The Journal of the Astronautical Sciences, Vol. 43, No. 3, July-September 1995, pp. 243–262

A New Parameterization of the Attitude Kinematics

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Abstract

We present a new method for describing the kinematics of the rotational motion of a rigid body. The new kinematic formulation provides a three-dimensional parameterization of the rotation group using two perpendicular rotations; thus it complements the Eulerian angles (three rotations) and Euler-Rodrigues parameters (one rotation). The differential equations can be described by two scalar equations. We show the connection of the new parameterization with the other classical parameterizations. The new kinematic formulation has potential applications in astrodynamics, attitude control, robotics and other fields.

Introduction

In recent years a considerable amount of effort has been devoted to the development of a comprehensive theory that will allow a better understanding of the complex dynamic behavior associated with the motion of rotating rigid bodies. A cornerstone in this effort is the development of alternative ways of describing the kinematics of this motion. As far as the *dynamics* of the rotational motion is concerned (i.e., the effect of external torques on the angular momentum or, equivalently, the angular velocity behavior), Euler's equations of motion provide a complete and well-defined framework. For the *kinematics* however, one has a certain degree of freedom, due to the fact that the rotation matrix (viz. the direction cosine matrix), which determines the relative orientation between two reference frames, can be parameterized in more than one way. By having available several different approaches for viewing the kinematics, more insight can be gained into a specific problem; in general, the best approach is clearly problem dependent.

The most commonly used parameterizations for the attitude kinematics are the Eulerian angles, the Euler-Rodrigues parameters (or quaternion formulation), the Cayley-Klein parameters, and the Cayley-Rodrigues parameters [1–5]. We will not elaborate on the advantages and disadvantages of these alternative descriptions for the rotational kinematics, since they vary with the particular application at

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hand, but we refer the interested reader to the excellent recent survey paper by Shuster [2]. For an earlier, albeit less thorough, treatment one may also peruse Stuelpnagel [3].

In this paper we propose a new approach for describing the kinematics of a rotating rigid body which appears to have certain advantages in the description of the attitude motion. In particular, we believe that the proposed kinematic formulation will be beneficial for attitude determination and control problems. This new formulation describes the relative orientation of two reference frames using two perpendicular rotations, thus complementing the Eulerian angle (three rotations) and the Euler-Rodrigues parameter (one rotation) descriptions. Although it uses three parameters to describe the motion, two of the parameters can be combined into a single complex variable, thus reducing the number of the differential equations required for the kinematics to *two*. (One, of course, still needs three real differential equations to describe the motion. The fact that two of these real equations can be effectively combined into a complex differential equation provides a convenient simplification of the equations.)

The complex coordinate is used to designate one of the two rotations and it is derived using stereographic projection; it describes the location of the "designated body-axis" (usually taken as the body 3-axis or the spin-axis for spinning vehicles), in the inertial frame. The real coordinate describes (loosely speaking) the rotation or the relative orientation about this axis. More precisely, it describes an *antecedent*—or initial—rotation about this axis in a way such that, along with the complex stereographic coordinate, provides a complete description of the attitude and forms a new coordinate set on SO(3). (SO(3) stands for the special orthogonal group of 3×3 rotation matrices.) The physical significance of these coordinates will become clear in the sequel. The use of an initial rotation has the advantage of making it an ignorable coordinate in the resulting differential equations. The fact that one of the coordinates in the proposed kinematic description is ignorable has important implications and advantages, and it has been very useful in deriving analytic solutions and control laws for spinning rigid bodies [6,7]. Roughly speaking, since the coordinate angle describing the initial rotation does not enter on the right hand side of the differential equations—and for cases when the acting torques do not depend on this angle—one is able to effectively decouple the kinematic equations. Such a decoupling is often a desirable feature, since for several problems it reduces the number of differential equations even further.

The structure of the paper is as follows. First we start with some preliminaries necessary to formulate the problem. We briefly state the desired requirements imposed on the proposed parameterization and we introduce the basic ideas on how to address these requirements. Next we give the main derivations and the essential results of the paper. The most interesting properties of the parameterization are then discussed. It is well-known that three is the minimum number of coordinates required to parameterize the rotation group, although every such three-dimensional parameterization necessarily involves a singularity [3]. We discuss the singularity associated with the proposed kinematic description.

The parameterization presented in this paper appears to be a new result in the vast literature of attitude representations [2], at least as far as the authors know. It is reminiscent of the Axis-Azimuth representation [2] in the sense that in this formulation one also describes the orientation of the body by the location of a selected body axis (or spin-axis) and the relative rotation about this axis. However, in this description the location of the body axis is given in terms of two angles (the polar coordinates of $\mathbb{R}^2$), instead of a complex variable. The introduction of the complex variable simplifies the equations significantly—thus avoiding the use of trigonometric functions—and at the same time it reduces the number of scalar differential equations required for the kinematics. More important, the azimuthal angle describing the relative rotation about the selected body-axis is taken about the "body spin-axis" instead of the "inertial spin-axis," i.e., it is a final rotation about the spin-axis instead of an initial rotation.

Recently we became aware of the Listing parameterization [8, 9] which is similar in spirit to the parameterization proposed in this paper. Walsh et al. [9] use this parameterization and they also introduce stereographic coordinates for the description of the orientation axis in order to solve the problem of spacecraft stabilization with two control torques. We will elaborate more on the connection of the new parameterization with the one used in Walsh et al., as well as the other traditional kinematic parameterizations.

Preliminaries

Let us consider the relative orientation between two reference frames, say, $(\hat{\mathbf{i}}_1, \hat{\mathbf{i}}_2, \hat{\mathbf{i}}_3)$ and $(\hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2, \hat{\mathbf{b}}_3)$. (Using this notation $\hat{\mathbf{i}}_i$ and $\hat{\mathbf{b}}_i$ denote the unit vectors along the respective coordinate axes for each of the two frames). Without loss of generality we will henceforth refer to the $\hat{\mathbf{i}}$ reference frame as the inertial reference frame and the $\hat{\mathbf{b}}$ as the body reference frame. The relative orientation between the two frames is completely determined by the *rotation matrix* $R \in SO(3)$. The elements of the rotation matrix R are just the direction cosines of the axes between the two reference frames. In other words, R is defined by the equation

$$\begin{bmatrix} \hat{\mathbf{b}}_1 \\ \hat{\mathbf{b}}_2 \\ \hat{\mathbf{b}}_3 \end{bmatrix} = R \begin{bmatrix} \hat{\mathbf{i}}_1 \\ \hat{\mathbf{i}}_2 \\ \hat{\mathbf{i}}_3 \end{bmatrix}. \tag{1}$$

If we consider a vector $\mathbf{n}$ having coordinates (n_1^b, n_2^b, n_3^b) and (n_1^i, n_2^i, n_3^i) with respect to the reference frames $\hat{\mathbf{b}}$ and $\hat{\mathbf{i}}$, respectively, we can write

$$\mathbf{n} = n_1^b \hat{\mathbf{b}}_1 + n_2^b \hat{\mathbf{b}}_2 + n_3^b \hat{\mathbf{b}}_3 = n_1^i \hat{\mathbf{i}}_1 + n_2^i \hat{\mathbf{i}}_2 + n_3^i \hat{\mathbf{i}}_3 \tag{2}$$

Using equation (1) it is easy then to establish the following relation between the *coordinates* of $\mathbf{n}$ in the two reference frames

$$\begin{bmatrix} n_1^b \\ n_2^b \\ n_3^b \end{bmatrix} = R \begin{bmatrix} n_1^i \\ n_2^i \\ n_3^i \end{bmatrix}. \tag{3}$$

If ${}^i\boldsymbol{\omega}^b := \omega_1 \hat{\mathbf{b}}_1 + \omega_2 \hat{\mathbf{b}}_2 + \omega_3 \hat{\mathbf{b}}_3$ denotes the angular velocity of $\hat{\mathbf{b}}$ with respect to the $\hat{\mathbf{i}}$ frame, (expressed in the $\hat{\mathbf{b}}$ frame), then the differential equation satisfied by $R(t)$, $t \geq 0$ is given by [1]

$$\dot{R}(t) = S({}^i\boldsymbol{\omega}^b(t))R(t), \qquad R(0) = R_0 \tag{4}$$

where R_0 denotes the initial orientation at $t = 0$ and where $S({}^i\boldsymbol{\omega}^b)$ is the skew-symmetric matrix

$$S({}^i\boldsymbol{\omega}^b) := \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix}. \tag{5}$$

The proof of equation (4) is fairly straightforward and can be found in several references [1, 2, 4]. An easy derivation of (4) can be obtained starting from the differential equations of the basis vectors $\hat{\mathbf{b}}_i$ in the inertial frame

$$\dot{\hat{\mathbf{b}}}_i = {}^i\boldsymbol{\omega}^b \times \hat{\mathbf{b}}_i, \qquad i = 1, 2, 3 \tag{6}$$

Our objective in this paper is to derive a parameterization of the matrix R in equation (1) using three parameters. Recall that owing to the constraint $RR^{\mathrm{T}} = I$, three is the minimum number of parameters required to describe R. (The superscript T here denotes the transpose.) In addition to the requirement of a three-dimensional parameterization, we also seek the following desirable properties:

i) The coordinates of the parameterization should have a physical description so that they can be easily visualized and conceptually interpreted.
ii) The parameterization should be physically realizable using *two* successive rotations from the initial to the final position. Thus, the proposed kinematic description would complement the classical Eulerian angle (three rotations) and the Euler parameter (one rotation) descriptions.
iii) The proposed parameterization should obey as simple and compact a set of differential equations as possible.
iv) It is desirable that the motion (e.g. the differential equations) be easily decomposed into its two rotations. In other words, one of the rotations should be decoupled from the other rotation, so that in problems not involving the former one should be able to work with a reduced set of kinematic equations.
v) The inherent singularity introduced by the parameterization should be as nonrestrictive as possible, in the sense that it should allow for as large a set of physical orientation configurations (namely, points on SO(3)) as possible.

In order to constructively derive the proposed parameterization we will use two basic ideas. The first idea uses the fact that an initial rotation in a set of successive rotations always introduces a coordinate which does not enter on the right hand side of the kinematic equations, i.e., an "ignorable" coordinate. (We use the term "ignorable" here rather loosely, as the precise definition in the theory of mechanics is somewhat different [5, 10]; for the purposes of illustrating the ideas in this paper however we choose to introduce and follow this terminology.) As a result, if one is not directly interested in the motion described by this ignorable coordinate, one may safely discard its differential equation, thus effectively reducing the system of kinematic equations. Such a reduction can be very helpful in a series of problems [6, 7, 11]. Note that this property is shared by the Eulerian angles as well, where the angle describing the first rotation is also always ignorable [1, 4].

The second idea is borrowed from the field of complex analysis. It uses the fact that a unit vector (i.e., a point on the unit sphere) can be completely characterized by a scalar (although complex) number, via the use of *stereographic projection.* The introduction of the stereographic coordinate simplifies the equations and effectively "reduces" the number of *scalar* equations needed for the kinematics. Although the application of stereographic coordinates to describe the orientation of a unit vector in $\mathbb{R}^3$ is not entirely new, the full potential of this observation has not been completely realized.

The New Parameterization

As mentioned in the previous sections, we wish to derive a parameterization of the matrix R in equation (1) using two successive rotations. This implies, in particular, that the matrix R should be decomposable as

$$R = R_2(w)R_1(z) \tag{7}$$

where, in anticipation of future results, we denote the parameters characterizing these two rotations as z and w. Clearly, it is assumed that $R_1, R_2 \in SO(3)$, that is, R_1 and R_2 are valid rotation matrices. Notice from the structure of equation (7) that this decomposition presumes that the kinematic variable z is the ignorable variable for the parameterization. This is obvious from equation (1) and the fact that $R_1(z)$ multiplies $R_2(w)$ on the right, so it represents an *initial* rotation from the inertial frame.

We now assume that $(\hat{\mathbf{i}}'_1, \hat{\mathbf{i}}'_2, \hat{\mathbf{i}}'_3)$ is the reference frame resulting from the rotation $R_1(z)$, that is,

$$\begin{bmatrix} \hat{\mathbf{i}}'_1 \\ \hat{\mathbf{i}}'_2 \\ \hat{\mathbf{i}}'_3 \end{bmatrix} = R_1(z) \begin{bmatrix} \hat{\mathbf{i}}_1 \\ \hat{\mathbf{i}}_2 \\ \hat{\mathbf{i}}_3 \end{bmatrix} \tag{8}$$

Although any rotation is equally valid for a decomposition such as (7), we will assume in the sequel that z represents a rotation about one of the body axes (which are initially coincident with the inertial axes); in particular, we will assume that it represents a (positive) rotation about the body z-axis, viz. the 3-axis (hence the obvious notation). Therefore, $R_1(z)$ is given by

$$R_1(z) := \begin{bmatrix} \cos z & \sin z & 0 \\ -\sin z & \cos z & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{9}$$

Since the coordinate z represents an initial rotation about one of the body axes (here taken to be the z-axis) it should be intuitively clear that the second rotation matrix $R_2(w)$ should somehow give information about the orientation of this axis in $\mathbb{R}^3$. It turns out that this is indeed the case, and moreover that this axis (more precisely the location of the unit vector designating this axis in $\mathbb{R}^3$) can be efficiently and elegantly described using stereographic coordinates.

In order to motivate the introduction of the stereographic coordinates, let us consider the two reference frames associated with the unit vectors $(\hat{\mathbf{i}}'_1, \hat{\mathbf{i}}'_2, \hat{\mathbf{i}}'_3)$ and $(\hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2, \hat{\mathbf{b}}_3)$. Based on equations (7) and (8) we have

$$\begin{bmatrix} \hat{\mathbf{b}}_1 \\ \hat{\mathbf{b}}_2 \\ \hat{\mathbf{b}}_3 \end{bmatrix} = R_2(w) \begin{bmatrix} \hat{\mathbf{i}}'_1 \\ \hat{\mathbf{i}}'_2 \\ \hat{\mathbf{i}}'_3 \end{bmatrix}. \tag{10}$$

Let us assume that the axes $(\hat{\mathbf{i}}'_1, \hat{\mathbf{i}}'_2, \hat{\mathbf{i}}'_3)$ and $(\hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2, \hat{\mathbf{b}}_3)$ are initially coincident and, in addition, that the $\hat{\mathbf{b}}_3$-axis is rotated from its original position (which is along $\hat{\mathbf{i}}'_3$) to its final position. We are interested in characterizing this rotation, i.e., its magnitude and the axis of rotation. In pursuing this task, let the location of the unit vector $\hat{\mathbf{i}}'_3$ in the $\hat{\mathbf{b}}$ reference frame be described by the direction cosines (a, b, c), i.e., let

$$\hat{\mathbf{i}}'_3 = a\hat{\mathbf{b}}_1 + b\hat{\mathbf{b}}_2 + c\hat{\mathbf{b}}_3 \tag{11}$$

(Notice from equation (10) that this implies that $[a, b, c]^{\mathrm{T}}$ is the third *column* of the matrix $R_2(w)$.) Clearly, the angle between $\hat{\mathbf{i}}'_3$ and $\hat{\mathbf{b}}_3$ is

$$\arccos(\hat{\mathbf{i}}'_3 \cdot \hat{\mathbf{b}}_3) = \arccos c \tag{12}$$

where $0 \leq \arccos c \leq \pi$. The direction of rotation is about a vector $\hat{\mathbf{u}}$ (positive by right hand rule) perpendicular to the plane defined by the unit vectors $\hat{\mathbf{i}}'_3$ and $\hat{\mathbf{b}}_3$ (see Fig. 1).

The unit vector $\hat{\mathbf{u}}$ can be computed by

$$\hat{\mathbf{u}} = \frac{\hat{\mathbf{i}}'_3 \times \hat{\mathbf{b}}_3}{\|\hat{\mathbf{i}}'_3 \times \hat{\mathbf{b}}_3\|}. \tag{13}$$

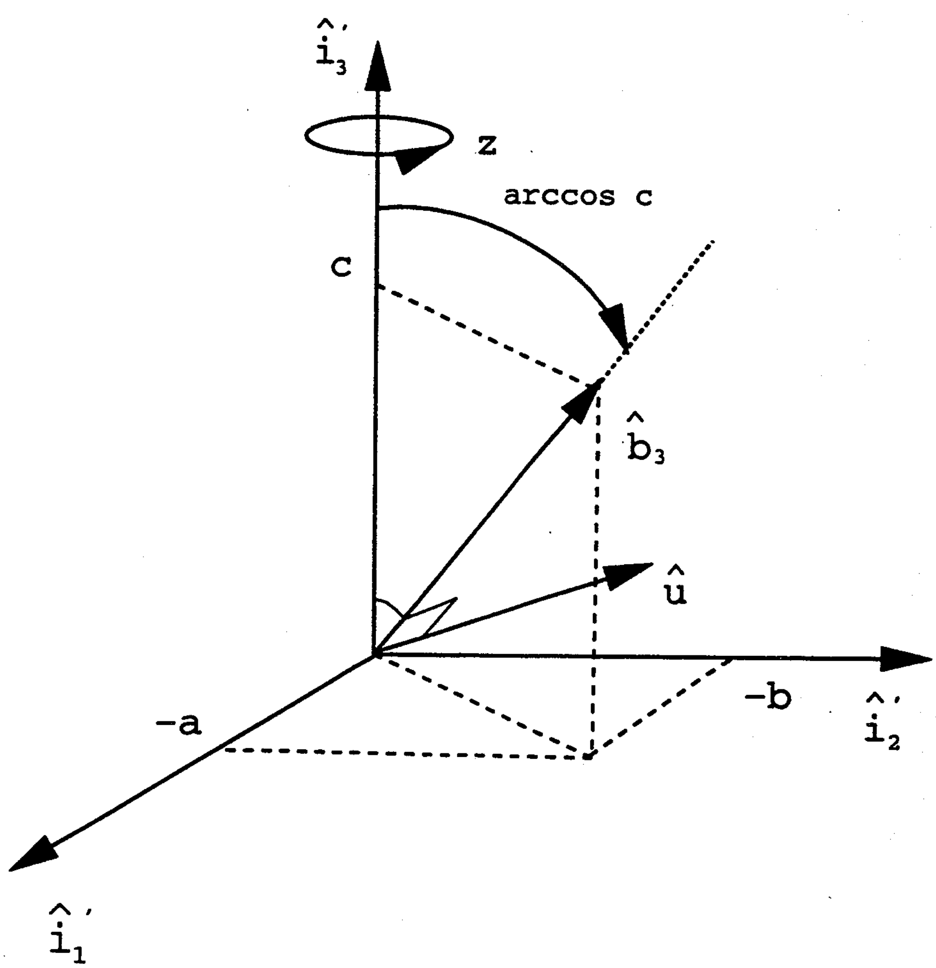

FIG. 1. Orientation Description in (w, z) Coordinates.

It is an easy exercise to show now that $(-a, -b, c)$ are the direction cosines of the $\hat{\mathbf{b}}_3$ in the $\hat{\mathbf{i}}'$ frame, i.e.,

$$\hat{\mathbf{b}}_3 = -a\hat{\mathbf{i}}'_1 - b\hat{\mathbf{i}}'_2 + c\hat{\mathbf{i}}'_3. \tag{14}$$

Using equation (13) we therefore have for the direction of rotation that

$$\hat{\mathbf{u}} = \frac{1}{\sqrt{a^2 + b^2}}(b\hat{\mathbf{i}}'_1 - a\hat{\mathbf{i}}'_2) \tag{15}$$

Using Euler's formula [2, 12] one can easily compute the rotation matrix which corresponds to a rotation by an angle arccos c about the unit vector $\hat{\mathbf{u}}$:

$$R_2(w) = I + \sin(\arccos c)S(\hat{\mathbf{u}}) + [1 - \cos(\arccos c)]S^2(\hat{\mathbf{u}}) \tag{16}$$

where S is defined by equation (5). Carrying out the necessary algebra, and noticing that for $0 \leq \arccos c \leq \pi$ we have $\sin(\arccos c) \geq 0$ and hence $\sin(\arccos c) = \sqrt{1 - c^2}$, we finally obtain that

$$R_2(w) = \begin{bmatrix} c + \dfrac{b^2}{1+c} & -\dfrac{ab}{1+c} & a \\ -\dfrac{ab}{1+c} & c + \dfrac{a^2}{1+c} & b \\ -a & -b & c \end{bmatrix} \tag{17}$$

The final form of the rotation matrix R relating the inertial and the body frames is given by performing the matrix multiplication in (7)

$$R(w,z) = \begin{bmatrix} \dfrac{c\cos z + ab\sin z + (b^2 + c^2)\cos z}{1+c} & \dfrac{c\sin z - ab\cos z + (b^2 + c^2)\sin z}{1+c} & a \\ -\dfrac{c\sin z + (c^2 + a^2)\sin z + ab\cos z}{1+c} & \dfrac{c\cos z + (c^2 + a^2)\cos z - ab\sin z}{1+c} & b \\ b\sin z - a\cos z & -b\cos z - a\sin z & c \end{bmatrix} \tag{18}$$

Note that both $R_1(z)$ and $R_2(w)$ are elements of SO(3) as required. The matrix $R_1(z) \in \mathrm{SO}(3)$ by its definition (9), and $R_2(w) \in \mathrm{SO}(3)$ since equation (16) always produces elements in SO(3). Thus, equation (18) is well-defined and represents a valid rotation matrix.

Although the previous matrix incorporates all the necessary information for the kinematic description, it is still redundant, in the sense that the elements a, b, c are not independent, but satisfy the constraint

$$a^2 + b^2 + c^2 = 1 \tag{19}$$

(Recall that a, b, c are components of a unit vector.) We can therefore eliminate one more parameter from the parameterization (18). There are many ways to achieve this reduction, but perhaps the most natural and elegant way is through the use of stereographic projection [13]. To use this method, notice that because of the constraint (19) a, b, and c represent coordinates on the unit sphere $S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$ in $\mathbb{R}^3$.

For $(a, b, c) \in S^2$, the stereographic projection $\sigma : S^2 \backslash \{(0, 0, -1)\} \to \mathbb{C}$ defined by (see Fig. 2)

$$w := \sigma(a, b, c) = \frac{b - ia}{1 + c} \tag{20}$$

introduces a complex variable w which includes the necessary information about the location of the $\hat{\mathbf{i}}_3'$ axis in the $\hat{\mathbf{b}}$ frame, or equivalently, the location of the $\hat{\mathbf{b}}_3$ axis in the $\hat{\mathbf{i}}'$ frame. (Recall the matrix (17) as well as the equations (11) and (14).)

The stereographic projection establishes a one-to-one correspondence between the unit sphere S^2 and the extended complex plane $\mathbb{C}_\infty \triangleq \mathbb{C} \cup \{\infty\}$. It can be easily verified then that the inverse map $\sigma^{-1} : \mathbb{C} \to S^2 \backslash \{(0, 0, -1)\}, w \mapsto (a, b, c)$ is given by

$$a = \frac{i(w - \overline{w})}{1 + |w|^2}, \qquad b = \frac{w + \overline{w}}{1 + |w|^2}, \qquad c = \frac{1 - |w|^2}{1 + |w|^2} \tag{21}$$

and can be used to find a, b and c once w is known. Here $|\cdot|$ denotes the absolute value of a complex number, i.e., $w\overline{w} = |w|^2$, $w \in \mathbb{C}$.

Finally, using (21) we can express $R_2(w)$ in terms of $w := w_1 + iw_2 \in \mathbb{C}$ alone as follows

$$R_2(w) = \frac{1}{1 + w_1^2 + w_2^2} \begin{bmatrix} 1 + w_1^2 - w_2^2 & 2w_1 w_2 & -2w_2 \\ 2w_1 w_2 & 1 - w_1^2 + w_2^2 & 2w_1 \\ 2w_2 & -2w_1 & 1 - w_1^2 - w_2^2 \end{bmatrix} \tag{22}$$

or more compactly,

$$R_2(w) = \frac{1}{1 + |w|^2} \begin{bmatrix} 1 + \mathrm{Re}(w^2) & \mathrm{Im}(w^2) & -2\,\mathrm{Im}(w) \\ \mathrm{Im}(w^2) & 1 - \mathrm{Re}(w^2) & 2\,\mathrm{Re}(w) \\ 2\,\mathrm{Im}(w) & -2\,\mathrm{Re}(w) & 1 - |w|^2 \end{bmatrix} \tag{23}$$

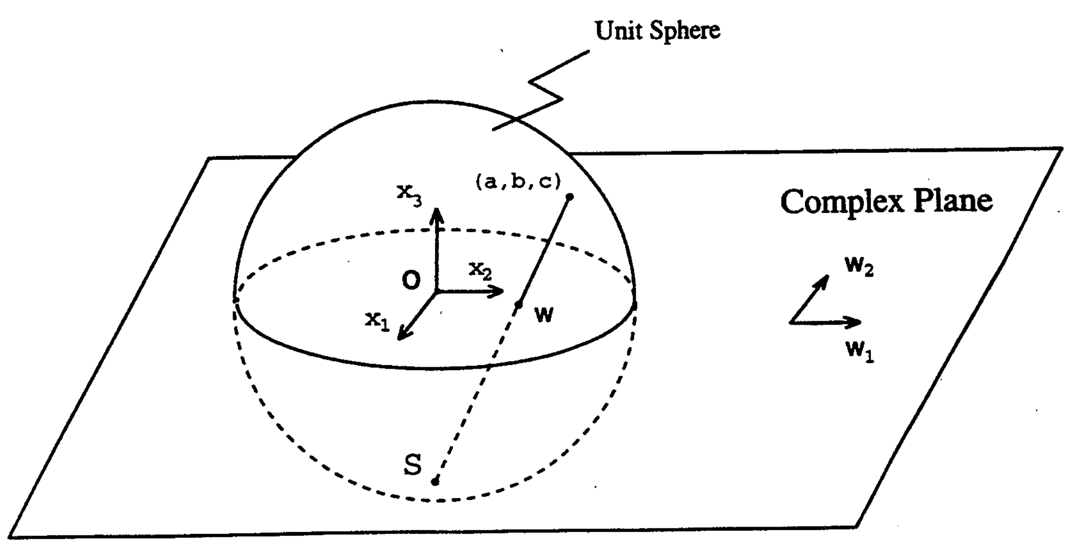

FIG. 2. Stereographic Projection.

where Re(·) and Im(·) denote the real and imaginary parts of a complex number. Thus, $R(w,z)$ is given in terms of w and z as follows

$$\frac{1}{1+w_1^2+w_2^2}\cdot\begin{bmatrix}(1+w_1^2-w_2^2)cz-2w_1w_2sz & (1+w_1^2-w_2^2)sz+2w_1w_2cz & -2w_2\\ 2w_1w_2cz-(1-w_1^2+w_2^2)sz & 2w_1w_2sz+(1-w_1^2+w_2^2)cz & 2w_1\\ 2w_2cz+2w_1sz & 2w_2sz-2w_1cz & 1-w_1^2-w_2^2\end{bmatrix} \tag{24}$$

where cz and sz denote $\cos z$ and $\sin z$, respectively. $R(w,z)$ can be written more compactly as

$$R(w,z)=\frac{1}{1+|w|^2}\begin{bmatrix}\mathrm{Re}[(1+w^2)e^{iz}] & \mathrm{Im}[(1+w^2)e^{iz}] & -2\,\mathrm{Im}(w)\\ \mathrm{Im}[(1-\overline{w}^2)e^{-iz}] & \mathrm{Re}[(1-\overline{w}^2)e^{-iz}] & 2\,\mathrm{Re}(w)\\ 2\,\mathrm{Im}(we^{iz}) & -2\,\mathrm{Re}(we^{iz}) & 1-|w|^2\end{bmatrix} \tag{25}$$

Therefore, the orientation of the body with respect to the inertial frame can be described by an initial rotation about the 3-axis of magnitude z and then by a rotation perpendicular to the 3-axis of magnitude arccos c (characterized by w), such that this axis points to the desired direction (see Fig. 1).

Kinematic Equations

In this section we derive the differential equations (kinematics of the attitude motion) which correspond to the kinematic parameters w and z introduced in the previous section. To derive the differential equations for w let us denote by $[a,b,c]^{\mathrm{T}}$ the third column of the matrix R (see equation (18)). Clearly from equations (4) and (5) one has that $[a,b,c]^{\mathrm{T}}$ satisfies the system of differential equations

$$\begin{bmatrix}\dot a\\ \dot b\\ \dot c\end{bmatrix}=\begin{bmatrix}0 & \omega_3 & -\omega_2\\ -\omega_3 & 0 & \omega_1\\ \omega_2 & -\omega_1 & 0\end{bmatrix}\begin{bmatrix}a\\ b\\ c\end{bmatrix}. \tag{26}$$

Recall from equation (20) that w is related to (a,b,c) by

$$w=\frac{b-ia}{1+c}. \tag{27}$$

Differentiating the last equation and using (26) and (21) we obtain the following differential equation for the complex quantity $w \in \mathbb{C}$

$$\dot w=-i\omega_3 w+\frac{\omega}{2}+\frac{\overline{\omega}}{2}w^2 \tag{28}$$

where $\omega := \omega_1 + i\omega_2$, the bar denotes complex conjugate, and $i := \sqrt{-1}$.

In terms of the real and imaginary parts of w, equation (28) can be rewritten as

$$\dot w_1=\omega_3 w_2+\omega_2 w_1 w_2+\omega_1(1+w_1^2-w_2^2)/2 \tag{29a}$$

$$\dot w_2=-\omega_3 w_1+\omega_1 w_1 w_2+\omega_2(1+w_2^2-w_1^2)/2 \tag{29b}$$

In order to find the differential equation for z we make use of equation (4) in the scalar form

$$\operatorname{tr}[\dot{R}(w,z)] = \operatorname{tr}[S({}^i\boldsymbol{\omega}^b)R(w,z)] \tag{30}$$

where $\operatorname{tr}(\cdot)$ denotes the trace of the matrix. Taking the trace of $\dot{R}(w,z)$ we have

$$\operatorname{tr}[\dot{R}(w,z)] = -\frac{2\dot{z}\sin z}{1 + w_1^2 + w_2^2} - \frac{4(1+\cos z)(w_1\dot{w}_1 + w_2\dot{w}_2)}{(1+w_1^2+w_2^2)^2} \tag{31}$$

Using equations (29) we have the relation

$$2\frac{w_1\dot{w}_1 + w_2\dot{w}_2}{1+w_1^2+w_2^2} = \omega_1 w_1 + \omega_2 w_2 \tag{32}$$

Substituting (32) into (31) we obtain

$$\operatorname{tr}[\dot{R}(w,z)] = -\frac{2}{1+w_1^2+w_2^2}[\dot{z}\sin z + (1+\cos z)(\omega_1 w_1 + \omega_2 w_2)] \tag{33}$$

Next, we find from equations (5) and (24) that

$$\operatorname{tr}[S({}^i\boldsymbol{\omega}^b)R(w,z)] = -\frac{2}{1+w_1^2+w_2^2}[(1+\cos z)(\omega_1 w_1 + \omega_2 w_2) + (\omega_3 - \omega_1 w_2 + \omega_2 w_1)\sin z] \tag{34}$$

Equating (34) with (33), we finally obtain that the initial angle z obeys the following differential equation

$$\dot{z} = \omega_3 - \omega_1 w_2 + \omega_2 w_1 \tag{35}$$

or equivalently,

$$\dot{z} = \omega_3 + i(\overline{\omega} w - \omega\overline{w})/2 \tag{36}$$

Equations (28) and (36) describe the kinematic equations in terms of the new (w, z) parameterization.

Finally, we notice that since

$$\frac{d}{dt}|w|^2 = 2\operatorname{Re}(\overline{w}\dot{w}) \tag{37}$$

the kinematic equations (28) and (36) can also take the convenient form

$$\frac{d}{dt}|w|^2 = (1+|w|^2)\operatorname{Re}(\omega\overline{w}) \tag{38a}$$

$$\dot{z} = \omega_3 + \operatorname{Im}(\omega\overline{w}) \tag{38b}$$

In equation (38b) only the imaginary part of the product $\omega\overline{w}$ appears, while in (38a) only the real part appears. This duality (or anti-symmetry) of equations (38a) and (38b) is desirable and can be used to derive stabilizing control laws for the system of equations (28) and (36) as in Tsiotras and Longuski [7] and Tisiotras et al. [14].

An equation similar to (28) first appeared in Darboux [15] in connection with some problems in classical differential geometry. However, its use in the description of attitude kinematics has been for the most part ignored, at least as far as the authors know. This is probably also due to the fact that w by itself is not enough for a complete description of the body orientation. In Tsiotras and Longuski [6], for example, equation (28) was derived via stereographic projection and was used for the kinematic description of the rotational motion of a rigid body. Such a description is however incomplete, since the complex variable w gives information only about a single column of the rotation matrix, or equivalently, about the orientation of only one of the body axes (the designated body-axis, or the spin-axis for the case of spinning bodies). No information can be gained from w about the relative orientation of the rigid body about this axis. Knowledge of w is therefore not enough to reconstruct the rotation matrix; one needs to introduce an additional coordinate to complete the kinematics. The introduction of the angle z provides the additional information required and complements the coordinate set. Here z is taken as the initial rotation about the 3-axis, although it is not restricted to the 3-axis in general. The only requirement is that the w variable represents the orientation of a body axis and that the z variable represents the initial rotation about this body axis.

Properties

Notice from equation (7) that we perform an initial rotation about the inertial 3-axis (the polar axis in Fig. 1) instead of a final rotation about the body 3-axis. Although the second choice might at first glance appear to be more natural, an initial rotation by an angle z has the property that it does not enter on the right-hand-side of equations (28) and (36), i.e., it is ignorable, since it is the *first* rotation. This is a property shared by the Eulerian angles as well, where the first angle is always ignorable [1,4]. This fact has some quite interesting and nontrivial consequences. First, in the case of analytic solutions of the system of equations (28) and (36) one can initially concentrate only on the search for analytic solutions of equation (28), which does not require knowledge of z, and then solve for z from (36) by quadrature. This procedure can be used whenever one has an analytic solution for the angular velocities at hand, e.g., for the case of body-fixed torques. Clearly, in the case of orientation-dependent torques one cannot decouple the kinematics from the dynamics and one has to treat the complete system of the differential equations. Analytic solutions for this latter problem are usually intractable. Even for the case of body-fixed torques, one cannot expect to be able to solve (28) exactly since ω and ω_3 are, in general, functions of time, but one can seek approximate solutions to (28). Notice, for example, that if the quadratic term (the only nonlinear term in the system (28) and (36)) vanishes, one can immediately solve for w by direct integration of the remaining linear (time-varying) differential equation. For cases when the term $\overline{\omega} w^2$ is small, such a methodology is expected to give very good results. For most cases, however, one needs to improve the solution using perturbation or successive approximation methods [11].

One might think that the quaternion formulation leading to a system of linear equations would be the ultimate formulation for analytic studies. Many researchers (including the authors of this paper) have found that the time-varying coefficients of this system of equations pose great difficulties in the search for analytic, closed-form solutions, although attempts towards this direction have been reported in the literature [6, 16–18]. Also, it is not easy to make reasonable simplifying assumptions with these equations since they have no physical-intuitive appeal. For example, we can't use small angle approximations in the quaternion formulation. On the other hand, the Eulerian angles, although leading to a highly nonlinear set of differential equations, provide a formulation which is most easily visualized and is reasonably tractable for analytic work, because the equations can be linearized using small angle approximations. This is the reason that most researchers pursuing analytic solutions prefer to work with the Eulerian angles [19–22].

The new formulation provides a very interesting compromise between the two previous extreme cases. It involves only quadratic nonlinearities. When the quadratic nonlinearity is dropped we end up with exactly the same equations as those found from applying small angle approximations to Euler's kinematic equations. Thus, if we find even an approximate solution for the quadratic term, we have a large angle analytic theory [11]. Hence, the new equations are ideally suited for analytic investigations.

For control problems on the other hand, the property of z being ignorable allows a natural decomposition of the control problem into one of controlling only w (for which no knowledge of z is required) and one of controlling z. Especially in problems where we are interested only in controlling a body axis, with the relative orientation of the body about this axis being irrelevant, such a decomposition is clearly favorable. In such problems the (w, z) framework offers an obvious advantage over other traditional kinematic formulations, where such a natural decomposition of the rotational motion is neither immediate nor clear [7, 14].

Singular Orientations

The (w, z) parameterization is a three-dimensional parameterization of the rotation group, and as such there are orientations where this set of parameters is not defined [3]. The singularity of the (w, z) parameterization is the one inherited by the stereographic projection. Recall from the definition of the stereographic projection that the base point of the projection is mapped to infinity (whereas the antipodal point is mapped to the origin). In equation (20) we have chosen the base point of the projection to be the point $(0, 0, -1) \in S^2$, i.e., the South pole. We could have chosen the point $(0, 0, 1)$, without any loss of generality. By choosing $(0, 0, -1)$ as the base point of the projection we map this point to infinity and the equilibrium $w = 0$ of equation (28) corresponds to the antipodal point $(0, 0, 1) \in S^2$. Therefore the singularity of the stereographic projection $w = \infty$ (and, consequently, of the parameterization) corresponds to an "upside-down" configuration of the rigid body, while the equilibrium $w = 0$ corresponds to the "up" configuration. Initial orientations corresponding to this singular "upside-down" configuration cannot, of course, be accommodated using this kinematic formulation. However, the stereographic projection moves the singularity as far away from the equilibrium point as possible, a property which has obvious advantages in the case of control problems, for example. Clearly, in the case of stabilizing the w equation one guarantees, in particular, that w remains finite for all time, and one need not worry about passing through the singular point $w = \infty$ [7].

Summarizing, we note that the (w, z) parameterization is favorably compared to the Eulerian angles as far as the domain of validity of the representations is concerned. Specifically, the (w, z) remains valid everywhere except at the South pole of the sphere, whereas the Eulerian angles, in general, either become singular at the equator, or they are singular at the equilibrium (the 3-1-3 set, for example)—clearly an undesirable situation.

Connection to Other Parameterizations

In this section we present the connection between the (w, z) coordinates introduced in the previous section and some of the other well-known parameterizations of the rotation group.

Eulerian Angles

Perhaps the easiest way to visualize and understand the new (w, z) parameterization is through the Eulerian angles. If we choose a 3-2-1 Euler angle sequence (ψ, θ, ϕ), for example, the orientation of the body-fixed reference frame with respect to the inertial reference frame is found by first rotating the body about its 3-axis through an angle ψ, then rotating about its 2-axis by an angle θ and finally rotating about its 1-axis by an angle ϕ.

We can determine the location of the 3 body-axis using the θ and ϕ angles (the angle ψ provides the initial rotation about this axis). According to the previous section, the coordinate w is determined uniquely by ϕ and θ only, while z is determined by all three angles. The domain of validity for these kinematic coordinates is given by $-\pi \leq \phi \leq \pi$, $-\pi \leq \psi \leq \pi$, $-\pi/2 < \theta < \pi/2$.

The rotation matrix, $R(\psi, \theta, \phi)$, associated with the 3-2-1 set of Eulerian angles is given by

$$R(\psi,\theta,\phi) = \begin{bmatrix} c\psi c\theta & s\psi c\theta & -s\theta \\ -s\psi c\phi + c\psi s\theta s\phi & c\psi c\phi + s\psi s\theta s\phi & c\theta s\phi \\ s\psi s\phi + c\psi s\theta c\phi & -c\psi s\phi + s\psi s\theta c\phi & c\theta c\phi \end{bmatrix} \tag{39}$$

Identifying the $[a, b, c]^{\mathrm{T}}$ with the third column of the rotation matrix, and using its expression in terms of (ψ, θ, ϕ) from (39) we get that

$$w = \frac{\sin\phi\cos\theta + i\sin\theta}{1 + \cos\phi\cos\theta} \tag{40}$$

or that

$$w_1 = \frac{\sin\phi\cos\theta}{1 + \cos\phi\cos\theta}, \qquad w_2 = \frac{\sin\theta}{1 + \cos\phi\cos\theta} \tag{41}$$

The expression for z in terms of (ψ, θ, ϕ) is computed using equations (18) and (39). A straightforward, but rather lengthy calculation, shows that

$$z = \psi + \arcsin(p\cos\phi) - \arcsin(p) \tag{42}$$

where $p = \alpha/\sqrt{1+\alpha^2}$ and $\alpha = \tan\theta/\sin\phi$. For $\phi = 0$, we take $z = \psi$. By letting $\alpha = \tan\tilde{\alpha}$, equation (42) can also be written as[3]

$$z = \psi + \arcsin(\sin\tilde{\alpha}\cos\phi) - \tilde{\alpha} \tag{43}$$

The details for the derivation of (42) are given in the appendix. Equation (42) was initially introduced in Tsiotras et al. [14], where it was utilized as an output for the construction of an invariant manifold for the kinematic equations subject to a linear feedback.

The determinant of the Jacobian of the transformation $(\phi,\theta,\psi) \mapsto (w_1, w_2, z)$ is given by

$$\frac{\cos\theta}{(1+\cos\phi\cos\theta)^2} \tag{44}$$

Thus, the transformation (40)–(42) is valid everywhere except at the pole of the stereographic projection, and at $\theta = \pm\pi/2$, i.e., the inherent singularities of the (w,z) parameterization and the Eulerian set itself.

Euler-Rodrigues Parameters

Euler's Principal Rotation Theorem [23] states that a completely general angular displacement between two reference frames can be accomplished by a single rotation through an angle Φ (the principal angle) about a unit vector $\hat{\mathbf{e}}$ (the principal vector), which is fixed in both reference frames. Using this result, we can define the Euler-Rodrigues parameters by

$$q_0 = \cos(\Phi/2), \qquad q_i = e_i \sin(\Phi/2), \qquad (i = 1,2,3) \tag{45}$$

where $\hat{\mathbf{e}} := (e_1, e_2, e_3)$ is the principal vector. For such a parameterization of SO(3) the rotation matrix is given by [1,4]

$$R(q_0,q_1,q_2,q_3) = \begin{bmatrix} q_0^2+q_1^2-q_2^2-q_3^2 & 2(q_1q_2+q_0q_3) & 2(q_1q_3-q_0q_2) \\ 2(q_1q_2-q_0q_3) & q_0^2-q_1^2+q_2^2-q_3^2 & 2(q_2q_3+q_0q_1) \\ 2(q_1q_3+q_0q_2) & 2(q_2q_3-q_0q_1) & q_0^2-q_1^2-q_2^2+q_3^2 \end{bmatrix} \tag{46}$$

Comparing elements of the two representations (18) and (46) for the rotation matrix R we obtain

$$a = 2(q_1q_3 - q_0q_2), \qquad b = 2(q_2q_3 + q_0q_1), \qquad c = q_0^2 - q_1^2 - q_2^2 + q_3^2 \tag{47}$$

Using equations (21) we get the following equations relating the (w,z) parameterization to the Euler-Rodrigues parameters

$$w_1 = \frac{q_2q_3 + q_0q_1}{q_0^2 + q_3^2}, \qquad w_2 = \frac{q_0q_2 - q_1q_3}{q_0^2 + q_3^2}, \qquad z = 2\arctan\frac{q_3}{q_0} \tag{48}$$

[3]We owe this observation to an anonymous reviewer.

Cayley-Rodrigues Parameters

Since equations (45) imply

$$q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1 \tag{49}$$

the Euler-Rodrigues parameters are once-redundant. One is then naturally led to the elimination of this constraint, thus reducing the number of coordinates from four to three. The Rodrigues parameters achieve this by defining

$$\rho_i = q_i/q_0, \qquad (i = 1,2,3) \tag{50}$$

with the associated rotation matrix representation, $R(\rho_1, \rho_2, \rho_3)$, given by

$$R(\rho_1,\rho_2,\rho_3) = \frac{1}{1 + \rho_1^2 + \rho_2^2 + \rho_3^2} \cdot \begin{bmatrix} 1 + \rho_1^2 - \rho_2^2 - \rho_3^2 & 2(\rho_1\rho_2 + \rho_3) & 2(\rho_1\rho_3 - \rho_2) \\ 2(\rho_1\rho_2 - \rho_3) & 1 - \rho_1^2 + \rho_2^2 - \rho_3^2 & 2(\rho_2\rho_3 + \rho_1) \\ 2(\rho_1\rho_3 + \rho_2) & 2(\rho_2\rho_3 - \rho_1) & 1 - \rho_1^2 - \rho_2^2 + \rho_3^2 \end{bmatrix} \tag{51}$$

Again, comparing elements of the two representations (18) and (51) or using equations (48) and (50) we obtain the following transformation between the (w, z) coordinates and the Cayley-Rodrigues parameters

$$w_1 = \frac{\rho_1 + \rho_2\rho_3}{1 + \rho_3^2}, \qquad w_2 = \frac{\rho_2 - \rho_1\rho_3}{1 + \rho_3^2}, \qquad z = 2\arctan\rho_3 \tag{52}$$

The determinant of the Jacobian of the transformation $(\rho_1, \rho_2, \rho_3) \mapsto (w_1, w_2, z)$ is given by

$$\frac{2}{(1 + \rho_3^2)^2} \tag{53}$$

The Jacobian thus becomes singular for $\rho_3 = -1$ which corresponds to the (South) pole of the stereographic projection. The inverse transformation $(w_1, w_2, z) \mapsto (\rho_1, \rho_2, \rho_3)$ is given by

$$\rho_1 = w_1 - \rho_3 w_2, \qquad \rho_2 = w_2 + \rho_3 w_1, \qquad \rho_3 = \tan(z/2) \tag{54}$$

As a last remark, we note that the elimination of the constraint (49) via the equations (50) may not be the most natural one. Perhaps a better approach is to achieve the elimination of the redundancy associated with the Euler-Rodrigues parameters using the stereographic projection. In this case we introduce the *Modified Cayley-Rodrigues Parameters* [2, 24–26]

$$\sigma_i = \frac{q_i}{1 + q_0}, \qquad (i = 1,2,3) \tag{55}$$

We will not elaborate on the advantages of using the Modified Cayley-Rodrigues parameters in this paper, but refer the interested reader to the literature [2, 24, 26, 27] for a more complete discussion. We merely state the obvious fact that the Cayley-Rodrigues parameters allow eigenaxis rotations of only up to 180 degrees, whereas the Modified Cayley-Rodrigues parameters allow eigenaxis rotations of up to 360 degrees. This can be seen from equations (45) and (50) where we have that

$$\rho_i = e_i \tan(\Phi/2), \qquad (i = 1,2,3) \tag{56}$$

whereas from (45) and (55) we have that

$$\sigma_i = e_i \tan(\Phi/4), \qquad (i = 1,2,3) \tag{57}$$

Clearly, equation (56) allows rotations for $0 \le \Phi < \pi$ whereas (57) allows rotations for $0 \le \Phi < 2\pi$.

Cayley-Klein Parameters

The new attitude parameters w and z have a very elegant connection with the Cayley-Klein parameters. Recall that the Cayley-Klein parameters $\lambda, \mu \in \mathbb{C}$ are complex numbers defined from the Euler-Rodrigues parameters via

$$\lambda := q_0 + iq_3, \qquad \mu := q_1 + iq_2 \tag{58}$$

One can easily compute that (w, z) are related to (λ, μ) by the simple transformation

$$w = \mu/\lambda, \qquad z = 2\angle\lambda \tag{59}$$

where $\angle x$ is the angle of the complex number $x = x_1 + ix_2 \in \mathbb{C}$ defined, as usual, by

$$\angle x := \arctan(x_2/x_1) \tag{60}$$

Axis-Azimuth Parameters

As mentioned in the introduction, the description of the rotational motion in terms of an axis and an angle is not new. The Axis-Azimuth and Listing's parameterizations use this idea, although the location of the spin axis is determined by means of the two polar (spherical) angles. It is interesting to note that even when Euler first presented his famous formula for the rotation matrix as a function of the axis and angle, he represented the axis not as a unit vector but, instead, in terms of two spherical angles. Although this is the most natural way to describe the motion, it should be obvious from the results of this paper that, by using the stereographic coordinates of the unit vector instead of the spherical coordinates, one simplifies the kinematic equations significantly.

Motivated by Tsiotras and Longuski [28] and Montgomery [29], Walsh et al. [9] also used stereographic coordinates for the attitude representation and control of a rigid satellite [9, 30]. In Walsh et al. the stereographic coordinates are introduced in lieu of the two polar coordinates of Listing's parameterization. The third coordinate (as in the Axis-Azimuth) description is a *final* rotation about the body-fixed 3-axis. Using Listing's parameterization the rotation matrix is given by

$$R(\phi, \theta, \psi) = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} c^2\theta(1 - c\phi) + c\phi & c\theta s\theta(1 - c\phi) & -s\phi s\theta \\ c\theta s\theta(1 - c\phi) & s^2\theta(1 - c\phi) + c\phi & s\phi c\theta \\ s\phi s\theta & -s\phi c\theta & c\phi \end{bmatrix} \tag{61}$$

(The angles ϕ, θ, ψ in this subsection should not be confused with the ones used in equation (39).) In equation (61) the angles θ and ϕ are the two spherical coordinates which determine the location of the body 3-axis in inertial space. The angle ψ is the rotation about this axis. Notice that ψ is a final rotation in this description. Following Walsh et al. we can use the stereographic projection in order to introduce the complex coordinate

$$h := h_1 + ih_2 = \frac{\sin\phi\,\sin\theta - i\,\sin\phi\,\cos\theta}{1 + \cos\phi} \tag{62}$$

in the place of the polar coordinates ϕ and θ. In this case the kinematic equations take the form

$$\dot{h} = -\frac{1 + |h|^2}{2}\,\omega e^{i(\psi+\pi/2)} \tag{63a}$$

$$\dot{\psi} = \omega_3 + \frac{2}{1 + |h|^2}\,\mathrm{Im}(\dot{h}\bar{h}) \tag{63b}$$

The angle ψ enters on the right-hand side of equations (63) and therefore is not ignorable. On the other hand, in equation (63a) the third component of the angular velocity ω_3 does not affect the equation for $\dot{h}$. This might be useful in some problems. Equation (62) corresponds to stereographically projecting the third *row* of the rotation matrix $R(\phi, \theta, \psi)$ in equation (61) (the one depending only on ϕ and θ). Since any row or column of the rotation matrix satisfies the constraint (19) one can apply, in principle, the stereographic projection to any row or column of the rotation matrix. The choice of the particular row or column will depend, of course, on the application at hand.

We note in passing that the advantage of the stereographic projection in eliminating constraints of the form (19) can also be demonstrated in the case of the Euler parameters where the Modified Rodrigues parameters, derived through stereographic projection, have a definite advantage over the traditional Cayley-Rodrigues parameters [2, 25–27], as discussed previously.

Concluding Remarks

We have presented a new formulation for describing the kinematics of the rotational motion of a rigid body. This formulation is derived by first stereographically projecting one of the columns of the rotation matrix (which lies on the unit sphere) on the complex plane. The complex coordinate then describes the location of the "designated body-axis" corresponding to the column of the rotation matrix used in the stereographic projection. To complete the parameterization we introduce an additional angle which describes an initial rotation about this axis, thus leading to a natural decomposition of the motion. Although a final rotation about this axis would seem more natural and obvious, the fact that the rotation is introduced first has the advantage of generating an ignorable coordinate in the kinematic equations, a desirable property for many applications.

It is shown that the new parameterization can be realized using two rotations about perpendicular axes. Thus, in some sense, the new parameterization "fills the gap" between the Euler-Rodrigues parameterization which can be realized through one rotation and the Eulerian angle parameterization which requires three successive rotations. We also give the relations of the new parameterization in terms of the other classical parameterizations of the rotation group. These connections shed new light on the physical significance of abstract mathematical quantities such as the Euler-Rodrigues parameters, the Cayley-Rodrigues parameters and the Cayley-Klein parameters.

Some of the advantages of the new parameterization in deriving analytic solutions and in designing control laws for the attitude motion have been already reported in the literature [7, 11, 14]. Finally, we hope that the new parameterization will be useful not only to control engineers and astrodynamicists, but also to those interested in robotics, attitude estimation, and related fields.

Acknowledgments

The authors wish to thank Professor Malcolm Shuster and Professor John Junkins for helpful discussions during the preparation of this work. The anonymous reviewers are also gratefully acknowledged for suggestions which contributed to the final version of this paper. This research has been supported by the National Science Foundation under Grant No. MSS-9114388.

Appendix

In order to show equation (42) consider the trace of the rotation matrix (18). One then obtains

$$\begin{aligned} \mathrm{tr}[R(w,z)] &= \frac{2c\cos z + (b^2 + 2c^2 + a^2)\cos z}{1+c} + c \\ &= \frac{2c\cos z + (1 + c^2)\cos z}{1+c} + c = \frac{(1+c)^2}{1+c}\cos z + c \\ &= (1+c)\cos z + c \end{aligned} \tag{A1}$$

Taking the trace of the rotation matrix (39), we have

$$\mathrm{tr}[R(\psi,\theta,\phi)] = \mathrm{c}\psi\mathrm{c}\theta + \mathrm{c}\psi\mathrm{c}\phi + \mathrm{s}\psi\mathrm{s}\theta\mathrm{s}\phi + \mathrm{c}\theta\mathrm{c}\phi \tag{A2}$$

Now using the fact that

$$c = \cos\theta\cos\phi \tag{A3}$$

and equating (A1) and (A2) we obtain

$$(1 + \mathrm{c}\phi\mathrm{c}\theta)\mathrm{c}z = \mathrm{c}\theta\mathrm{c}\psi + \mathrm{c}\phi\mathrm{c}\psi + \mathrm{s}\phi\mathrm{s}\theta\mathrm{s}\psi \tag{A4}$$

or that

$$\cos z = \frac{\mathrm{c}\theta\mathrm{c}\psi + \mathrm{c}\phi\mathrm{c}\psi + \mathrm{s}\phi\mathrm{s}\theta\mathrm{s}\psi}{1 + \mathrm{c}\phi\mathrm{c}\theta} \tag{A5}$$

One could also use equation (A5) as a definition of z in terms of the Eulerian angles (ψ, θ, ϕ).

In order to show that (42) is equivalent to (A5), we take the cosine of both sides of (42). Using the formula

$$\begin{aligned} \cos(A + B + C) = \mathrm{c}(A)\mathrm{c}(B)\mathrm{c}(C) &- \mathrm{c}(A)\mathrm{s}(B)\mathrm{s}(C) \\ &- \mathrm{s}(A)\mathrm{s}(B)\mathrm{c}(C) - \mathrm{s}(A)\mathrm{c}(B)\mathrm{s}(C) \end{aligned} \tag{A6}$$

and the fact that p can be written as

$$p = \frac{\sin\theta}{(1 - \cos^2\theta\cos^2\phi)^{1/2}} \tag{A7}$$

one obtains that

$$\begin{aligned} \cos z &= \frac{\mathrm{c}\psi\mathrm{c}\phi + \mathrm{c}\psi\mathrm{c}\theta\mathrm{s}^2\phi - \mathrm{c}^2\theta\mathrm{c}\psi\mathrm{c}\phi - \mathrm{c}\theta\mathrm{s}\theta\mathrm{s}\psi\mathrm{s}\phi\mathrm{c}\phi + \mathrm{s}\phi\mathrm{s}\psi\mathrm{s}\theta}{1 - \mathrm{c}^2\phi\mathrm{c}^2\theta} \\ &= \frac{(1 - \mathrm{c}\phi\mathrm{c}\theta)(\mathrm{c}\theta\mathrm{c}\psi + \mathrm{c}\phi\mathrm{c}\psi + \mathrm{s}\phi\mathrm{s}\theta\mathrm{s}\psi)}{1 - \mathrm{c}^2\phi\mathrm{c}^2\theta} \end{aligned} \tag{A8}$$

Dividing the numerator and denominator by $(1 - \cos\phi\cos\theta)$ (this is allowable since $\cos\phi\cos\theta \neq 1$ within the domain of definition of the angles ϕ and θ) we finally obtain that

$$\cos z = \frac{\mathrm{c}\theta\mathrm{c}\psi + \mathrm{c}\phi\mathrm{c}\psi + \mathrm{s}\phi\mathrm{s}\theta\mathrm{s}\psi}{1 + \mathrm{c}\phi\mathrm{c}\theta} \tag{A9}$$

as required.

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