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REAL SOLUTIONS FOR THE ATTITUDE MOTION OF A SELF-EXCITED RIGID BODY†

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Abstract—Approximate real analytic solutions are derived for the motion of a near-symmetric rigid body subject to constant body-fixed moments about three axes. The solution for Euler's equations of motion is expressed in terms of Fresnel integrals and is exact for symmetric bodies. An approximate solution for the Eulerian angles is found in terms of Fresnel integrals and sine and cosine integrals. Although the expressions for the Eulerian angles are complicated, the behavior of the angular momentum vector in inertial space exhibits a simple spiral path. Numerical examples reveal that the solutions are very accurate when applied to typical spinning spacecraft.

1. INTRODUCTION

In the past several years there has been much interest in analytic solutions for the motion of spinning spacecraft. In[1] an exact solution is obtained for the free motion of a dual-spin spacecraft. Reference[2] presents a closed-form solution for linearized equations in which transverse torques appear, but the spin rate is constant. An exact analytic solution for Euler's equations of motion for a symmetric rigid body subject to constant body-fixed torques about three axes is given in[3] and[4], however the "exact solution", presented for the orientation of the body in inertial space, is incorrect in these references for reasons explained in[5]. Reference[6] provides very useful observations for limiting cases for symmetric bodies, but does not give explicit solutions for the attitude motion. In[7] a perturbation solution for attitude motion under body-fixed torques is derived, based on the ratio of transverse to spin rotation rate as the small parameter. The method is limited to selected time intervals, so it has short term validity.

In this paper, explicit real solutions are derived for the attitude motion of a self-excited rigid body. The organization is as follows. First, an approximate analytic solution of Euler's equations of motion is presented for a near-symmetric rigid body subject to arbitrary constant body-fixed torques. This solution reduces to Bödewadt's exact solution[3] and[4] when the body is symmetric. Second, the corresponding analytic solution for the Eulerian angles is derived. The main restrictions are that two of the Eulerian angles are small and the parameter $|\dot{\omega}_z|/\omega_z^2$ must remain small. Third, the applications of the analytic solutions to such problems as the behavior[8] and control[9] of the angular momentum vector in inertial space and the inertial velocity imparted to a rigid body subject to body-fixed forces[10] are briefly discussed. Finally, a numerical example demonstrates the accuracy of the analytical solutions for a practical problem (a spinning interplanetary spacecraft).

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2. SOLUTION OF EULER'S EQUATIONS OF MOTION

Euler's equations of motion for a rigid body with principal axes at the center of mass are

$$M_x = I_x\dot{\omega}_x + (I_z - I_y)\omega_y\omega_z \tag{1a}$$

$$M_y = I_y\dot{\omega}_y + (I_x - I_z)\omega_z\omega_x \tag{1b}$$

$$M_z = I_z\dot{\omega}_z + (I_y - I_x)\omega_x\omega_y. \tag{1c}$$

These are coupled, nonlinear differential equations with no general analytical solution. Assume that an analytical solution is available for ω_z as a known function of time

$$\omega_z = f(t). \tag{2}$$

Then (1) is reduced to

$$\dot{x} + \lambda_1 f(t) y = c \tag{3a}$$

$$\dot{y} - \lambda_2 f(t) x = d \tag{3b}$$

where

$$x = \omega_x, \quad y = \omega_y, \quad \lambda_1 = (I_z - I_y)/I_x,$$

$$\lambda_2 = (I_z - I_x)/I_y, \quad c = M_x/I_x, \quad d = M_y/I_y.$$

The homogeneous equation in matrix form is

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & -\lambda_1 f \\ \lambda_2 f & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{4}$$

or

$$\dot{\mathbf{X}} = \mathbf{AX}. \tag{5}$$

The matrix A commutes with its integral. The system is integrable and can be transformed into a linear system with constant coefficients.

Using the time transformation[11]

$$\tau = \sqrt{\lambda_1\lambda_2}\int_0^t f(\xi)\,\mathrm{d}\xi \tag{6}$$

the system (3) becomes

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} + \sqrt{\frac{\lambda_1}{\lambda_2}}y = \frac{c}{\sqrt{\lambda_1\lambda_2}f(t)} \tag{7a}$$

$$\frac{\mathrm{d}y}{\mathrm{d}\tau} - \sqrt{\frac{\lambda_2}{\lambda_1}}x = \frac{d}{\sqrt{\lambda_1\lambda_2}f(t)}. \tag{7b}$$

So far no assumptions have been made about the transverse torques M_x and M_y. Assuming they are constants, then c and d are constants and (7) simplifies to the harmonic oscillator equation with forcing function F.

$$\frac{\mathrm{d}^2x}{\mathrm{d}\tau^2} + x = -\frac{d}{\lambda_2 f} - \frac{c\dot{f}}{\lambda_1\lambda_2 f^3} = F(\tau). \tag{8}$$

The equation in y is unnecessary since

$$y = -\sqrt{\frac{\lambda_2}{\lambda_1}}\frac{\mathrm{d}x}{\mathrm{d}\tau} + \frac{c}{\lambda_1 f(t)}. \tag{9}$$

Also, from (3) it is clear that (3a) permutes to (3b) by $x \to y$ and $y \to -(\lambda_2/\lambda_1)x$. Applying the same rule to (3b) provides (3a) where it is noted that d becomes $-(\lambda_2/\lambda_1)c$.

Up to this point no comment has been made as to the nature of $f(t)$. For a constant torque about the z body axis, M_z, the spin rate is nearly linear for a near-symmetric rigid body ($I_x \approx I_y$)

$$\omega_z \approx at + b = f(t) \tag{10}$$

where

$$a = M_z/I_z, \quad b = \omega_{z0}.$$

For stable vehicles (such as rockets and spacecraft) spinning about a principal axis of minimum or maximum moment of inertia this assumption is very reasonable since the Euler velocities ω_x and ω_y tend to remain small so that the last term of (1c) can be ignored.

Without loss of generality, assume that

$$a > 0. \tag{11}$$

For the case of negative torque, M_z, the direction of the z axis can be reversed to maintain (11). The sign of the initial spin rate, b, is of course, arbitrary.

With (10), the independent variable, τ, becomes

$$\tau = \sqrt{\lambda_1\lambda_2}(1/2\,at^2 + bt) \tag{12}$$

from (6). The forcing function (8) is

$$F(\tau) = -\frac{d}{\lambda_2(at+b)} - \frac{ca}{\lambda_1\lambda_2(at+b)^3}. \tag{13}$$

Putting

$$s = \tau + \sqrt{\lambda_1\lambda_2}\frac{b^2}{2a} = \sqrt{\lambda_1\lambda_2}\left(1/2\,at^2 + bt + \frac{b^2}{2a}\right) \tag{14}$$

gives

$$\frac{2as}{\sqrt{\lambda_1\lambda_2}} = (at+b)^2. \tag{15}$$

Care must be exercised in writing $at + b$ in terms of s because the quantity inside the parenthesis of (15) may be negative. Let

$$at + b = u\sqrt{\frac{2as}{\sqrt{\lambda_1\lambda_2}}} = u\sqrt{\frac{2as}{\lambda}} \tag{16}$$

where

$u = 1$ for spin up (a and b same sign),
$u = -1$ for spin down (a and b opposite sign) and only for $0 \leqslant t \leqslant -b/a$, and
$\lambda = \sqrt{\lambda_1\lambda_2}$.

The forcing function can now be written as a function of s and (8) becomes

$$\frac{\mathrm{d}^2x}{\mathrm{d}s^2} + x = u\left(\frac{A}{\sqrt{s}} + \frac{B}{\sqrt{s^3}}\right) \tag{17}$$

where

$$A = \frac{-d}{\sqrt{2a\lambda}}\sqrt{\frac{\lambda_1}{\lambda_2}}, \; B = \frac{-c}{2\sqrt{2a\lambda}}.$$

The solution takes the form

$$x(s) = A_x\cos\alpha(s-s_0) + B_x\sin\alpha(s-s_0) + \frac{1}{\alpha}\int_{s_0}^{s}F(\xi)\sin\alpha(s-\xi)\,\mathrm{d}\xi \tag{18}$$

where

$$s_0 = \lambda b^2/2a \quad \text{and} \quad \alpha = 1.$$

In evaluating the solution, the initial condition for $\dot{x}$ is found from (3a) at $t = 0$

$$\dot{x}(t=0) = c - \lambda_1 f(0)y(0). \tag{19}$$

The final solution for x is

$$\begin{aligned} x(t) = {} & x_0\cos\tau - \sqrt{\frac{\lambda_1}{\lambda_2}}y_0\sin\tau \\ & + \frac{u}{\sqrt{2a\lambda}}\tilde{S}\left[\sqrt{\frac{\lambda_1}{\lambda_2}}d\cos s + c\sin s\right] \\ & + \frac{u}{\sqrt{2a\lambda}}\tilde{C}\left[c\cos s - \sqrt{\frac{\lambda_1}{\lambda_2}}d\sin s\right] \end{aligned} \tag{20}$$

where

$$\tilde{C} = \int_{s_0}^{s}\frac{\cos t}{\sqrt{t}}\,\mathrm{d}t = \sqrt{2\pi}\left[C\left(\sqrt{\frac{2}{\pi}s}\right) - C\left(\sqrt{\frac{2}{\pi}s_0}\right)\right]$$

$$\tilde{S} = \int_{s_0}^{s}\frac{\sin t}{\sqrt{t}}\,\mathrm{d}t = \sqrt{2\pi}\left[S\left(\sqrt{\frac{2}{\pi}s}\right) - S\left(\sqrt{\frac{2}{\pi}s_0}\right)\right]$$

and

$$C(z) = \int_0^z \cos\left(\frac{\pi}{2}t^2\right)\mathrm{d}t, \; S(z) = \int_0^z \sin\left(\frac{\pi}{2}t^2\right)\mathrm{d}t.$$

The Fresnel integrals C and S are discussed in[12]. For a symmetric rigid body ($I_x = I_y$), the solution (20) reduces to Bödewadt's exact solution[3, 4]. To obtain the explicit solution for y, all terms in x are permuted to $y(x_0 \to y_0, c \to d)$ and all terms in y are permuted to $-\lambda_2 x/\lambda_1 (y_0 \to -\lambda_2 x_0/\lambda_1, d \to -\lambda_2 c/\lambda_1)$.

3. SOLUTION FOR THE EULERIAN ANGLES

A Type 1:3–1–2 Euler angle sequence is used for the kinematic equations[13]. This means that the Eulerian angles (ϕ_x, ϕ_y, ϕ_z) are defined by successive rotations by angles ϕ_z, ϕ_x, and ϕ_y about the z, x' and y'' coordinate axes. The resulting kinematic equations are

$$\dot{\phi}_x = \omega_x \cos\phi_y + \omega_z \sin\phi_y \tag{21a}$$

$$\dot{\phi}_y = \omega_y - (\omega_z \cos\phi_y - \omega_x \sin\phi_y)\tan\phi_x \tag{21b}$$

$$\dot{\phi}_z = (\omega_z \cos\phi_y - \omega_x \sin\phi_y)\sec\phi_x. \tag{21c}$$

No exact general solution exists for these nonlinear equations in the case of the self-excited rigid body. However, for many applications, small angle approximations for ϕ_x and ϕ_y are appropriate, giving

$$\dot{\phi}_x = \omega_x + \phi_y\omega_z \tag{22a}$$

$$\dot{\phi}_y = \omega_y - \phi_x\omega_z \tag{22b}$$

$$\dot{\phi}_z = \omega_z - \phi_y\omega_x. \tag{22c}$$

It will also be assumed that $\phi_y\omega_x$ is small compared to ω_z so that (22c) becomes

$$\dot{\phi}_z = \omega_z = at + b. \tag{23}$$

Integrating (23)

$$\phi_z = \tfrac{1}{2}at^2 + bt + \phi_{z0} = \tau/\lambda + \phi_{z0}. \tag{24}$$

Note that (22a) and (22b) are independent of ϕ_z. If a more precise solution for ϕ_z is desired, it may be possible to reinstate the ignored term $-\phi_y\omega_x$ in (22c) as a perturbation.

Using the independent variable, s, introduced in (14) reduces the system (22a) and (22b) to

$$\frac{d^2\phi_x}{ds^2} + \frac{1}{\lambda^2}\phi_x = \frac{u}{\sqrt{2a\lambda}}\left\{\frac{\omega_y(s)}{\lambda\sqrt{s}} + \frac{d}{ds}\left[\frac{\omega_x(s)}{\sqrt{s}}\right]\right\}. \tag{25}$$

As before, the equation for ϕ_y is unnecessary because, from (22a)

$$\phi_y = (\dot{\phi}_x - \omega_x)/\omega_z. \tag{26}$$

The equation in ϕ_y can be obtained from (25) by permuting terms in $x \to y$ and terms in $y \to -x$.

The forcing function of the harmonic oscillator of (25) is more complicated than that of (17), because the terms $\omega_x(s)$ and $\omega_y(s)$ appear, which are composed of Fresnel integrals. The solution takes the form of (18) with $\alpha = 1/\lambda$. The initial conditions link ϕ_x and ϕ_y through (22a) and (22b). After integration by parts and evaluation of the initial conditions, the form of the solution for $\phi_x(t)$ becomes

$$\begin{aligned}\phi_x(t) &= \phi_{x0}\cos(\tau/\lambda) + \phi_{y0}\sin(\tau/\lambda) \\ &+ \frac{u}{\sqrt{2a\lambda}}\int_{s_0}^{s}\frac{\omega_y(\xi)}{\sqrt{\xi}}\sin\frac{1}{\lambda}(s-\xi)\,d\xi \\ &+ \frac{u}{\sqrt{2a\lambda}}\int_{s_0}^{s}\frac{\omega_x(\xi)}{\sqrt{\xi}}\cos\frac{1}{\lambda}(s-\xi)\,d\xi\end{aligned} \tag{27}$$

where ϕ_{x0} and ϕ_{y0} refer to initial conditions at $t = 0$.

The evaluation of the integrals in (27) results in certain terms which cannot be readily integrated and so they are evaluated by asymptotic expansions. In order to make the labor systematic, the problem is divided into those integrals that are known and those that are unknown. This is seen most clearly by replacing the Fresnel integrals, which appear in ω_x and ω_y, by the auxiliary f and g functions[12]

$$C(x) = \tfrac{1}{2} + f(x)\sin(\tfrac{1}{2}\pi x^2) - g(x)\cos(\tfrac{1}{2}\pi x^2) \tag{28a}$$

$$S(x) = \tfrac{1}{2} - f(x)\cos(\tfrac{1}{2}\pi x^2) - g(x)\sin(\tfrac{1}{2}\pi x^2). \tag{28b}$$

Then the expressions for $\omega_x(\xi)$ and $\omega_y(\xi)$ can be written

$$\omega_x(\xi) = k_{x1}c_1 + k_{x2}s_1 + k_{x3}c_2 + k_{x4}s_2 + k_{x5}f + k_{x6}g \tag{29a}$$

$$\omega_y(\xi) = k_{y1}c_1 + k_{y2}s_1 + k_{y3}c_2 + k_{y4}s_2 + k_{y5}f + k_{y6}g \tag{29b}$$

where

$$k_{x1} = \omega_{x0}, \quad k_{x2} = -\sqrt{\frac{\lambda_1}{\lambda_2}}\omega_{y0}$$

$$k_{x3} = u\sqrt{\frac{\pi}{a}}\left[\sqrt{\frac{\lambda_1}{\lambda_2}}(\tfrac{1}{2} - S_0) + c(\tfrac{1}{2} - C_0)\right]$$

$$k_{x4} = u\sqrt{\frac{\pi}{a}}\left[c(\tfrac{1}{2} - S_0) - \sqrt{\frac{\lambda_1}{\lambda_2}}(\tfrac{1}{2} - C_0)\right]$$

$$k_{x5} = -u\sqrt{\frac{\pi}{a}}\left[\sqrt{\frac{\lambda_1}{\lambda_2}}d\right], \quad k_{x6} = -u\sqrt{\frac{\pi}{a}}c$$

$$k_{y1} = \omega_{y0}, \quad k_{y2} = \sqrt{\frac{\lambda_1}{\lambda_2}}\omega_{x0}$$

$$k_{y3} = u\sqrt{\frac{\pi}{a}}\left[-\sqrt{\frac{\lambda_2}{\lambda_1}}(\tfrac{1}{2} - S_0) + d(\tfrac{1}{2} - C_0)\right]$$

$$k_{y4} = u\sqrt{\frac{\pi}{a}}\left[d(\tfrac{1}{2} - S_0) + \sqrt{\frac{\lambda_2}{\lambda_1}}(\tfrac{1}{2} - C_0)\right]$$

$$k_{y5} = u\sqrt{\frac{\pi}{a}}\left[\sqrt{\frac{\lambda_2}{\lambda_1}}c\right], k_{y6} = -u\sqrt{\frac{\pi}{a}}d$$

$$c_1 = \cos(\xi - s_0), s_1 = \sin(\xi - s_0)$$

$$c_2 = \cos\xi, s_2 = \sin\xi$$

$$C_0 = C\left(\sqrt{\frac{2}{\pi}s_0}\right), \quad S_0 = S\left(\sqrt{\frac{2}{\pi}s_0}\right)$$

$$g = g\left(\sqrt{\frac{2}{\pi}\xi}\right), \quad f = f\left(\sqrt{\frac{2}{\pi}\xi}\right)$$

$$s_0 = \lambda b^2/2a.$$

The k_ys can be deduced from the k_xs, but are included here for the reader's convenience.

From (27), the following integrals are identified for further evaluation

$$W_{xc}(s) = \int_{s_0}^{s} \frac{\omega_x(\xi)}{\sqrt{\xi}} \cos\frac{1}{\lambda}(s-\xi)\,\mathrm{d}\xi \tag{30a}$$

$$W_{ys}(s) = \int_{s_0}^{s} \frac{\omega_y(\xi)}{\sqrt{\xi}} \sin\frac{1}{\lambda}(s-\xi)\,\mathrm{d}\xi. \tag{30b}$$

Inspection of (29) and (30) reveals that the following integrals must be found

$$J_{cc}(s, k_1, k_2, k_3, k_4) = \int_{s_0}^{s} \frac{\cos(k_1\xi + k_2)\cos(k_3\xi + k_4)}{\sqrt{\xi}}\,\mathrm{d}\xi \tag{31a}$$

$$J_{cs}(s, k_1, k_2, k_3, k_4) = \int_{s_0}^{s} \frac{\cos(k_1\xi + k_2)\sin(k_3\xi + k_4)}{\sqrt{\xi}}\,\mathrm{d}\xi \tag{31b}$$

$$J_{ss}(s, k_1, k_2, k_3, k_4) = \int_{s_0}^{s} \frac{\sin(k_1\xi + k_2)\sin(k_3\xi + k_4)}{\sqrt{\xi}}\,\mathrm{d}\xi \tag{31c}$$

$$F_c(s) = \int_{s_0}^{s} \frac{f\left(\sqrt{\frac{2}{\pi}\xi}\right)\cos\frac{1}{\lambda}(s-\xi)}{\sqrt{\xi}}\,\mathrm{d}\xi \tag{32a}$$

$$F_s(s) = \int_{s_0}^{s} \frac{f\left(\sqrt{\frac{2}{\pi}\xi}\right)\sin\frac{1}{\lambda}(s-\xi)}{\sqrt{\xi}}\,\mathrm{d}\xi \tag{32b}$$

$$G_c(s) = \int_{s_0}^{s} \frac{g\left(\sqrt{\frac{2}{\pi}\xi}\right)\cos\frac{1}{\lambda}(s-\xi)}{\sqrt{\xi}}\,\mathrm{d}\xi \tag{32c}$$

$$G_s(s) = \int_{s_0}^{s} \frac{g\left(\sqrt{\frac{2}{\pi}\xi}\right)\sin\frac{1}{\lambda}(s-\xi)}{\sqrt{\xi}}\,\mathrm{d}\xi. \tag{32d}$$

The integrals W_{xc} and W_{ys} of (30) can be written in terms of the Js, Fs and Gs of (31) and (32).

$$\begin{aligned} W_{xc} = {} & k_{x1}J_{cc}(s, 1, -s_0, -1/\lambda, s/\lambda) \\ & + k_{x2}J_{cs}(s, -1/\lambda, s/\lambda, 1, -s_0) \\ & + k_{x3}J_{cc}(s, 1, 0, -1/\lambda, s/\lambda) \\ & + k_{x4}J_{cs}(s, -1/\lambda, s/\lambda, 1, 0) \\ & + k_{x5}F_c(s) + k_{x6}G_c(s) \end{aligned} \tag{33a}$$

$$\begin{aligned} W_{ys} = {} & k_{y1}J_{cs}(s, 1, -s_0, -1/\lambda, s/\lambda) \\ & + k_{y2}J_{ss}(s, 1, -s_0, -1/\lambda, s/\lambda) \\ & + k_{y3}J_{cs}(s, 1, 0, -1/\lambda, s/\lambda) \\ & + k_{y4}J_{ss}(s, 1, 0, -1/\lambda, s/\lambda) \\ & + k_{y5}F_s(s) + k_{y6}G_s(s). \end{aligned} \tag{33b}$$

Define the integrals L_c and L_s:

$$L_c(s, h_1, h_2) = \frac{1}{2}\int_{s_0}^{s} \frac{\cos(h_1\xi + h_2)}{\sqrt{\xi}}\,\mathrm{d}\xi \tag{34a}$$

$$L_s(s, h_1, h_2) = \frac{1}{2}\int_{s_0}^{s} \frac{\sin(h_1\xi + h_2)}{\sqrt{\xi}}\,\mathrm{d}\xi. \tag{34b}$$

Then, by well-known trigonometric identities

$$\begin{aligned} J_{cc}(s, k_1, k_2, k_3, k_4) = {} & L_c(s, k_1 + k_3, k_2 + k_4) \\ & + L_c(s, k_1 - k_3, k_2 - k_4) \end{aligned} \tag{35a}$$

$$\begin{aligned} J_{cs}(s, k_1, k_2, k_3, k_4) = {} & L_s(s, k_1 + k_3, k_2 + k_4) \\ & + L_s(s, k_3 - k_1, k_4 - k_2) \end{aligned} \tag{35b}$$

$$\begin{aligned} J_{ss}(s, k_1, k_2, k_3, k_4) = {} & -L_c(s, k_1 + k_3, k_2 + k_4) \\ & + L_c(s, k_1 - k_3, k_2 - k_4). \end{aligned} \tag{35c}$$

The integrals L_c and L_s can be expressed solely in terms of the Fresnel integrals C_2 and S_2

$$\begin{aligned} L_c = \sqrt{\frac{\pi}{2|h_1|}} \{ & \cos h_2[C_2(|h_1|s) - C_2(|h_1|s_0)] \\ & - \operatorname{sgn} h_1 \sin h_2[S_2(|h_1|s) - S_2(|h_1|s_0)]\} \end{aligned} \tag{36a}$$

$$\begin{aligned} L_s = \sqrt{\frac{\pi}{2|h_1|}} \{ & \cos h_2 \operatorname{sgn} h_1[S_2(|h_1|s) - S_2(|h_1|s_0)] \\ & + \sin h_2[C_2(|h_1|s) - C_2(|h_1|s_0)]\} \end{aligned} \tag{36b}$$

where

$$C_2(x) = \frac{1}{\sqrt{2\pi}}\int_0^x \frac{\cos t}{\sqrt{t}}\,\mathrm{d}t$$

$$S_2(x) = \frac{1}{\sqrt{2\pi}}\int_0^x \frac{\sin t}{\sqrt{t}}\,\mathrm{d}t$$

and

$$C_2\left(\frac{\pi}{2}x^2\right) = C(x)$$

$$S_2\left(\frac{\pi}{2}x^2\right) = S(x).$$

Thus, the integrals J_{cc}, J_{cs}, and J_{ss} are known integrals, since they are explicit functions of the Fresnel integrals.

Next, the unknown integrals F_c, F_s, G_c and G_s are evaluated by asymptotic expansions since they cannot be expressed explicitly in terms of known functions.

The asymptotic expansions of the f and g functions are[12]

$$\pi z f(z) \sim 1 + \sum_{m=1}^{\infty} (-1)^m \frac{1 \cdot 3 \cdot \ldots (4m-1)}{(\pi z^2)^{2m}} \tag{37a}$$

$$\pi z g(z) \sim \sum_{m=0}^{\infty} (-1)^m \frac{1 \cdot 3 \cdot \ldots (4m+1)}{(\pi z^2)^{2m+1}}. \tag{37b}$$

Only two terms of the expansions will be used

$$f(z) \approx \frac{1}{\pi z} - \frac{3}{\pi^3 z^5} \tag{38a}$$

$$g(z) \approx \frac{1}{\pi^2 z^3} - \frac{15}{\pi^4 z^7}. \tag{38b}$$

Substitution of (38) into (32) yields the approximations for F_c, F_s, G_c, and G_s:

$$F_c(s) \approx \frac{1}{\sqrt{2\pi}} \left\{ \cos\frac{s}{\lambda} \cos_1\left(\frac{s_0}{\lambda}, \frac{s}{\lambda}\right) + \sin\frac{s}{\lambda} \sin_1\left(\frac{s_0}{\lambda}, \frac{s}{\lambda}\right) - \frac{3}{(2\lambda)^2}\left[\cos\frac{s}{\lambda}\cos_3\left(\frac{s_0}{\lambda}, \frac{s}{\lambda}\right) + \sin\frac{s}{\lambda}\sin_3\left(\frac{s_0}{\lambda}, \frac{s}{\lambda}\right)\right]\right\} \tag{39a}$$

$$F_s(s) \approx \frac{1}{\sqrt{2\pi}} \left\{ \sin\frac{s}{\lambda} \cos_1\left(\frac{s_0}{\lambda}, \frac{s}{\lambda}\right) - \cos\frac{s}{\lambda} \sin_1\left(\frac{s_0}{\lambda}, \frac{s}{\lambda}\right) - \frac{3}{(2\lambda)^2}\left[\sin\frac{s}{\lambda}\cos_3\left(\frac{s_0}{\lambda}, \frac{s}{\lambda}\right) - \cos\frac{s}{\lambda}\sin_3\left(\frac{s_0}{\lambda}, \frac{s}{\lambda}\right)\right]\right\} \tag{39b}$$

$$G_c(s) \approx \frac{1}{2\lambda\sqrt{2\pi}} \left\{ \cos\frac{s}{\lambda} \cos_2\left(\frac{s_0}{\lambda}, \frac{s}{\lambda}\right) + \sin\frac{s}{\lambda} \sin_2\left(\frac{s_0}{\lambda}, \frac{s}{\lambda}\right) - \frac{15}{(2\lambda)^2} \times \left[\cos\frac{s}{\lambda}\cos_4\left(\frac{s_0}{\lambda}, \frac{s}{\lambda}\right) + \sin\frac{s}{\lambda}\sin_4\left(\frac{s_0}{\lambda}, \frac{s}{\lambda}\right)\right]\right\} \tag{39c}$$

$$G_s(s) \approx \frac{1}{2\lambda\sqrt{2\pi}} \left\{ \sin\frac{s}{\lambda} \cos_2\left(\frac{s_0}{\lambda}, \frac{s}{\lambda}\right) - \cos\frac{s}{\lambda} \sin_2\left(\frac{s_0}{\lambda}, \frac{s}{\lambda}\right) - \frac{15}{(2\lambda)^2}\left[\sin\frac{s}{\lambda}\cos_4\left(\frac{s_0}{\lambda}, \frac{s}{\lambda}\right) - \cos\frac{s}{\lambda}\sin_4\left(\frac{s_0}{\lambda}, \frac{s}{\lambda}\right)\right]\right\} \tag{39d}$$

where the following definitions have been used

$$\cos_m(x_0, x_1) \equiv \int_{x_0}^{x_1} \frac{\cos t}{t^m} \, dt \tag{40a}$$

$$\sin_m(x_0, x_1) \equiv \int_{x_0}^{x_1} \frac{\sin t}{t^m} \, dt. \tag{40b}$$

The terms $\cos_m(x_0, x_1)$ and $\sin_m(x_0, x_1)$ can be reduced to $\cos_1(x_0, x_1)$ and $\sin_1(x_0, x_1)$ by repeated application of the formulas:

$$\cos_m(x_0, x_1) = \frac{1}{1-m}[x_1^{1-m}\cos x_1 - x_0^{1-m}\cos x_0] + \frac{1}{1-m}\sin_{m-1}(x_0, x_1) \tag{41a}$$

$$\sin_m(x_0, x_1) = \frac{1}{1-m}[x_1^{1-m}\sin x_1 - x_0^{1-m}\sin x_0] - \frac{1}{1-m}\cos_{m-1}(x_0, x_1). \tag{41b}$$

The functions $\cos_1(x_0, x_1)$ and $\sin_1(x_0, x_1)$ are expressible in terms of the sine and cosine integrals S_i and C_i[12]

$$\cos_1(x_0, x_1) = C_i(x_1) - C_i(x_0) \tag{42a}$$

$$\sin_1(x_0, x_1) = S_i(x_1) - S_i(x_0) \tag{42b}$$

where

$$C_i(z) \equiv \gamma + \ln(z) + \int_0^z \frac{\cos t - 1}{t} \, dt$$

$$S_i(z) \equiv \int_0^z \frac{\sin t}{t} \, dt$$

and

$$\gamma = \text{Euler's constant}(= 0.57721\ldots).$$

Thus, the final approximate solution for the kinematic eqns (21) when ϕ_x, ϕ_y and $\phi_y \omega_x$ are small is

$$\phi_x(t) = \phi_{x0}\cos\Delta\phi_z + \phi_{y0}\sin\Delta\phi_z + \frac{u}{\sqrt{2a\lambda}}[W_{xc}(s) + W_{ys}(s)] \tag{43a}$$

$$\phi_z(t) = \tfrac{1}{2}at^2 + bt + \phi_{z0} = \Delta\phi_z + \phi_{z0} \tag{43b}$$

where

$$s = \frac{\lambda}{2a}\omega_z^2(t)$$

and $W_{xc}(s)$ and $W_{ys}(s)$ are defined by relations (30)–(42). The solution for ϕ_y is found by symmetry with ϕ_x by permuting all terms in x to terms in y and all terms in y to terms in $-x$. Because of the asymptotic expansions used for the F and G integrals, the solution is limited to cases in which the parameter $|\dot{\omega}_z|/\omega_z^2$ is small compared to unity.

4. APPLICATIONS

4.1. Solution for the angular momentum vector

The solution for the components of the angular momentum vector in inertial space (H_X, H_Y, H_Z) is obtained from the analytical solutions for the Euler velocities $(\omega_x, \omega_y, \omega_z)$ and the Euler angles (ϕ_x, ϕ_y, ϕ_z) as follows:

$$\begin{pmatrix} H_X \\ H_Y \\ H_Z \end{pmatrix} = A \begin{pmatrix} I_x\omega_x \\ I_y\omega_y \\ I_z\omega_z \end{pmatrix} \tag{44}$$

where, for a Type 1: 3–1–2 Euler angle sequence, the matrix $\mathbf{A}$ is

$$\mathbf{A} = \begin{bmatrix} c\phi_z c\phi_y - s\phi_z s\phi_x s\phi_y & -s\phi_z c\phi_x & c\phi_z s\phi_y + s\phi_z s\phi_x c\phi_y \\ s\phi_z c\phi_y + c\phi_z s\phi_x s\phi_y & c\phi_z c\phi_x & s\phi_z s\phi_y - c\phi_z s\phi_x c\phi_y \\ -c\phi_x s\phi_y & s\phi_x & c\phi_x c\phi_y \end{bmatrix}$$

where c and s denote cosine and sine. This analytic solution has lead to an understanding of the behavior of the angular momentum vector during spinning-up maneuvers[8].

By assuming that ϕ_x and ϕ_y are small in (44) and replacing all Fresnel integrals and sine and cosine integrals in the analytical solutions for the Euler velocities and angles by first order asymptotic expansions, eqn (44) becomes

$$H_X \approx \frac{M_x}{\omega_z}\sin\phi_z + \frac{M_y}{\omega_z}\cos\phi_z - \frac{M_y\omega_z}{\omega_{z0}^2} \tag{45a}$$

$$H_Y \approx -\frac{M_x}{\omega_z}\cos\phi_z + \frac{M_y}{\omega_z}\sin\phi_z + \frac{M_x\omega_z}{\omega_{z0}^2} \tag{45b}$$

$$H_z \approx I_z\omega_z \tag{45c}$$

where the initial conditions have been assumed

$$H_Z(0) = I_z\omega_{z0},\ \omega_{x0} = \omega_{y0} = \phi_{x0} = \phi_{y0} = \phi_{z0} = 0.$$

This remarkably simple behavior in the form of a spiral in inertial space is depicted in Fig. 1 for the case of spinning up with $M_y = 0$.

In Fig. 1 the bias angle ρ_0 is a constant given by

$$\tan\rho_0 = \frac{\sqrt{M_x^2 + M_y^2}}{I_z\omega_{z0}^2} \tag{46}$$

for the general case ($M_x \neq 0$, $M_y \neq 0$). The angle θ of the spiral is given by the Euler angle, ϕ_z:

$$\theta = \phi_z. \tag{47}$$

An alternate derivation of (45) directly from Euler's equation $\mathbf{M} = \dot{\mathbf{H}}$ is presented in[9].

The insight gained from (45) leads to a new method of controlling the behavior of the angular momentum vector which is also described in[9]. For cases in which transverse body-fixed torques perturb the angular momentum vector from its initial orientation in inertial space, it is possible to perform the spin-up or thrusting maneuver in two segments with an intervening coast period. The coast period allows the orientation of body-fixed thrusters to change while the angular momentum vector remains fixed in inertial space. The second segment of the maneuver forces the angular momentum vector to encircle the original orientation position in space, instead of the offset orientation of angle ρ_0 in Fig. 1.

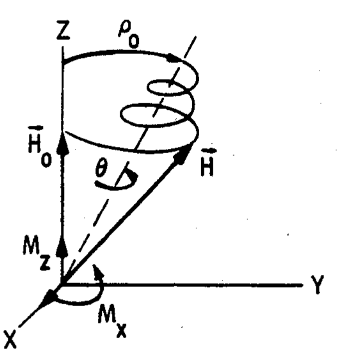

Fig. 1. Behavior of the angular momentum vector during spinning up.

4.2. Solution for the change of velocity

When body-fixed forces (f_x, f_y, f_z) are present, the rigid body accelerates according to

$$\begin{pmatrix} a_X \\ a_Y \\ a_Z \end{pmatrix} = \mathbf{A}\begin{pmatrix} f_x/m \\ f_y/m \\ f_z/m \end{pmatrix} \tag{48}$$

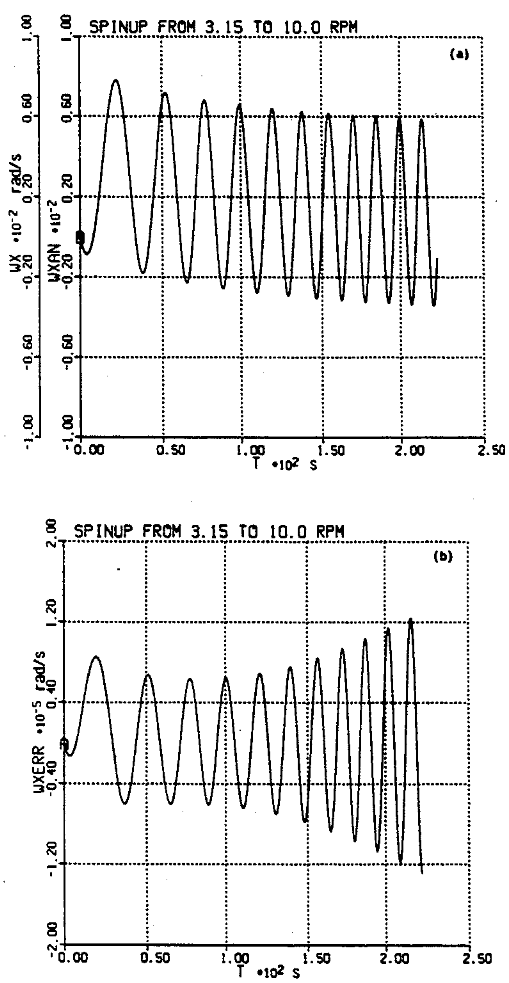

Fig. 2. (a) Exact and analytic solutions for $\omega_x(t)$. (b) Exact solution minus analytic solution.

where a_X, a_Y, a_Z are acceleration components in inertial space and matrix A is given in (44). When ϕ_x and ϕ_y are small, (48) becomes

$$a_X = \frac{f_x}{m}\cos\phi_z - \frac{f_y}{m}\sin\phi_z - \frac{f_z}{m}(\phi_y\cos\phi_z + \phi_x\sin\phi_z) \quad (49a)$$

$$a_Y = \frac{f_x}{m}\sin\phi_z + \frac{f_y}{m}\cos\phi_z + \frac{f_z}{m}(\phi_y\sin\phi_z - \phi_x\cos\phi_z) \quad (49b)$$

$$a_Z = -\frac{f_x}{m}\phi_y + \frac{f_y}{m}\phi_x + \frac{f_z}{m}. \quad (49c)$$

When f_x and f_y are constant and $f_z = 0$, the solution for the change of velocity in inertial space is approximated by[10]

$$\Delta V_X \approx (f_x/m)\bar{C} - (f_y/m)\bar{S} \quad (50a)$$

$$\Delta V_Y \approx (f_x/m)\bar{S} + (f_y/m)\bar{C} \quad (50b)$$

$$\Delta V_Z \approx 0 \quad (50c)$$

where

$$\bar{C} = \int_0^t \cos(\tfrac{1}{2}a\xi^2 + b\xi + \phi_{z0})\,d\xi$$

$$= \cos\left(\phi_{z0} - \frac{b^2}{2a}\right)\left\{\sqrt{\frac{\pi}{a}}\left[C\left(\sqrt{\frac{a}{\pi}}\left(t + \frac{b}{a}\right)\right) - C\left(\sqrt{\frac{a}{\pi}}\frac{b}{a}\right)\right]\right\} - \sin\left(\phi_{z0} - \frac{b^2}{2a}\right)\left\{\sqrt{\frac{\pi}{a}} \times \left[S\left(\sqrt{\frac{a}{\pi}}\left(t + \frac{b}{a}\right)\right) - S\left(\sqrt{\frac{a}{\pi}}\frac{b}{a}\right)\right]\right\}$$

$$\bar{S} = \int_0^t \sin(\tfrac{1}{2}a\xi^2 + b\xi + \phi_{z0})\,d\xi$$

$$= \cos\left(\phi_{z0} - \frac{b^2}{2a}\right)\left\{\sqrt{\frac{\pi}{a}}\left[S\left(\sqrt{\frac{a}{\pi}}\left(t + \frac{b}{a}\right)\right) - S\left(\sqrt{\frac{a}{\pi}}\frac{b}{a}\right)\right]\right\} + \sin\left(\phi_{z0} - \frac{b^2}{2a}\right)\left\{\sqrt{\frac{\pi}{a}} \times \left[C\left(\sqrt{\frac{a}{\pi}}\left(t + \frac{b}{a}\right)\right) - C\left(\sqrt{\frac{a}{\pi}}\frac{b}{a}\right)\right]\right\}$$

and where the initial conditions have been assumed

$$\omega_{x0} = \omega_{y0} = \phi_{x0} = \phi_{y0} = 0.$$

Using first order terms from the asymptotic expansions for the Fresnel integrals, reduces the transverse velocities to

$$\Delta V_X \approx \frac{f_x}{m}\left(\frac{\sin\phi_z}{\omega_z} - \frac{\sin\phi_{z0}}{\omega_{z0}}\right) - \frac{f_y}{m}\left(\frac{\cos\phi_z}{\omega_z} + \frac{\cos\phi_{z0}}{\omega_{z0}}\right) \quad (51a)$$

$$\Delta V_Y \approx -\frac{f_x}{m}\left(\frac{\cos\phi_z}{\omega_z} + \frac{\cos\phi_{z0}}{\omega_{z0}}\right) + \frac{f_y}{m}\left(\frac{\sin\phi_z}{\omega_z} - \frac{\sin\phi_{z0}}{\omega_{z0}}\right). \quad (51b)$$

For the case in which $f_z \neq 0$, secular terms appear in all three velocity components. It becomes important to retain the f_z terms in all of eqns (49) because ϕ_x and ϕ_y contain terms in $\cos\phi_z$ and $\sin\phi_z$ which lead to constants on the right-hand side. The secular solution for the velocity components is

$$\Delta V_X = \frac{-f_z}{m}\frac{M_y}{I_z\omega_{z0}^2}t \quad (52a)$$

$$\Delta V_Y = \frac{f_z}{m}\frac{M_x}{I_z\omega_{z0}^2}t \quad (52b)$$

$$\Delta V_Z = \frac{f_z}{m}t. \quad (52c)$$

Thus, the secular behavior of the velocity vector follows the bias angle of the angular momentum vector, due to the constant terms in (45a) and (45b). This is the reason that the control scheme for the angular momentum vector bias[9] also corrects the velocity pointing errors arising from the secular terms (52).

5. NUMERICAL EXAMPLE

The spin-up maneuver of the Galileo, an interplanetary spacecraft bound for Jupiter, is used as a numerical example. The following numbers are representative for the mass properties and body-fixed moments:

$$I_x = 2985\ \text{kg-m}^2,\ I_y = 2729\ \text{kg-m}^2,\ I_z = 4183\ \text{kg-m}^2 \quad (53a)$$

$$M_x = -1.253\ \text{N-m},\ M_y = -1.494\ \text{N-m},\ M_z = 13.5\ \text{N-m}. \quad (53b)$$

Normally a spacecraft is equipped with thruster couples in order to provide torque along the spin axis. In the case of the Galileo spacecraft only one thruster is available for spin-up about the z axis, and so, depending on the position of the center-of-mass (which varies along the z axis according to propellant loading), there may be transverse moments, M_x and M_y. This particular example has been an inspiration to the author to analyze the attitude motion of a self-excited rigid body.

To represent the spin-up maneuver, the following initial conditions are assumed

$$\omega_x(0) = 0,\ \omega_y(0) = 0,\ \omega_z(0) = 3.15\ \text{rpm} \quad (54a)$$

$$\phi_x(0) = \phi_y(0) = \phi_z(0) = 0 \quad (54b)$$

and the final spin rate is

$$\omega_z(t_f) = 10\ \text{rpm}. \quad (55)$$

The analytic solutions for the attitude motion are compared to "exact" solutions which are found by very precise numerical integrations of Euler's equations of motion (1) and the kinematic eqns (21).

Figure 2 compares the exact solution for $\omega_x(t)$ with the analytic solution (20). The Fresnel integrals were

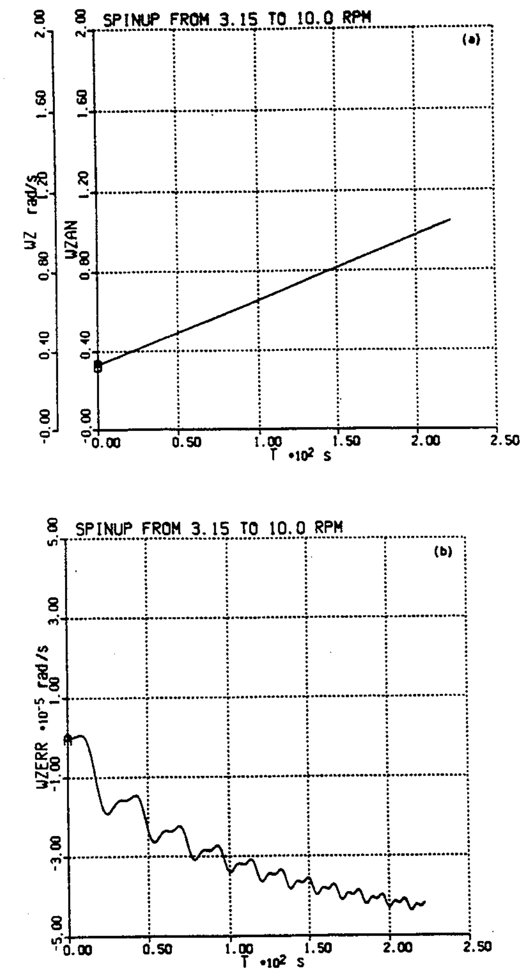

Fig. 3. (a) Exact and analytic solutions for $\omega_z(t)$. (b) Exact solution minus analytic solution.

computed from (28) and the two-term asymptotic expansions (38). In Fig. 2(a) both exact and analytic solutions are displayed, but the solutions are indistinguishable from one another.

In Fig. 2(b), the difference between exact and analytic solutions is presented (exact minus analytic). From the plot it is clear that the analytic solution for ω_x deviates from the exact solution by only about 0.1%. Bödewadt's solution[3, 4], which only applies to symmetric rigid bodies, gives an error of a few percent for this example[5]. Figure 3 demonstrates that the assumption that ω_z is a linear function of time (10) is reasonable, since the error indicates a discrepancy of only about 0.01% from the exact solution.

Although the Galileo spacecraft does not perform a spin-up manuever starting from an initial spin rate of zero, the analytic solutions (10) and (20) still apply and have been tested for this hypothetical case. Of course the asymptotic relations for the Fresnel integrals are no longer valid. Instead, rational approximations for the f and g functions may be used from[12], when the argument is zero or near zero. The results are nearly as accurate as those of Fig. 2. In this case it was noted, however, that most of the error was due to errors in computing the Fresnel integrals.

In Fig. 4, the exact solution for ϕ_x is compared to the analytic solution (43a), for conditions (54) and (55). Rational approximations for the sine and cosine integrals (42) were used from[12] in the analytic solution. The discrepancy between the exact and analytic solutions is not apparent in Fig. 4(a), but Fig. 4(b), which displays the difference (exact minus analytic) reveals that the error is of the order 0.5%. In Fig. 5 the analytic solution for ϕ_z (43b) is shown to be within about 0.01% of the exact solution.

The accuracy of the analytic solutions for the angular momentum vector (45) and the secular solution for change of velocity (52) is discussed in[9]. The solution for the change of transverse velocity (51) is discussed in[10].

6. CONCLUSIONS

Highly accurate analytic solutions for the attitude motion of a self-excited rigid body have been presented. These solutions can be useful in two ways. First, they provide a basis for computational algorithms which allow parametric studies to be

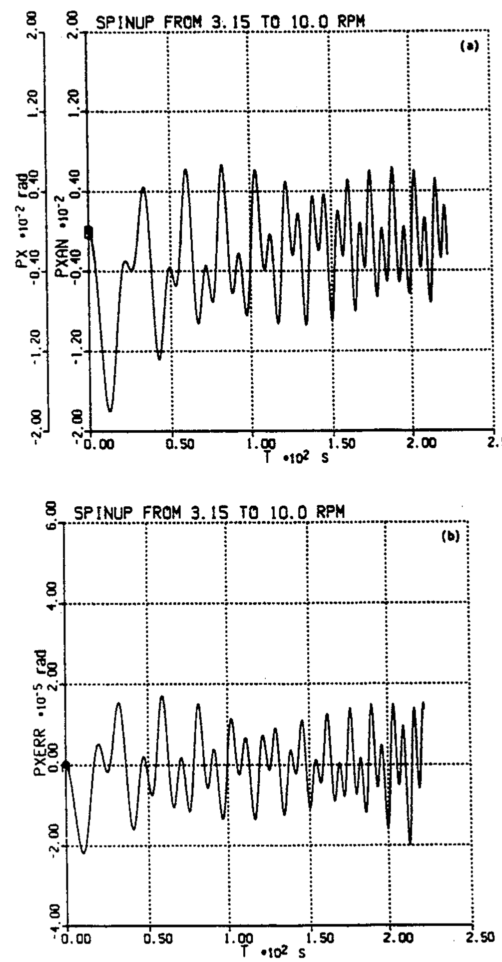

Fig. 4. (a) Exact and analytic solutions for $\phi_x(t)$. (b) Exact solution minus analytic solution.

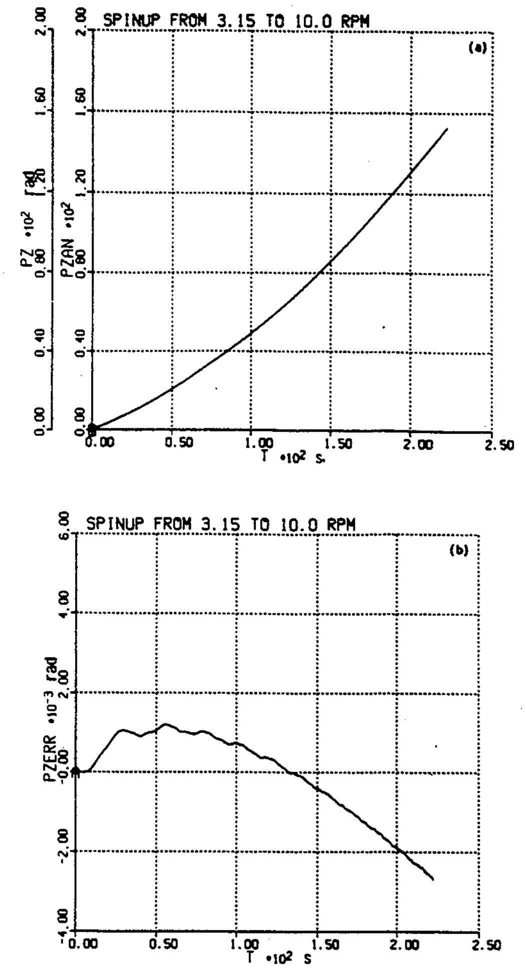

Fig. 5. (a) Exact and analytic solutions for $\phi_z(t)$. (b) Exact solution minus analytic solution.

performed. Second, they provide insight into the behavior of the attitude motion of a rigid body, subject to moments about all three axes. An obvious example is the understanding gained in the behavior of the angular momentum vector and new ideas about controlling it.

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