TRAJECTORY ARCS WITH LUNAR ENCOUNTERS FOR TRANSFERS TO SMALL AMPLITUDE LISSAJOUS ORBITS

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In support of current and future libration point missions, analysis concerning trajectories that use a lunar encounter to reach a libration point orbit is ongoing. The design process to compute transfer trajectories, via a lunar encounter, to small amplitude Lissajous orbits in the vicinity of the $L_1$ and $L_2$ libration points, may involve a time-consuming shooting procedure. Additional insight and an improved understanding of the solution space will eventually translate into a more efficient and flexible design process. This paper summarizes a recent preliminary investigation into the design of such transfer trajectories, specifically to the vicinity of $L_2$. The phase at insertion into the Lissajous orbit is pre-specified to be consistent with an “opening pattern,” that is, a phase when the orbit is expanding. The addition of phasing loops is also briefly considered.

INTRODUCTION

As the number of missions to the vicinity of the Sun-Earth/Moon libration points increases, wider variations are introduced into the design of the trajectory as well as the transfer path. For example, there are some advantages that accrue to the mission if the libration point trajectory is smaller in amplitude than the periodic halo orbits. However, a smaller amplitude orbit typically requires a larger insertion cost. To reduce the magnitude of the insertion maneuver for fuel-limited missions, a lunar gravity assist can be exploited. Such transfer arcs have been successfully generated for a number of different missions in recent years.$^{1-8}$ However, as baseline and contingency requirements become more complex, greater flexibility and an extended range of options are necessary. Therefore, in support of trajectory design, a process is sought that enables fast and efficient design of transfer trajectories that include a lunar encounter. The baseline solution in this problem must, at least, satisfy the governing differential equations in the three-body problem and, ultimately, an arc in the four-body problem is required. A preliminary investigation into this problem offers insight into the solution space and its

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use in generating an initial approximation for the path; a subsequent differential corrections scheme can modify the first guess to produce a viable transfer path in an ephemeris model of the system.

**BACKGROUND**

For application to the Sun-Earth/Moon system, the goal in this study is an efficient strategy for the design of a trajectory that uses a lunar encounter to reach a specified libration point orbit. Spacecraft have successfully completed missions in this regime, so this is evidently not a new problem. Rather, more complex scenarios and more challenging constraints create a more difficult design task with any methodology that employs shooting procedures. In the alternative formulation of the problem that is described here, a number of steps are involved to produce a solution, but, essentially, various arcs are determined numerically and patched together for a continuous end-to-end trajectory. One of those arcs includes the lunar encounter.

**Libration Point Orbits**

Lissajous trajectories are three-dimensional quasi-periodic solutions in the six-dimensional phase space associated with the three-body problem. Under certain conditions, a similar three-dimensional solution also emerges that is precisely periodic; these periodic families of orbits have been labeled halo orbits. Precisely periodic halo orbits do not exist in a general ephemeris model with perturbations such as solar radiation pressure and non-periodic primary motion. However, Lissajous trajectories can still be generated numerically and at least two different approaches are available. One procedure generates these solutions, in conjunction with the transfer paths, using straightforward integration from Earth launch conditions. In an alternative approach, a particular Lissajous trajectory is identified or designed, perhaps even a collection of such quasi-periodic trajectories, as solutions that closely match the specifications of the mission of interest. Then, the best transfer path from Earth, or the most useful trajectory arc to/from another point/orbit in this region, can be determined. Previous efforts have resulted in a numerical process based on this strategy that successfully generates such a Lissajous solution. A sample quasi-periodic Lissajous trajectory, in the Sun-Earth/Moon system, appears in Figure 1. It has been computed in a model that uses ephemeris information for the locations of the Sun, Earth, and Moon. Three planar projections of the orbit are shown with the origin in each plot corresponding to the L libration point. The three axes in the figure are defined consistent with the rotating frame typically used in the restricted three-body problem. Thus, the x-axis is directed from the larger primary (Sun) to the smaller (Earth/Moon barycenter), the y-axis is defined in the plane of motion of the primaries and 90° from the x-axis, and, finally, the z-axis completes the right-handed frame. The trajectory that appears in the figure is computed with an approximate out-of-plane amplitude \( A_x = 145,000 \text{ km} \), and with the corresponding approximate in-plane amplitude \( A_y = 230,000 \text{ km} \). The duration along each revolution is approximately six months. Note that the spacecraft moves along its path about L in a clockwise direction as viewed in the y–z projection.
Invariant Manifolds and Transfer Paths

Once a periodic or quasi-periodic orbit is computed, a transfer path from the Earth to the specified orbit is sought. To most efficiently determine a transfer path, the characteristics of the flow in the phase space is critical.\textsuperscript{10,12,13} In the restricted three-body problem, the special solutions that include periodic and quasi-periodic orbits serve as examples of one of the fundamental models for the phase space, i.e., invariant manifolds. Accepting a manifold as an $m$-dimensional analog of a two-dimensional surface in $\mathbb{R}^n$, a manifold will be identified as an $m$-dimensional surface for the purposes of this discussion. So, an invariant manifold can be described as a surface ($m$-dimensional) such that orbits originating on the surface remain on the surface throughout the course of their dynamical evolution. Thus, an invariant manifold is a collection of orbits that form a surface. These bounded periodic and quasi-periodic motions exist as invariant manifolds in the center subspace of the phase space. Additionally, the set of orbits that approach or depart an invariant manifold asymptotically are also invariant manifolds (under certain conditions) and are denoted stable and unstable manifolds, respectively. In the context of
the three-body problem, the libration points, halo orbits, and the tori to which a Lissajous trajectory is confined are all invariant manifolds. The hyperbolic nature of these libration point orbits can be exploited by using the associated stable and unstable manifolds to generate transfer trajectories. For example, a stable manifold that originates in the vicinity of the Earth, crosses the lunar orbit — timed such that the Moon is near — and, then, asymptotically approaches a Lissajous trajectory along an expanding revolution, well represents the desired transfer path from Earth to libration point orbit.

If the system is modeled as a three-body problem (Sun, Earth, and spacecraft), approximations that represent the stable manifolds are computed as initial states in the vicinity of the Lissajous trajectory. Then, the state is propagated backwards in time toward the lunar orbit. The gravity of the Moon (a fourth body) will bend the trajectory back toward the Earth. If the system model includes four bodies (Sun, Earth, Moon, and spacecraft), then quasi-periodic solutions in the four-body problem can be calculated, as well as approximations to the associated stable manifolds. These surfaces incorporate the fourth body and the corresponding bending of the path. The former option is employed in this preliminary study. It is simpler and the four-body relationships remain in development. Thus, given a specified quasi-periodic Lissajous trajectory and an appropriate expanding (“opening”) revolution, the focus of the design effort is the isolation and computation of the stable manifolds that best represent candidate trajectory arcs for the Earth-to-Lissajous transfer. These arcs include a lunar encounter and are sufficiently accurate such that a differential corrections scheme can deliver an end-to-end solution that satisfies all the constraints.

ANALYSIS

For a given libration point mission, a number of different scenarios may necessitate a design approach that includes a small amplitude Lissajous orbit. Hardware or a communications link may constrain the distance from the x-axis, for example. Then, a simple, direct transfer to the vicinity of the libration point will imply a large Lissajous Orbit Insertion (LOI) maneuver; however, a lunar encounter can reduce the LOI significantly, possibly to zero, assuming that a small amplitude orbit is desired. Or, perhaps, because of some specified constraint conditions for a particular mission, the launch options result in a transfer path with insufficient energy and lunar gravity is required to increase the spacecraft energy level and reshape the arc to reach the vicinity of L1 or L2. Launch contingency options may include a lunar flyby as well. In any case, for the spacecraft to arrive in the vicinity of L1 or L2 with a state vector that is consistent with an asymptotic approach to an expanding revolution along a Lissajous trajectory, the vehicle requires a state corresponding to a stable manifold for the quasi-periodic orbit. Thus, the lunar pass must somehow coincide with such a stable manifold. So the search for a solution to the transfer problem is essentially an investigation of the lunar pass conditions that satisfy the requirements for a stable manifold.

Considerations of the Two-Body Problem and Conics

A quick review of a two-body problem (2BP) approximation offers some background concerning values of the turn angle, δ, and the equivalent Δv that can reasonably be obtained from the Moon. Both turn angle and Δv can then be related to the
magnitude of the arrival velocity relative to the Moon, that is, $v_\infty$. This facilitates the identification of a range of possible locations for the lunar encounter. Define the elongation angle $\Psi$ as the $L_2$-Earth-Moon angle and measure it clockwise in the ecliptic plane. A typical, currently utilized range for $\Psi$ is about $-30^\circ$ to $-50^\circ$. Thus, the angle is measured from the rotating Sun-Earth line and is positive counterclockwise. A corresponding angle can be measured for $L_1$ missions as an $L_1$-Earth-Moon angle. Other larger magnitude angles are possible but require more turn and, thus, a closer approach to the Moon. As mentioned, the stable invariant manifolds represent the flow from the vicinity of the Earth towards the particular libration point orbit that has been specified. Their use in the process to determine a transfer path allows multiple lunar positions, i.e., a range of viable dates, for the lunar flyby. The manifolds, however, generally do not meet the altitude constraint at the final perigee (prior to the lunar encounter) and/or at the TTI (Transfer Trajectory Insertion) point. Nevertheless, to meet any particular altitude constraint, these manifolds can be used as initial guesses in a differential corrections scheme. To isolate the regions of the phase space that will serve to generate acceptable transfer arcs, information concerning the conditions during the lunar pass is helpful.

As is well known, the effects of the Moon can be quantified if the close approach is modeled in terms of a two-body problem. In general, by modeling the flyby as a selenocentric hyperbolic path, the equivalent $\Delta v$ from the flyby is computed as

$$\Delta v = 2v_\infty \sin \left( \frac{\delta}{2} \right),$$

(1)

where $v_\infty$ is the velocity of the spacecraft at infinity relative to the encounter body of interest, and $\delta$ is the turn angle due to the close approach. Solving the equation for $\delta$ yields

$$\delta = 2 \sin^{-1} \left( \frac{\Delta v_{fb}}{2v_\infty} \right).$$

(2)

Suppose that $v_\infty$, and the close approach radial distance, $Q_p$, is known. Then,

$$\Delta v_{fb} = \frac{2v_\infty}{1 + \left( \frac{v_\infty^2 Q_p}{\mu} \right)}.$$  

(3)

With these relationships, the turn angle (and the $\Delta v_{fb}$) can be plotted as a function of $Q_p$ and $v_\infty$. (See Figure 2.) In this figure, note that $Q_p = 2,000$ km represents a lunar flyby altitude of just 262 km. As demonstrated later, trajectories representing the stable manifolds result in a value of $v_\infty$ approximately equal to 1 km/s for a Lissajous orbit with a size similar to that in Figure 1. Suppose there is a periselene altitude requirement of 1,000 km. Then, with a $v_\infty$ approximately equal to 1 km/s, the maximum turn angle that can be obtained from the lunar flyby is about $80^\circ$. ($\Delta v_{fb} \approx 1.28$ km/s). Given this information concerning trajectory states along a path representing a stable manifold, the turn angle that is required to meet “appropriate” $P_f$ altitudes (altitudes at the last perigee
prior to encounter) can easily be estimated. To quantify this value, an asymptote approximation is developed.

**Approximate Asymptote Analysis**

To determine an appropriate one-dimensional transfer path, it is necessary to isolate a particular trajectory arc along the stable manifold, i.e., an arc that is sufficiently bent. The direction corresponding to the vector from the Earth to the spacecraft at the lunar crossing can approximate the incoming asymptote direction; denote it $\hat{S}$. Then, the velocity direction at lunar orbit crossing in the direction of the outgoing asymptote is denoted $\hat{O}$. From a specified, pre-determined expanding revolution along the Lissajous path, the state can be propagated backwards in time along the stable manifold roughly targeting the Earth. The turn angle, denoted $\delta$, that is required to achieve this objective
Figure 3 Approximate Asymptotes

serves as an approximation of the manifold that meets the appropriate $P_f$ altitude requirement. (See Figure 3.) Furthermore, let the angle between $\hat{O}$ and the Sun-Earth line be denoted by $\xi$.

For manifolds sampled between $-32.1^\circ < \Psi < -91.1^\circ$, the following quantities are plotted in Figure 4: Earth radial distance, $R_p$, inclination (Earth Equatorial − Mean of J2000), $\Psi$, $\delta$, and $\xi$. The “case id” number is an arbitrary tag; due to the numerical method of approximating and globalizing manifolds, and different manifolds are tagged to points along the Lissajous revolution. In the x-y view, the case id numbers originate at the “top” of the orbit and proceed clockwise. It is probably not surprising to note that the approximate turn angle, $\delta$, is a nearly linear function of the elongation angle at lunar orbit crossing, since the distance from the Earth to $L_2$ is much greater than the distance to the Moon. Therefore, to maintain the turn angle between $30^\circ$ and $50^\circ$, the elongation angle must be in the range from $-30^\circ$ to $-50^\circ$. Two sample manifold trajectories are marked as green and blue circles in the figure. Now, notice the first “spike” in the $R_p$ distance in Figure 4. This indicates the presence of a lunar encounter.
Of course, the out-of-plane excursion of the Moon is critical and identifying the Moon position relative to the ecliptic is necessary. In Figure 5, the lunar out-of-plane position components appear for the months of December 2001 to May 2001. The plot includes values as the Moon moves through the fourth quadrant along its orbit.

**MANIFOLD OPTIONS**

Now consider the Lissajous orbit from Figure 1 with the following amplitudes: \( A_y = 230,000 \text{ km} \) and \( A_z = 145,000 \text{ km} \). This size meets sample
requirements that excursions are bounded within 0.5° and 10.0° off the Sun-Earth line. The flow toward the quasi-periodic orbit can be represented by the stable manifolds. These manifolds can be approximated by assuming that the orbit is periodic and that each state along the path is a fixed point.\textsuperscript{10,13} (Periodicity is one of the fundamental characteristics that is exploited when applying the theory of dynamical systems to the three- and four-body problems. Periodicity no longer exists, of course, in a model that incorporates ephemeris positions for the locations of the planets and moons. However, the baseline Lissajous is quasi-periodic and the approximations simply supply an initial guess to a numerical process that produces integrated solutions. Thus, the assumption that the trajectory is sufficiently close to periodic for successful application of the procedure is acceptable. Also, along a periodic orbit, and associated with each fixed point, the computation of the states that correspond to a stable manifold, requires the availability of the monodromy matrix. Because of the Earth orbit eccentricity, and the fact that the time interval along one revolution of a Lissajous trajectory is about six months, the assumption of periodicity is better over two revolutions of the quasi-periodic libration point orbit. Thus, the approximate monodromy matrix is evaluated over two revolutions.)

A state that is assumed to be on the stable manifold is now computed near each fixed point. To globalize the manifold and numerically determine the flow toward the orbit, the states are numerically integrated backwards in time toward the vicinity of the Earth and Moon. Since the periodic (quasi-periodic) orbit is continuous, this computation is accomplished for a large number of points along one revolution of the path. Then, for a particular Lissajous revolution, the manifold can be represented in configuration space as a two-dimensional surface formed by “meshing” the individual one-dimensional trajectories. Some of these curves, representing the flow toward fixed
points along one complete opening/expanding revolution of the Lissajous trajectory, appear — as projected into configuration space — in Figure 6. For the specified size of the Lissajous trajectory, the flow arrives from the vicinity of the lunar orbit. In fact, the set of all such trajectories lies on a surface that sweeps toward the Lissajous orbit; this surface can be visually described as a tube in configuration space. All paths remain on the tube and asymptotically approach the Lissajous orbit. The plots in Figure 6 show representative paths that lie on the tube surface. In the $x$-$y$ (ecliptic) view, the tube appears to cross the lunar orbit at various elongation angles within a range from $-32.1°$ to $-91.1°$. Therefore, for all of these manifolds, the lunar encounter occurs in the fourth quadrant. (Consistent with basic trigonometry, the quadrants are defined counterclockwise in the ecliptic plane with the center at the Earth.) In the $x$-$y$ (ecliptic) projection, the tube is bounded by the green and blue arcs. As the flow evolves toward the libration point, the manifold highlighted in green possesses a small loop just prior to the Lissajous orbit. (This observation reflects the winding of the one-dimensional manifold/trajectory that actually encircles the tube surface as it also flows downstream.) Denote this type of manifold, a type I. Now, note the manifold highlighted in blue. The small loop is not apparent. Denote these manifolds as type II. Note that for the type I manifold, the trajectory passes below the ecliptic in the $x$-$z$ projection. Suppose that

![Figure 6 Manifolds Associated with a Lissajous Orbit](image-url)
a Moon encounter is desired, but the Moon is above the ecliptic. Changing the type of the transfer would remedy this situation. Changing the type essentially implies a different manifold corresponding to a different phase as the trajectories wind around the tube.

As is apparent in Figure 6, the three-dimensional nature of the surfaces is critical. The trajectories representing the stable manifold associated with one revolution along a Lissajous trajectory are plotted as a surface in Figure 7. The three-dimensional tube surface is seen in green; the red curve represents the evolution of the lunar orbit over six months. Therefore, the intersections of the lunar orbit with the surface identify times and locations of potential lunar encounters with trajectories that will asymptotically flow into this libration point orbit without any insertion maneuver. Of course, an intersection of the lunar orbit with the tube in this figure does not necessarily imply that the point on the surface and the Moon co-exist at this position in space at the same time. However, it is apparent that a limited number of potential encounters occur to be investigated. The intersections also indicate that multiple options may exist and may represent a variety of departure altitudes and inclinations near the Earth. But, of course, the inclination and $R_p$ are not independent. Not surprisingly, the size, shape, and orientation of the tube at the lunar orbit is dependent on the size, shape, and orientation of the libration point orbit (or the particular revolution along the Lissajous trajectory) from which the stable manifolds are computed. This information should be useful in a numerical process to determine a transfer path. But, recall also that the manifolds (tube surface) actually exist in a higher dimensional space and Figure 7 is a projection onto configuration space. Nevertheless, the challenge is to ultimately determine the relationships between these parameters.

**ALGORITHM**

Given the libration point orbit specifications for a mission, assume that a candidate Lissajous trajectory is available and that a transfer path is sought. With a conceptual understanding of the issues associated with the search for a transfer arc, the steps in the process are converted to an algorithm for implementation. Computing manifolds that are bent sufficiently by the Moon is the key for obtaining approximations that are useful in further numerical processes. The path is sufficiently bent if acceptable Earth launch conditions are satisfied; some number of phasing loops may be specified. It is noted that most of the algorithms employed here are automated.

**Approximating Manifolds**

Consider the following approach. First, the manifold and the Moon must possess nearly the same elongation angle at the time of the encounter. Thus, the manifold trajectories that pass closest to the Moon must be isolated. Certain revolutions along the Lissajous trajectory are first computed. The numerically determined trajectory arcs that represent the stable manifolds are then calculated by integrating their respective initial conditions backwards in time. Many trajectories can be quickly generated for fixed points along a complete Lissajous revolution. To generate a specific set of initial conditions that correspond to the stable manifold ($W^s$) associated with one of these fixed points along the path, an algorithm has been employed that was developed to find both the stable and
Figure 7 From Lunar Orbit, Flow Towards the Lissajous Trajectory
and unstable manifolds of a second-order system. The algorithm, however, does not possess any inherent limit to the order of the system, and has been used successfully here. Near the fixed point $X^F$, the stable manifold is determined, to first order, by the six-dimensional stable eigenvector $Y^{W_S}$. Then, globalizing the manifold requires an initial six-dimensional state that is integrated backwards in time. It also requires an initial state that is on $W_S$ but not on the Lissajous path. To determine such an initial state, the position of the spacecraft is displaced from the orbit in the direction of $Y^{W_S}$ by some distance $d$ such that the new initial state, denoted $X_0^F$, is calculated as

$$X_0^F = X^F + d \ Y^{W_S}. \quad (4)$$

Higher order expressions for $X_0^F$ are available but not necessary. The magnitude of the scalar $d$ should be small enough to avoid violating the linear estimate, yet not so small that the time of flight becomes too large to due the asymptotic nature of the stable manifold. This investigation is conducted with a nominal value of 200 km for $d$. This value of the parameter also serves to tag certain manifolds as associated with certain fixed points. Using a specified value of $d$ for each fixed point, all of the one-dimensional curves taken together create a two-dimensional surface (or tube) arriving at the orbit. Note that changing the value of $d$ simply identifies another trajectory along the same surface.

**Criteria for Potential Solutions**

Of all the trajectory arcs, i.e., stable manifolds, that are generated, recall from Figure 6, that only a small percentage will intersect the lunar orbit and therefore, indicate a close encounter with the Moon. In fact, most will be hundreds of thousands of kilometers away from the Moon at lunar orbit crossing. For a direct transfer from an Earth parking orbit, it is desirable to avoid the Moon and the large number of arcs without an encounter suggests many potential options. For those other missions that plan to exploit lunar gravity, the limited number of intersections that do exist need to be isolated. At this point, it is only necessary to propagate the trajectories backward from the Lissajous orbit to the lunar orbit crossing and then determine potential candidates for subsequent analysis. The criterion used to numerically identify these promising cases requires two steps that are easy to implement. First, the difference in the elongation angle between the Moon and spacecraft at lunar orbit crossing must be relatively small, such as 0.5°. This implies that the ecliptic $x$ and $y$ components corresponding to the positions of the Moon and spacecraft are very close in value. Then, the distance between the Moon and the spacecraft in the ecliptic $z$-direction must also be relatively small. Those cases with a small difference in the $z$ components of the Moon and spacecraft (as represented by the state along the stable manifold) are therefore near the intersection points in Figure 6. These are the trajectories of interest, since they will be influenced the most by the Moon’s gravity. More importantly, their neighboring solutions will also pass close to the Moon.
The next phase of the analysis involves further investigation of the promising trajectories by varying their $d$ values from Equation (4). Recall that slight variations in the $d$ value shift the initial state to a different one-dimensional manifold, as is apparent in Figure 8, and this nearby arc possesses very similar characteristics. Adjusting this parameter allows much “finer” control over the modifications in the state at the lunar pass. By varying the $d$ value, the asymptotic nature of the manifold can be utilized to change the time of flight and, thus, effectively modify the lunar encounter conditions and, subsequently, the Earth arrival dates. For each value of $d$, the associated trajectory, representing a stable manifold, is integrated back to perigee. The $d$ value is differentially adjusted until the perigee altitude shifts above some specified level. A typical increment in the value of $d$ is 50 m. These subtle changes in the value of the $d$ parameter offers insight into the amount of “bend” supplied by the Moon.

**Phasing Loops**

At this point, phasing loops may be added simply by targeting the solution backward through the desired number of perigees. Of course, phasing loops offer advantages in terms of an extended phase for corrective action prior to the lunar encounter, as well as a wider launch window. However, as is well known, the lunar influence on the phasing loops cannot be dismissed and the periods of the loops may require adjustment to once again avoid or exploit the lunar gravity. In any case, it is almost certain that the end result will not meet the desired mission specifications at launch. However, the desired launch constraints can be specified and a numerical, two-level differential corrections scheme will adjust the manifold trajectory to meet mission requirements. The “control” parameters for the corrector includes the date that identifies the Lissajous trajectory. When the quasi-periodic solution is originally created, a date is specified as the start date for the Lissajous path, corresponding to a given size, that is, the amplitudes $A_y$ and $A_z$, as well as a set of phase angles. The date change generates a different Lissajous trajectory, but one with the same size and initial phase. However, because it is a different orbit, the trajectories that represent the stable manifolds are also slightly modified. Considering one manifold trajectory at a time, the timing is then varied.

![Figure 8 Various Manifolds Asymptotically Approaching the Orbit](image)
until the spacecraft reaches periselene, within a prescribed angle of the Moon. This angle can change dramatically with a small change in date, so the \( d \) value is also varied to better refine this angle.

**PRELIMINARY MISSION DESIGN**

A small study has been completed to investigate the lunar options available in 2001 and to compare the results, based on Lissajous trajectories that are two different sizes. For each size, both classes of the Lissajous orbit are included. As it appears in Figure 9, one quasi-periodic solution is characterized by a large out-of-plane amplitude but smaller in-plane excursions, such that \( A_y \) and \( A_z \) amplitudes equal 110,000 km and 230,000 km, respectively. The second Lissajous represents the opposite case, that is, with \( A_y \) and \( A_z \) amplitudes of 230,000 km and 110,000 km, respectively. Both Class I trajectories appear in Figure 9. For the preliminary analysis, the first two opening revolutions of the Lissajous are investigated. Each revolution is discretized into

![Figure 9a Lissajous Trajectory: \( A_y > A_z \)](image_url)
a prescribed set of points that are then defined as the fixed points for the computation of the trajectories that represent the stable manifolds. The points are tagged for identification such that case #1 is defined at the point that is the farthest point from the Earth on the Sun-Earth line. The points are numbered subsequently by proceeding clockwise when viewed in the ecliptic plane and tagging a point every 2° along the path, relative to the libration point, that is, 2° between each tag. Thus, the first revolution consists of points denoted as cases 1-181 and along the second revolution, the points are labeled 181-361. There are 13 time periods in 2001 when the Moon is in the proper location. The following results are for lunar encounters from 6/27/2001 to 7/02/2001. (See Figure 10.) These figures represent information corresponding to the spacecraft at periselene. In the figure, the start dates are defined in terms of Julian Date. The Julian Date 2452088 corresponds to June 27, 2001 and 2452093 indicates the calendar date July 2, 2001. In Figure 9, different case id numbers denote the computation of the stable manifold at different fixed points over one revolution. From the figure, a number of cases

Figure 9b Lissajous Trajectory: $A_y < A_z$
can be identified for subsequent investigation. Recall that the first criterion is an elongation angle such that the Moon and spacecraft are within 0.5°. Then, the out-of-plane $z$ component is evaluated and, in the lower left plot in Figure 10, the intersections identify trajectories for which the Moon and spacecraft are less than 2,000 km apart. These trajectories are further analyzed.

To make the analysis more applicable, three phasing loops are added and the trajectories are integrated back to TTI, that is, the departure perigee. The goal is the determination of a manifold/trajectory with an acceptable final perigee altitude. It is also necessary to verify that all of the periapsis altitudes are acceptable. For the four quasi-periodic trajectories that are studied here, seven promising manifolds/trajectories are identified. Some of these trajectories actually pass below the Earth’s surface at the final perigee. For each of these cases, proper adjustment of the $d$ value can yield any desired altitude; such a modification is usually possible. The desired result is a trajectory with a final perigee altitude of 185.0 km and an inclination of 28.5°. The seven cases identified through the previous process are summarized in Table 1 where the altitude and inclination are evaluated at the final perigee.
**TABLE 1**

**Natural Solutions**

<table>
<thead>
<tr>
<th>Lissajous Trajectory</th>
<th>Case #</th>
<th>$d$ Value (km)</th>
<th>Periselene Elongation (deg)</th>
<th>Altitude (km)</th>
<th>Inclination (deg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Orbit S, Class I</td>
<td>47</td>
<td>201.4764</td>
<td>-62.04</td>
<td>186.90</td>
<td>33.47</td>
</tr>
<tr>
<td>Orbit S, Class I</td>
<td>141</td>
<td>201.0680</td>
<td>-36.00</td>
<td>196.20</td>
<td>21.12</td>
</tr>
<tr>
<td>Orbit S, Class II</td>
<td>134</td>
<td>201.2000</td>
<td>-47.40</td>
<td>699.87</td>
<td>44.35</td>
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<tr>
<td>Orbit T, Class I</td>
<td>52</td>
<td>199.1130</td>
<td>-64.04</td>
<td>183.62</td>
<td>54.91</td>
</tr>
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<td>Orbit T, Class I</td>
<td>138</td>
<td>201.8585</td>
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</tr>
<tr>
<td>Orbit T, Class II</td>
<td>64</td>
<td>198.7386</td>
<td>-72.12</td>
<td>184.17</td>
<td>10.07</td>
</tr>
<tr>
<td>Orbit T, Class II</td>
<td>134</td>
<td>201.0986</td>
<td>-50.27</td>
<td>183.53</td>
<td>18.35</td>
</tr>
</tbody>
</table>

**TABLE 2**

**Corrected Transfer Trajectories**

<table>
<thead>
<tr>
<th>Case Name</th>
<th>TTI Date</th>
<th>TTI Cost (m/s)</th>
<th>Lunar Encounter Date</th>
<th>Lunar Altitude (km)</th>
<th>LOI Cost (m/s)</th>
<th>Total Cost (m/s)</th>
</tr>
</thead>
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<tr>
<td>Orbit S, I, 47</td>
<td>5/20/01</td>
<td>3139.90</td>
<td>6/26/01</td>
<td>17084.52</td>
<td>0.78</td>
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<td>Orbit S, I, 141</td>
<td>5/22/01</td>
<td>3140.39</td>
<td>7/02/01</td>
<td>11999.28</td>
<td>5.04</td>
<td>3145.43</td>
</tr>
<tr>
<td>Orbit S, II, 134</td>
<td>5/22/01</td>
<td>3128.06</td>
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Obviously, the constraints on both altitude and inclination are not met. However, a differential corrections process is employed to modify the trajectory to meet the desired constraints at TTI. This is automatically accomplished for all seven of the manifold trajectories. The results are summarized in Table 2. The “Case Name” is just an abbreviation for the “Lissajous Case” and “Case Number” from the previous table. Transfer trajectory insertion is assumed to occur from a circular parking orbit. The LOI cost is the magnitude of the Lissajous orbit insertion maneuver. Two of the seven trajectories appears in Figure 11 and 12.

Note that these results are not optimized. Potentially, the cost can be furthered reduced. Each of these cases requires a LOI maneuver because the original manifold did not meet all of the desired constraints. Neighboring manifolds may actually possess altitude and inclination combinations that are closer to the desired values, and possibly a lower total cost after correction. Ideally, a trajectory exists that naturally meets the desired TTI constraints and then incurs a TTI maneuver but no LOI.
Observations

Upon completion of this preliminary investigation, a number of observations may prove useful. An increase in the $A_y$ amplitude appears to increase the range of elongation angles. This would be primarily useful when considering a case that does not naturally possess a sufficiently high elongation angle. Increasing the $A_y$ amplitude could be used to essentially widen the tube surface at the lunar orbit to ensure that lunar encounters at more desirable elongation angles can be determined. A correlation is also apparent between the $A_z$ amplitude of the Lissajous trajectory and the ecliptic $z$ excursion of the spacecraft trajectory (i.e., the tube surface) at the lunar orbit crossing. The out-of-plane component at the lunar orbit crossing is bounded and approximately equals half the distance corresponding to the $A_z$ amplitude of the quasi-periodic orbit. Thus, the first transfer passes by lunar orbit with a $z$ component of magnitude approximately half the value of the $A_z$ amplitude. The manifolds associated with Class I Lissajous trajectories passes above the ecliptic plane, while the stable manifolds from Class II libration point
solutions pass under the ecliptic, that is, with a negative $z$-component. Considering this fact, as well as the variations in the manifolds along a particular revolution of the same orbit, as long as the Lissajous trajectory $A_z$ amplitude is sufficiently large, every trajectory arc computed as representing a stable manifold will include intersections with the lunar orbit. The out-of-plane component of the lunar orbit is roughly bounded by ±40000 km during this time period. For this reason, smaller $A_z$ amplitudes will likely produce more opportunities for lunar assistance. This observation does not imply that these cases will be better but that more manifolds/trajectories will pass close to the Moon. Also note that, for the larger $A_z$ amplitudes, the intersections of the Moon’s orbit and manifolds (surface) are nearly perpendicular. Thus, only a very narrow band of trajectories actually pass close enough to the Moon to be useful.

One large factor in determining whether the Moon can provide a “bend” to accomplish the objective is the elongation angle of the encounter. The elongation angles for the cases examined in the two tables range from almost $-30^\circ$ to below $-90^\circ$. Elongation angles above $-70^\circ$ typically offer the most successful passes. If the Moon is closer to $-90^\circ$, the characteristics of the manifold trajectories frequently cannot be
influenced enough by the lunar gravity to reach the required Earth perigee altitudes directly after the encounter. One notable exception involves the addition of phasing loops. When there are phasing loops, the lunar gravity can reduce the perigees over a number of revolutions.

CONCLUDING REMARKS

This work supports trajectory design for libration point missions. The focus here is an improved understanding of the solution space to allow the determination of transfer trajectories when a lunar encounter is exploited to allow insertion into a small amplitude Lissajous orbit for reasonable cost — perhaps zero. Such transfers have been successfully computed for a number of missions. However, for the more complex requirements of potential future missions, the computational process requires improvement. So, a primary goal in this ongoing effort is the development of a more automated procedure. The results reported in this paper are all computed as part of an updated algorithm that uses insights from the dynamics to isolate solutions to the problem. Analysis and the development of computational tools are continuing.

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