# STOCHASTIC HILL'S EQUATIONS FOR THE STUDY OF ERRANT ROCKET BURNS IN ORBIT

MARTIN OSTOJA-STARZEWSKI
*Department of Materials Science and Mechanics, Michigan State University, East Lansing, MI 48824-1226, U.S.A.*

and

JAMES M. LONGUSKI
*School of Aeronautics and Astronautics, Purdue University, West Lafayette, IN 47907, U.S.A.*

(Received 30 January, 1991; accepted 15 May, 1992)

**Abstract.** The problem of errant rocket burns in low Earth orbit is of growing interest, especially in the area of safety analysis of nuclear powered spacecraft. The development of stochastic Hill's equations provides a rigorous mathematical tool for the study of such errant rocket maneuvers. These equations are analyzed within the context of a theory of linear dynamical systems driven by a random white noise. It is established that the trajectories of an errant rocket are realizations of a Gauss-Markov process, whose mean vector is given by the solution of a deterministic rocket problem. The time-dependent covariance matrix of the process is derived in an explicit form.

**Key words:** Hill's equations, stochastic differential equations, diffusion processes, rocket maneuvers, rendezvous.

## 1. Introduction

The uncertainty inherent in performance of any means of propulsion is the key consideration that leads to the present study. One typical situation in spacecraft dynamics where such uncertainty is of great importance is the execution of a rendezvous between two vehicles in orbit. The propulsion performance – or the firing of the engines involved in this maneuver – is considered to be a white noise vector random process. The maneuver is modeled by Hill's equations (Kaplan, 1976)

$$
\begin{aligned}
&\ddot{x} - 2n\dot{y} - 3n^2x = f_x \\
&\ddot{y} + 2n\dot{x} = f_y \\
&\ddot{z} + n^2z = f_z
\end{aligned}
\tag{1.1}
$$

where $x$, $y$, $z$ are the coordinates of a rendezvous vehicle in a reference frame fixed at the target vehicle. The target vehicle is rotating in a circular orbit at an angular velocity, $n$. The $x$ axis is along the radial direction from the Earth and $y$ is along the target orbital path. The third axis, $z$, is normal to the orbit. Both $x$ and $z$ components of relative motion are assumed to be small, but $y$ may be arbitrarily large.

In accordance with our foregoing discussion, the thrust, or equivalently the acceleration $\underset{\sim}{a}(t)$, is a random process, and hence we must deal with a stochastic nonlinear differential equation. The question we ask is this: how different are solutions of the system (1.1) when the temporal randomness of thrust is taken into account from the solutions when $\underset{\sim}{a}(t)$ is a deterministic process? The theoretical method we use is based on the fact expressed succinctly by Arnold (1974): *the stochastic differential equations and diffusion processes represent essentially the same classes of processes.*

## 2. General Formulation for Long Burns on Circular Orbits

For the case of maneuvers in which the deterministic thrust is inertially fixed, Hill's equations become (Longuski and McRonald, 1988)

$$\begin{aligned} &\ddot{x} - 2n\dot{y} - 3n^2 x = a_x \cos nt + a_y \sin nt \\ &\ddot{y} + 2n\dot{x} = a_x \cos nt - a_y \sin nt \\ &\ddot{z} + n^2 z = a_z \end{aligned} \tag{2.1}$$

where the rendezvous vehicle of mass $M$, is firing thrusters resulting in three forces, $Ma_x$, $Ma_y$, $Ma_z$. Each of these accelerations is taken as a sum of a mean part $\bar{a}_i$ constant in time and a zero-mean temporal fluctuation $a'_i$ $(i = x, y, z)$

$$a_i = \bar{a}_i + a'_i \,. \tag{2.2}$$

Henceforth, we assume all three fluctuations $a'_x$, $a'_y$, $a'_z$ to be components of a three-dimensional vector random process of Gaussian white noise type. Thus*

$$\begin{aligned} &E\{a'_i(t)\} = 0 \\ &E\{a'_i(t)\, a'_j(t+\tau)\} = D_{ij}\delta(\tau) \end{aligned} \tag{2.3}$$

where $E\{\cdot\}$ is the expectation operator, $D_{ij}$ is an element of a (symmetric) $3 \times 3$ correlation matrix $\underset{\approx}{D}$, and $\delta(\tau)$ is the Dirac delta function.

We define six state variables as follows

$$X_1 = x\,,\ \ X_2 = \dot{x}\,,\ \ X_3 = y\,,\ \ X_4 = \dot{y}\,,\ \ X_5 = z\,,\ \ X_6 = \dot{z}\,. \tag{2.4}$$

System (2.1) is now written as a vector linear stochastic differential equation

* In this paper the symbols $\underset{\sim}{\cdot}$ and $\underset{\approx}{\cdot}$ denote tensors of first (vectors) and second rank, respectively.

$$\frac{d}{dt} \underset{\sim}{X} = \underset{\approx}{A} \underset{\sim}{X} + \underset{\sim}{a}(t) + \underset{\approx}{\sigma}(t) \underset{\sim}{a}'(t) \tag{2.5}$$

in which

$$\underset{\approx}{A} = \begin{bmatrix} 0, & 1, & 0, & 0, & 0, & 0 \\ 3n^2, & 0, & 0, & 2n, & 0, & 0 \\ 0, & 0, & 0, & 1, & 0, & 0 \\ 0, & -2n, & 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0, & 0, & 1 \\ 0, & 0, & 0, & 0, & -n^2, & 0 \end{bmatrix} \tag{2.6}$$

$$\underset{\sim}{a}(t) = \begin{bmatrix} 0 \\ \bar{a}_x \cos nt \\ 0 \\ \bar{a}_y \sin nt \\ 0 \\ \bar{a}_z \end{bmatrix} \tag{2.7}$$

and

$$\underset{\approx}{\sigma}(t) = \begin{bmatrix} 0, & 0, & 0 \\ \cos nt, & \sin nt, & 0 \\ 0, & 0, & 0 \\ -\sin nt, & \cos nt, & 0 \\ 0, & 0, & 0 \\ 0, & 0, & 1 \end{bmatrix} . \tag{2.8}$$

Evidently, $\underset{\sim}{X}(t)$ is a Markov random process driven by a vector random process with components specified in (2.3)

$$\underset{\sim}{a}'(t) = \begin{bmatrix} a'_x(t) \\ a'_y(t) \\ a'_z(t) \end{bmatrix} . \tag{2.9}$$

The process $\underset{\sim}{X}(t)$ takes values $\underset{\sim}{x}$ and $\underset{\sim}{y}$ in the state space $X = I\!R^6$.

The solution to (2.5) can only be described probabilistically, and this will be done here using the diffusion equations. To that end we introduce a transition probability function

$$P(x, \underset{\sim}{x}, t, E) = P\{\underset{\sim}{X}(t) \in E \mid \underset{\sim}{X}(s) = \underset{\sim}{x}\} \tag{2.10}$$

where $E$ is any set in the $\sigma$-algebra $\mathcal{F}$ of the state space $X$, while $P$ is a probability measure on $(X, \mathcal{F})$. We now assume that the transition density $p \equiv p(s, \underset{\sim}{x}, t, \underset{\sim}{y})$ is derivable from $P(s, \underset{\sim}{x}, t, E)$ and satisfies all the usual restrictions, so that its behavior is governed by the backward and forward (or Fokker-Planck) Kolmogorov diffusion equations ($m = 6$)

$$\frac{\partial p}{\partial s} = -\sum_{i=1}^{m} a_i(s) \frac{\partial p}{\partial x_i} - \sum_{i,j=1}^{m} A_{ij} x_j \frac{\partial p}{\partial x_i} - \frac{1}{2} \sum_{i,j=1}^{m} B_{ij} \frac{\partial^2 p}{\partial x_i \partial x_j} \tag{2.11}$$

$$\frac{\partial p}{\partial t} = -\sum_{i=1}^{m} a_i(t) \frac{\partial p}{\partial y_i} - \sum_{i,j=1}^{m} A_{ij} \frac{\partial}{\partial y_i} (y_j p) + \frac{1}{2} \sum_{i,j=1}^{m} \frac{\partial^2}{\partial y_i \partial y_j} (B_{ij} p) . \tag{2.12}$$

The vector $\underset{\sim}{a}(t)$ and the matrix $\underset{\approx}{A}$ are identical with those given in (2.7) and (2.6), respectively – they describe the local drift and the process $\underset{\sim}{X}(t)$. The matrix $\underset{\approx}{B}$ which is found from (superscript T indicates a transpose)

$$\underset{\approx}{B}(t) = \underset{\approx}{\sigma}(t) \underset{\approx}{D} \underset{\approx}{\sigma}^{\mathrm{T}}(t) \tag{2.13}$$

describes the mean square deviation of $\underset{\sim}{X}(t)$ from the original position $\underset{\sim}{x}$ during a short time interval from $t$ to $t + \Delta t$. This matrix is found to have the following form

$$\underset{\approx}{B} = \begin{bmatrix} 0, & 0 & , 0, & 0 & , 0, & 0 \\ 0, & B_{22} & , 0, & B_{24} & , 0, & B_{26} \\ 0, & 0 & , 0, & 0 & , 0, & 0 \\ 0, & B_{42} & , 0, & B_{44} & , 0, & B_{46} \\ 0, & 0 & , 0, & 0 & , 0, & 0 \\ 0, & B_{62} & , 0, & B_{64} & , 0, & B_{66} \end{bmatrix} \tag{2.14}$$

in which

$$\begin{aligned}
B_{22} &= D_{11} \cos^2 nt + 2D_{12} \sin nt \cos nt + D_{22} \sin^2 nt \\
B_{44} &= D_{11} \sin^2 nt - 2D_{12} \sin nt \cos nt + D_{22} \cos^2 nt \\
B_{66} &= D_{33} \\
B_{42} &= B_{24} = -D_{11} \sin nt \cos nt + D_{12}(\cos^2 nt - \sin^2 nt) + \qquad (2.14') \\
&\qquad + D_{22} \cos nt \sin nt \\
B_{62} &= B_{26} = D_{13} \cos nt + D_{23} \sin nt \\
B_{64} &= B_{46} = -D_{13} \sin nt + D_{23} \cos nt
\end{aligned}$$

Both Kolmogorov diffusion equations are subject to

$$p(t, \underset{\sim}{x}, t, \underset{\sim}{y}) = \delta(\underset{\sim}{x} - \underset{\sim}{y}) = \prod_{i=1}^{6} \delta(x_i - y_i) \tag{2.15}$$

which plays the role of a final condition of (2.11) for all times $s \leq t$, and the role of an initial condition for (2.12) for all times $t \geq s$.

Let $\underset{\sim}{X}_0 = \underset{\sim}{X}(t_0)$ indicate the initial condition for all sample trajectories of the process $\underset{\sim}{X}(t)$, $t \in [t_0, \infty]$. It follows from the theory of stochastic differential equations that the solution of (2.5), for $\underset{\sim}{X}_0$ having a Gaussian or Dirac delta density, is a Gauss-Markov process. Thus, the transition density $p(s, \underset{\sim}{x}, t, \underset{\sim}{y})$ is a six-dimensional normal density

$$p(s, \underset{\sim}{x}, t, \underset{\sim}{y}) = N(\underset{\sim}{m}_t(s, \underset{\sim}{x}), \underset{\approx}{K}_t(s, \underset{\sim}{x})) \tag{2.16}$$

where $\underset{\sim}{m}_t(s, \underset{\sim}{x})$ is a conditional expectation vector

$$\underset{\sim}{m}_t(s, \underset{\sim}{x}) = \int_X \underset{\sim}{y}\, p(s, \underset{\sim}{x}, t, \underset{\sim}{y}) d\underset{\sim}{y} \tag{2.17}$$

and $\underset{\approx}{K}_t(s, \underset{\sim}{x})$ is a $6 \times 6$ covariance matrix

$$\underset{\approx}{K}_t(s, \underset{\sim}{x}) = \int_X (\underset{\sim}{y} - \underset{\sim}{m}_t(s, \underset{\sim}{x}))(\underset{\sim}{y} - \underset{\sim}{m}_t(s, \underset{\sim}{x}))^{\mathrm{T}} p(s, \underset{\sim}{x}, t, \underset{\sim}{y}) d\underset{\sim}{y}\,. \tag{2.18}$$

Solutions to (2.17) and (2.18) are to be found, in principle, with help of the fundamental matrix $\underset{\sim}{\Phi}(t, s)$, or propagator, of the homogeneous matrix equation

$$\frac{d}{dt}\underset{\sim}{\Phi}(t) = \underset{\approx}{A}\,\underset{\sim}{\Phi}(t)\,, \qquad \underset{\sim}{\Phi}(0) = \underset{\sim}{I} \tag{2.19}$$

where $\underset{\sim}{I}$ is a unit vector. However, in view of the linearity of the system (2.5) we have

$$\frac{d}{dt}\underset{\sim}{m}_t(s, \underset{\sim}{x}) = \underset{\approx}{A}\,\underset{\sim}{m}_t(s, \underset{\sim}{x}) + \underset{\sim}{a}(t)\,, \qquad \underset{\sim}{m}_0(0, \underset{\sim}{x}) = \underset{\sim}{x}_0 \tag{2.20}$$

which is the same as the linearized Hill's equations in a deterministic case (Longuski and McRonald, 1988; Longuski, 1987, 1992), i.e. when random noises are absent.

Thus, the solution to this latter differential equation system may directly be adopted here; for completeness of presentation and a reference it is given in Appendix I.

The covariance matrix is to be calculated formally from (see (9.2.12) in Arnold (1974)

$$\underset{\approx}{K}_t(s, \underset{\sim}{x}) =$$

$$= \underset{\sim}{\Phi}(t, s) \int_s^t \underset{\sim}{\Phi}^{-1}(u, s)\, \underset{\approx}{\sigma}(u)\, \underset{\approx}{D}\, \underset{\approx}{\sigma}^{\mathrm{T}}(u)\, (\underset{\sim}{\Phi}^{\mathrm{T}}(u, s))^{-1} du\, \underset{\sim}{\Phi}^{\mathrm{T}}(t, s)\,. \tag{2.21}$$

Now, since $\underset{\approx}{A}$ is not a function of $t$

$$\underset{\sim}{\Phi}(t, s) = e^{\underset{\approx}{A}(t-s)}\,. \tag{2.22}$$

Moreover, since $\underset{\approx}{K}_t(s, \underset{\sim}{x})$ does not, according to (2.21), depend on the vector $\underset{\sim}{a}(t)$, we can use another result from Arnold (1974) pertaining to the covariance matrix in the autonomous case – $\underset{\approx}{A}(t) = \underset{\approx}{A}$, $\underset{\sim}{a}(t) = \underset{\sim}{a}$, $\underset{\approx}{B}(t) = B$ – although our case is not autonomous

$$\underset{\approx}{K}_{s+t}(s, \underset{\sim}{x}) = \int_0^t e^{\underset{\sim}{A}(t-u)}\, \underset{\approx}{B}\, e^{\underset{\sim}{A}^{\mathrm{T}}(t-u)}\, du\,. \tag{2.23}$$

This depends only on $t$ and $\underset{\sim}{x}$ since the transition function is time-homogeneous.

Thus, we find

$$K_{s+t}(s, \underset{\sim}{x}) =$$

$$= \begin{bmatrix} B_{22}e^{2(t-u)} & , B_{24}e^{(1+2n)(t-u)}, & B_{24}e^{2(t-u)} & , B_{22}e^{(1-2n)(t-u)}, & B_{26}e^{2(t-u)} & ,0 \\ B_{42}e^{(1+2n)(t-u)}, & B_{44}e^{4n(t-u)} & , B_{44}e^{2n(t-u)} & , B_{42} & , B_{46}e^{(2n+1)(t-u)} & ,0 \\ B_{42}e^{2(t-u)} & , B_{44}e^{(1+2n)(t-u)} & , B_{44}e^{(1+2n)(t-u)} & , B_{42}e^{(1-2n)(t-u)}, & B_{46}e^{2(t-u)} & ,0 \\ B_{22}e^{(1-2n)(t-u)}, & B_{24} & , B_{24}e^{(1-2n)(t-u)} & , B_{22}e^{(1-2n)(t-u)} & , B_{26}e^{(1-2n)(t-u)} & ,0 \\ B_{62}e^{2(t-u)} & , B_{64}e^{(1+2n)(t-u)} & , B_{64}e^{(1+2n)(t-u)} & , B_{62}e^{(1-2n)(t-u)}, & B_{66}e^{2(t-u)} & ,0 \\ 0 & , 0 & , 0 & , 0 & , 0 & ,0 \end{bmatrix} \tag{2.24}$$

The components of $\underset{\approx}{K}_{s+t}(s, \underset{\sim}{x})$ are calculated more explicitly in the special case when fluctuations in thrust are independent random processes – $D_{12} = D_{13} =$

$D_{23} = 0$ – and when $D_{11} = D_{22}$; they are listed in Appendix II.

## 3. Conclusions

The principal conclusions of our study are as follows:

(i) the transition density function, and hence the probability density of the state $\underset{\sim}{X}$ governed by linearized Hill's equations is, under a Gaussian or deterministic initial condition, a normal distribution for both long and short burns;

(ii) the mean of this normal distribution is in either case the same as the solution of the corresponding deterministic noise-free problem;

(iii) the covariance matrix is the same in both cases; it is derived in an explicit form.

Also, we note that these conclusions remain valid when the random noise is treated as a Markov vector process rather than the simple white noise vector process, see e.g. Soong (1973). Such a choice of a model would have, of course, to be justified by a study of a given propulsion system. The pertinent formulas describing the stochastic evolution could easily be obtained by generalizing the analysis of this paper.

Finally, we note that the study presented here provides a basis for analysis of a nonlinear case under random forcing, that is presently under our consideration.

## Appendix I

In the following we list the components of mean position and velocity of the rocket (vide (2.4)); $m_{it}$ and $x_{i0}$ denote the $i$th components of $\underset{\sim}{m}_t(s, \underset{\sim}{x})$ and $\underset{\sim}{x}_0$, respectively.

$$\begin{aligned} m_{1t}(s,\underset{\sim}{x}) = {} & \left(-3x_{10} - \frac{2}{n}\,x_{40} + \frac{2}{n^2}\,\bar{a}_x\right) \cos nt + \\ & + \left(\frac{1}{n}\,x_{20} + \frac{3}{2}\,\frac{\bar{a}_y}{n^2}\right) \sin nt - \\ & - \frac{3}{2}\,\frac{\bar{a}_y}{n}\,t \cos nt + \frac{3}{2}\,\frac{\bar{a}_x}{n}\,t \sin nt + \\ & + \frac{2}{n}\,(x_{40} + 2nx_{10}) - 2\,\frac{\bar{a}_x}{n^2} \end{aligned} \tag{I.1}$$

$$m_{2t}(s,\underset{\sim}{x}) = \dot{m}_{1t}(s,\underset{\sim}{x}) \tag{I.2}$$

$$m_{3t}(s,\underset{\sim}{x}) = 2\left(3x_{10} + \frac{2}{n}\,x_{40} - \frac{3}{n^2}\,\bar{a}_x\right) \sin nt -$$

$$- \left(\frac{2}{n} x_{20} + \frac{5\bar{a}_y}{n^2}\right)(1 - \cos nt) +$$

$$+ 3 \frac{\bar{a}_y}{n} t \sin nt + 3 \frac{\bar{a}_x}{n} t \cos nt +$$

$$- 3\left(2nx_{10} + x_{40} - \frac{\bar{a}_x}{n}\right) t + x_{30} \tag{I.3}$$

$$m_{4t}(s, \underset{\sim}{x}) = \dot{m}_{3t}(s, \underset{\sim}{x}) \tag{I.4}$$

$$m_{5t}(s, \underset{\sim}{x}) = \left(x_{50} - \frac{\bar{a}_z}{n^2}\right) \cos nt + \frac{x_{60}}{n} \sin nt + \frac{\bar{a}_z}{n^2} \tag{I.5}$$

$$m_{6t}(s, \underset{\sim}{x}) = \dot{m}_{5t}(s, \underset{\sim}{x}) . \tag{I.6}$$

## Appendix II

In the following we list the components of $\underset{\approx}{K}_{s+t}(s, \underset{\sim}{x})$ calculated for the special case $D_{11} = D_{22}$ and $D_{12} = D_{13} = D_{23} = 0$

$$\underset{\approx}{K}_{s+t}(s, \underset{\sim}{x}) = \begin{bmatrix} K_{11}, & 0, & 0, & K_{14}, & 0, & 0 \\ 0, & K_{22}, & K_{23}, & 0, & 0, & 0 \\ 0, & K_{32}, & 0, & 0, & 0, & 0 \\ K_{41}, & 0, & 0, & K_{44}, & 0, & 0 \\ 0, & 0, & 0, & 0, & K_{55}, & 0 \\ 0, & 0, & 0, & 0, & 0, & 0 \end{bmatrix} \tag{II.1}$$

in which

$$\begin{aligned} K_{11} &= \frac{D_{11}}{2}(e^{2t} - 1) \\ K_{22} &= \frac{D_{11}}{4n}(e^{4nt} - 1) \\ K_{44} &= \frac{D_{11}}{-4n}(e^{-4nt} - 1) \\ K_{55} &= \frac{D_{33}}{2}(e^{2t} - 1) \\ K_{14} = K_{41} &= \frac{D_{11}}{(1 - 2n)}(e^{(1-2n)t} - 1) \\ K_{23} = K_{32} &= \frac{D_{11}}{(1 - 2n)}(e^{(1+2n)t} - 1) . \end{aligned} \tag{II.1$'$}$$

## References

Kaplan, M.H.: 1976, *Modern Spacecraft Dynamics and Control*, John Wiley & Sons, New York.

Arnold, L.: 1974, *Stochastic Differential Equations*, John Wiley & Sons, New York.

Longuski, J.M. and McRonald, A.D.: 1988, An Analytic-Geometric Model of the Effect of Spherically Distributed Injection Errors for Galileo and Ulysses Spacecraft: The Multi-Stage Problem, *AIAA/AAS Astrodynamics Conference*, Minneapolis, MN, AIAA Paper No. 88-4245-CP.

Longuski, J.M.: 1987, An Analytic-Geometric Model of the Effect of Spherically Distributed Injection Errors for the Galileo and Ulysses Spacecraft, *AAS/AIAA Astrodynamics Specialists Conference*, Kalispell, MT, AAS Paper No. 87-487.

Longuski, J.M.: 1992, Probabilities of Escape, Reentry and Orbit Decay Due to Misdirected Injection Maneuvers, *Journal of Guidance, Control and Dynamics* **15**, 410.

Soong, T.T.: 1973, *Random Differential Equations in Science and Engineering*, Academic Press, New York.