- - \vec{\lambda}^+_i.
\end{align*}
$$

(64)

The “$-”$ superscript indicates that the state and costate are obtained at the termination of the direct integration of $\vec{X}^+_p (t_{i-1})$ and $\vec{\lambda}^+_p (t_{i-1})$ over the time interval $t_{i-1}$ to $t_i$. If the length of the
state vector is \( p \) (where \( p = 7 \) here), and the time at each interior node is fixed, then it is apparent that direct multiple shooting adds \( 2p \times (N - 2) \) design variables at the expense of \( 2p \times (N - 2) \) constraints. Similar to single shooting, the cost is evaluated as follows,

\[
F(\vec{S}_{ms}) = -m_f \quad \text{or} \quad F(\vec{S}_{ms}) = c_{ms}^{T}(\vec{S}_{ss})c_{ms}(\vec{S}_{ss}),
\]

where the feasible solution now includes the total error at each interior node. The advantage to the multiple shooting approach is that the individual design variable sensitivity is reduced, allowing for more typically robust convergence, the freedom to add additional constraints to the nodes, and the possibility of parallel processing. The main disadvantage to the multiple shooting approach is additional function evaluations introduced by adding many new design variables and constraints (especially with finite difference gradients). Although not attempted here, some of this penalty may be offset by exploiting the sparsity of the Jacobian matrix. In summary, the function generator for the multiple shooting approach is as follows:

**Direct Multiple Shooting**

Input: \( \vec{S}_{ms} \)

Do for each segment \( i = 0, N - 1 \)

if \( i = 0 \)

\[ \text{Initialize: } \vec{X}_0 \text{ via eqs. (15)-(16) (scale } m_0 = 1), \text{ and } \vec{z} \]

\[ \text{Initialize: } \vec{\lambda}_0 \text{ via eq. (41), eqs. (42)-(56) (} \lambda_{m0} = 1) \]

else

\[ \text{Initialize: } \vec{X}^+_{pi}, \vec{\lambda}^+_{ti}, t_i, t_{i+1} \]

Propagate: States (eq. (9)), costates (eqs. (23-25)) using control laws of eq. (39), eq. (41), and \( P = P_{\text{max}} \) from \( t = t_i \) to \( t = t_{i+1} \) to obtain \( \vec{X}^-_{pi+1}, \vec{\lambda}^-_{i+1} \)

if \( i < N - 1 \)

\[ \text{Initialize: } \vec{X}^+_{P_{i+1}}, \vec{\lambda}^+_{i+1} \]

Compute interior node constraints: \( \vec{c}_s(t_{i+1}) \) and \( \vec{c}_a(t_{i+1}) \) via eq. (64)

End Do

Terminate trajectory

Compute Objective and Final Constraints: \( F(\vec{S}_{ms}) \) via eq. (65) and \( \vec{c}_{ss}(\vec{S}_{ms}) \) via eq. (17)

**MISSION SCENARIO: TRANSFERS TO LUNAR COVERAGE ORBITS**

One location of interest for future space exploration is the region near the lunar south pole. From a range of orbits, various trajectories exposed in the CR3BP are potentially applicable in mission design of lunar relay communication satellites for coverage of the south pole due to the fixed geometry in the rotating frame and line-of-sight capability. For example, \( L_1 \) and \( L_2 \) southern halo orbits possess a line-of-sight with the lunar south pole over the majority of the orbital period, and a line-of-sight with the Earth along the entire orbit. Thus, two spacecraft phased in such orbits yield continuous coverage of the lunar south pole. Additional orbital information on the periods and stability indices aids in the selection of specific orbits. From the study by Grebow et al., sample orbits are selected from the following families: \( L_1 \) and \( L_2 \) halo orbits, \( L_1 \) and \( L_2 \) vertical orbits, and \( L_2 \) “butterfly” orbits (see Figure 5).
Feasible transfer trajectories with a large convergence radius, i.e., with a high initial thrust are easily determined from continuation procedures. Then, the initial thrust magnitude is iteratively lowered until the magnitude resembles a realistic value. For this study, the maximum allowable thrust is fixed at 850 mN. The assumed departure orbit and engine constants are listed in Table 1. For consistency, each manifold tube is parameterized such that $\theta_M = [1, \ldots, 50]$, and $\tau_M = [1, \ldots, 5000]$. No restriction is placed upon the insertion location along the manifolds, however, all pre-generated data is propagated such that a maximum of four Earth periapses are included. Direct single shooting is used in all examples to develop transfers, and unless otherwise specified, is the basis for the number of iterations. Direct multiple shooting is attempted in the first example, transfer to a halo orbit, but since parallel processing is not available, the increase in total function evaluations does not justify the decrease in optimization iterations. Time history data is discussed for the first halo orbit transfer, and the time history data for all other transfers are available in Appendix A.

### Table 1: Departure Orbit and Engine Constants

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_0$</td>
<td>20000</td>
<td>km</td>
</tr>
<tr>
<td>$m_0$</td>
<td>1500</td>
<td>kg</td>
</tr>
<tr>
<td>$P_{max}$</td>
<td>10</td>
<td>kw</td>
</tr>
<tr>
<td>$m^*$</td>
<td>6.04680403834987 x 10^{15}</td>
<td>kg</td>
</tr>
<tr>
<td>$t^*$</td>
<td>385692.5</td>
<td>km</td>
</tr>
<tr>
<td>$t$</td>
<td>377084.152667039</td>
<td>s</td>
</tr>
<tr>
<td>Earth $GM$</td>
<td>398600.432896939</td>
<td>km^{3}/s^{2}</td>
</tr>
<tr>
<td>Moon $GM$</td>
<td>4902.80058214776</td>
<td>km^{3}/s^{2}</td>
</tr>
<tr>
<td>Earth Radius</td>
<td>6378.14</td>
<td>km</td>
</tr>
<tr>
<td>Moon Radius</td>
<td>1737.4</td>
<td>km</td>
</tr>
</tbody>
</table>
Three \( L_1 \) Halo Orbit Transfers

Two 12-day \( L_1 \) halo orbits with different \( z \)-amplitudes are initially selected to demonstrate the construction of transfer trajectories. The orbit with the lower \( z \)-amplitude (\( A_z = 13,200 \) km) is the focus of the first example (i) since it possesses the smaller out-of-plane excursion, and (ii) the manifolds retain their “tube-like” shape, thus the gradients of \( \theta_M \) are well-behaved with respect to the design variables. The resulting transfer appears in Figure 6. It is plotted in both the Earth-centered rotating frame and the Earth-centered inertial frame. The black portion along the trajectory indicates the powered phase, the blue asterisk represents the insertion point, and the coasting phase is green. The orbit color coincides with that in Figure 6. In the inertial frame, the dotted line is the path of the moon. To obtain a feasible solution, only 28 iterations are necessary (20 with a multiple shooting approach that uses 5 segments), but the optimal solution requires 90 additional iterations. (See Tables 2-4 for a detailed list of the performance data for the optimizing procedure.) Only 50 iterations are required for the optimal solution using multiple shooting with 5 segments. This disparity in number of iterations between the feasible and the optimal solutions is evident in almost every example, since the VSI engine continues to exhibit higher efficiency at lower values of thrust. Thus, the optimizer typically extends the transfer time slowly and gradually until a locally maximal amount of final spacecraft mass is obtained. In implementation, the optimal solution exhibits a savings of about 1-10 kg over the feasible solution. The oscillatory structure of the control histories are typical of all transfers, and an example appears in Figure 7. In Figure 7, the angle \( \zeta \) represents the angular displacement of the thrust vector from the velocity vector. Thus, even though the thrust direction oscillates, it is closely aligned with the velocity vector during the initial spiral out, and then shifts direction to meet the state at the match point. Both the in-plane (\( \alpha \)), and out-of-plane (\( \beta \)) oscillations similarly reflect this behavior. Since the Hamiltonian is an integral of the powered motion, its time history is used as an independent check for numerical accuracy. Tolerances are input as \( 1 \times 10^{-10} \) nondimensional units. The 182.8 day transfer is composed of 168.1 days of thrusting and 14.7 days of coasting along the manifold surface. The trajectories on the stable manifold that are initially perturbed in the direction toward the moon may also be considered. Such a transfer enables the possibility of a close lunar pass prior to insertion. (See Figure 8.) From the two locally optimal solutions, no conclusions are drawn concerning the global behavior.

The third example is a transfer to a 12-day \( L_1 \) halo orbit, but the out-of-plane amplitude is now \( A_z = 55,700 \) km. For computation of the manifold associated with this halo orbit, the manifold trajectories are spaced much further apart than those for the previous 12-day orbit. Thus, the cubic spline approximation across \( \theta_M \) degrades in comparison to the previous example. While this degradation may be offset by increasing the density of the manifold trajectories, i.e., the total integer value range of \( \theta_M \), the fixed demonstration value consistent with the previous example proved sufficient despite a higher iteration count for a maximal solution. (See Figure 9.) A 155.5 day transfer is obtained, with 131.2 days of thrusting, and coasting for 24.3 days. The slightly higher thrust values result in less overall conservation of mass propellant, as reflected in Table 3. The current formulation assumes a thrust arc followed by a coast arc. No \( L_2 \) halo orbits have yet been examined for low-thrust transfers.

Transfers to \( L_1 \) and \( L_2 \) Vertical Orbits

The second class of orbits that are targeted for transfers are \( L_1 \) and \( L_2 \) vertical orbits. The \( L_1 \) vertical orbit possesses an out-of-plane \( z \)-amplitude \( A_z = 57,000 \) km. The associated stable manifold passes more than 150,000 km from Earth, that is, approximately 50,000 km further than any insertion point on the stable manifold corresponding to either of the two halo orbits. The manifolds are also significantly out-of-plane, but the shape of the tube is well preserved by the trajectories, as in the sample transfer for the first 12-day halo orbit. The potential sensitivity to this out-of-plane behavior requires a a departure orbit with a higher inclination. A feasible solution is obtained in 44 iterations. The optimal solution then emerges after 74 iterations. (See Figure 10 and Tables
The next example is based on an $L_2$ vertical orbit possessing a period of 16 days and $A_z = 57,000$ km. As $\theta_M$ varies during the pregeneration phase of the stable manifold, it is apparent that a smooth, tubular structure is not a good mathematical assumption for this manifold. The lack of a well-defined surface impacts the numerical gradients corresponding to the cost and constraints with respect to $\theta_M$, therefore the variable is completely removed from the problem by fixing a trajectory of interest, i.e., fixing $\theta_M$. Despite the elimination of a design variable, a solution is still obtained, yet it is only a local minimum for a single trajectory associated with the fixed point. A feasible trajectory is obtained in 28 iterations, however, and a locally optimal solution in 198 iterations. It is also
apparent in Figure 11 that the converged trajectory exhibits a close passage of the moon to reach the translunar periodic orbit. The resulting 176.4 day transfer includes 132.5 days of thrusting and a 44.0 day coast.

Transfer to an $L_2$ Butterfly Orbit

The final orbit from Figure 5 for demonstration of the design of low-thrust transfers in this problem is a 14-day $L_2$ butterfly orbit. The initially targeted insertion state along the stable manifold, similar to the sample transfers to the vertical orbits, is over 150,000 km (actually 162,000 km) from Earth. Again, similar to the 16-day $L_2$ vertical orbit, a smooth surface is not formed when $\theta_M$ is varied during the initial manifold generation. This behavior is actually observed in several $L_2$ orbits due to the often large regions of exclusion that can prevent individual manifold trajectories from passing within the vicinity of the Earth. As a result, the angle-like parameter, $\theta_M$, is again fixed. The near-stability of this orbit also correlates to a longer coasting time along the manifold.
SUMMARY

The restricted three-body problem is used to construct preliminary designs for low-thrust transfers to libration point orbits. These transfers are based on the existence of predefined reference coast arcs computed as trajectories on invariant manifolds associated with the periodic orbit. A calculus of variations approach yields control laws that are parameterized in terms of costate variables for a variable specific impulse engine model. Additional optimality constraints also prove to be useful.
in establishing the design variables (via the adjoint control transformation). The ACT allows the specification of costates without a start-up value. Nonlinear programming with explicit numerical integration via direct single shooting and direct multiple shooting is used as a basis to obtain transfers that locally maximize the final spacecraft mass. These direct methods are successfully applied to a mission scenario that utilizes Earth-moon libration point orbits for potential lunar south pole
Table 4: Converged Design Variable Values for Orbit Examples

<table>
<thead>
<tr>
<th></th>
<th>12-Day L1 Halo 1</th>
<th>12-Day L1 Halo 1(^a)</th>
<th>12-Day L1 Halo 2</th>
<th>14-Day L1 Vertical</th>
<th>14-Day L2 Vertical(^a)</th>
<th>14-Day L2 Butterfly</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\Omega) (deg)</td>
<td>227.972</td>
<td>115.534</td>
<td>-132.492</td>
<td>70.255</td>
<td>41.473</td>
<td>22.389</td>
</tr>
<tr>
<td>(i) (deg)</td>
<td>4.43307</td>
<td>-8.85154</td>
<td>20.738</td>
<td>10.9948</td>
<td>11.581</td>
<td>22.190</td>
</tr>
<tr>
<td>(\theta) (deg)</td>
<td>209.378</td>
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<td>181.936</td>
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<td>(\alpha_0) ((\times 10^{-2}) rad)</td>
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<td>-1.76465</td>
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<td>(\dot{\alpha}_0) ((\times 10^{-6}) rad/s)</td>
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</tr>
<tr>
<td>(\beta_0) ((\times 10^{-2}) rad)</td>
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<td>(T_0) (mN)</td>
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<td>(t_f) (days)</td>
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<td>(\tau_M)</td>
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<td>3079.45</td>
<td>3501.65</td>
<td>4214.50</td>
<td>4401.13</td>
</tr>
</tbody>
</table>

\(^a\) Lunar Flyby

Feasible transfers are systematically generated with a continuation method that iteratively lowers the initial thrust value until the problem constraints are satisfied. Maximization via NLP results in gradual increases in the final mass with respect to the feasible solution.

The departure orbit under consideration may also be slightly modified to include more general classes of conics defined by an initial set of complete orbital elements. Such initial conditions may be useful, for example, if a spacecraft is placed into a geosynchronous transfer orbit (GTO). An example of a low-thrust transfer from GTO to a 12-day \(L_1\) halo orbit (\(A_z = 13,200\) km) appears in Figures A.6-A.7. Future work will more rigorously consider the type of departure orbit within the context of the potential mission application. Ongoing efforts also involve further investigation of low-thrust transfers to conic orbits via a dynamical systems approach, their globally maximal solutions, and of course, transition of current results into higher fidelity models.

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**REFERENCES**


APPENDIX A

Time history data is generated for the remaining orbit examples, and detailed in Figures A.1-A.5. An example transfer considering a departure from GTO appears in Figures A.6-A.7.

Figure A.1: Time History Data for Low-Thrust Transfer to a 12-day, $L_1$ Halo Orbit ($A_z = 13,200$ km) Using a Lunar Flyby
Figure A.2: Time History Data for Low-Thrust Transfer to a 12-day, $L_1$ Halo Orbit ($A_z = 55,700$ km)

Figure A.3: Time History Data for Low-Thrust Transfer to a 14-day, $L_1$ Vertical Orbit
Figure A.4: Time History Data for Low-Thrust Transfer to a 16-day, $L_2$ Vertical Orbit

Figure A.5: Time History Data for Low-Thrust Transfer to a 14-day, $L_2$ Butterfly Orbit
Figure A.6: Low-Thrust Transfer to a 12-day, $L_1$ Halo Orbit ($A_z = 13, 200$ km) via a GTO Departure Condition

Figure A.7: Time History Data for Low-Thrust Transfer to a 12-day, $L_1$ Halo Orbit ($A_z = 13, 200$ km) via a GTO Departure Condition