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On the Attitude Motion of a Self-Excited Rigid Body[1]

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Abstract

In 1965, Leimanis published a monograph containing both the Bödewadt solution of Euler's equations of motion of a symmetric rigid body subject to constant body-fixed torques, and the corresponding solution for the Eulerian angles of rotation. These analytic results, if accurate, could provide important tools for the analysis of currently planned spacecraft. Attempts to apply these solutions in the performance assessment of the Galileo spacecraft during spin up and spin down maneuvers have been made. Unfortunately, due to the fact that Galileo is not precisely symmetric, the solution for Euler's equations of motion fails to achieve the desired accuracy for useful analysis. In the case of the Eulerian angles, the analytic results are generally incorrect for theoretical reasons. To remedy this situation, analytic solutions have been developed which satisfy the criterion of high accuracy and provide a useful analytic tool for the performance assessment and maneuver analysis of the Galileo spacecraft and many similar nearly symmetric spinning spacecraft. Simulation results provide a comparison of the relative accuracies of the Bödewadt solution for Euler's equations of motion and the author's solution, and the restricted regions of validity of the Bödewadt solution for the Eulerian angles are clearly indicated.

Introduction

In 1965, Leimanis published [1], which contains Bödewadt's solution of Euler's equations of motion for a symmetric rigid body subject to a time-independent, self-excitement in a body-fixed direction, together with Bödewadt's solution for the corresponding angles of rotation. While the solution of Euler's equations of motion can be considered exact, it provides a poor approximation to the problem of nearly symmetric rigid bodies when a high degree of accuracy (less than 1 percent error) is required. A much more accurate approximate solution is provided in [2]. In the case of the Eulerian angles, the result given in [1] for arbitrary constant torques is incorrect. The approxi-

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mate solution given in [2], however, is extremely accurate and suitable for computer design work and error analysis of spacecraft performance.

In this paper the solutions are reviewed and the specific regions of validity of the solution for the Eulerian angles are pointed out.

Bödewadt's Solution of Euler's Equations of Motion

Euler's equations of motion for a rigid body with principal axes at the center of mass are

$$M_x = I_x\dot{\omega}_x + (I_z - I_y)\omega_y\omega_z \tag{1}$$

$$M_y = I_y\dot{\omega}_y + (I_x - I_z)\omega_z\omega_x \tag{2}$$

$$M_z = I_z\dot{\omega}_z + (I_y - I_x)\omega_x\omega_y \tag{3}$$

In 1952 Bödewadt obtained an exact analytic solution of these equations for a symmetric rigid body subject to arbitrary constant torques [3]. The solution allows for variable spin rate and is given in terms of the Fresnel integrals. Unfortunately, for nearly symmetric rigid bodies, such as the Galileo spacecraft, this solution does not provide great enough accuracy for performance analysis. This is especially true when the solution for the angular velocity components is used in the kinematic equations in order to find the Eulerian angles. This problem can be circumvented by the following assumption for nearly symmetric bodies:

$$\omega_z \approx (M_z/I_z)t + \omega_{z0} \tag{4}$$

When $I_x = I_y$, equation (4) is exact as in the Bödewadt solution. When $I_x \approx I_y$ the approximation provides very useful accurate solutions, particularly when ω_x and ω_y are small, which is usually the case for spin stabilized spacecraft. The solution for ω_x is [2]

$$\begin{aligned}\omega_x(t) = {} & \omega_{x0}\cos\tau_1 - (\lambda_1/\lambda_2)^{1/2}\omega_{y0}\sin\tau_1 \\ & + u \operatorname{sgn} a\, S|2a|^{-1/2}[(\lambda_1/\lambda_2)^{1/2} d\cos\tau_2 + c\operatorname{sgn} a \sin\tau_2] \\ & + C|2a|^{-1/2}[c\operatorname{sgn} a\cos\tau_2 - (\lambda_1/\lambda_2)^{1/2} d\sin\tau_2]\end{aligned} \tag{5}$$

where

$$\lambda_1 = (I_z - I_y)/I_x, \qquad \lambda_2 = (I_z - I_x)/I_y \tag{6}$$

$$a = (\lambda_1\lambda_2)^{1/2}M_z/I_z, \qquad b = (\lambda_1\lambda_2)^{1/2}\omega_{z0} \tag{7}$$

$$c = M_x/I_x, \qquad d = M_y/I_y \tag{8}$$

$$C = \int_{\tau_0}^{\tau_2} t^{-1/2}\cos t\,dt, \qquad S = \int_{\tau_0}^{\tau_2} t^{-1/2}\sin t\,dt \tag{9}$$

$$\tau_0 = b^2(2|a|)^{-1}, \qquad \tau_1 = (at^2/2) + bt \tag{10}$$

$$\tau_2 = [(at^2/2) + bt + b^2/(2a)]\operatorname{sgn} a \tag{11}$$

$$u = 1 \text{ for spin up } (a \text{ and } b \text{ same sign}) \tag{12}$$

$$u = -1 \text{ for spin down } (a \text{ and } b \text{ opposite signs) and only for } 0 \le t \le -b/a \tag{13}$$

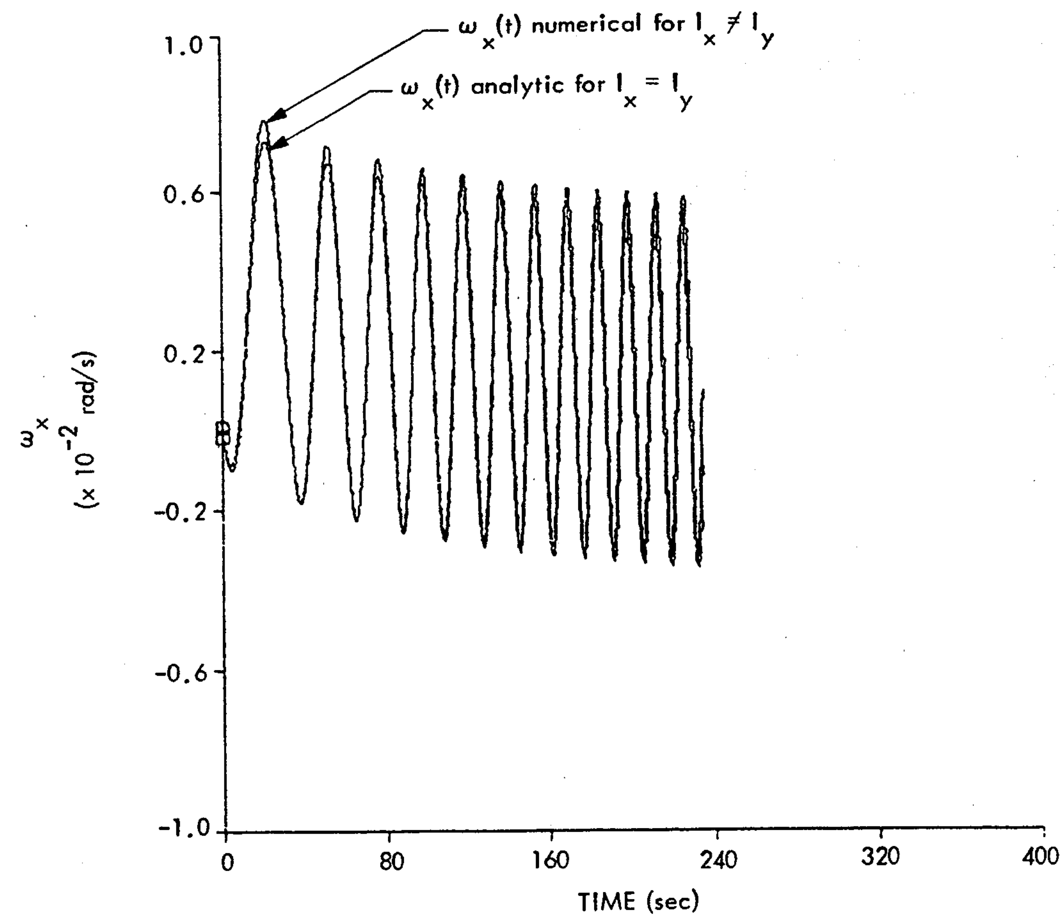

FIG. 1. Bödewadt Solution for Symmetric Body versus Numerical Solution for Nearly Symmetric Body.

The solution for ω_y is obtained from the solution for ω_x by interchanging $(\lambda_1/\lambda_2)^{1/2}$ and $-(\lambda_2/\lambda_1)^{1/2}$, c and d, and ω_{x0} and ω_{y0}. Note that these solutions retain the distinction between I_x and I_y, which is a trivial extension mathematically, but one which has important consequences numerically as shown in Figs. 1 and 2. It is also interesting to note the difference in the derivation of equation (5) in [2] from that of [1]. A simple nonlinear time transformation is used in [2] to decouple the linear first order differential equations with time varying coefficients into two separate linear second order differential equations with constant coefficients and time varying forcing functions. In [1], the Bödewadt solution is derived using the familiar complex variable approach with the definition

$$\omega = \omega_x + i\omega_y \tag{14}$$

The relative accuracy of the Bödewadt solution when applied to the Galileo spacecraft spin up maneuver is of the order of a few percent as shown in Fig. 1. The exact solution, ω_x numerical, was found by numerical integration using ACSL (Advanced Continuous Simulation Language) for the Galileo spacecraft parameters given in Table I.

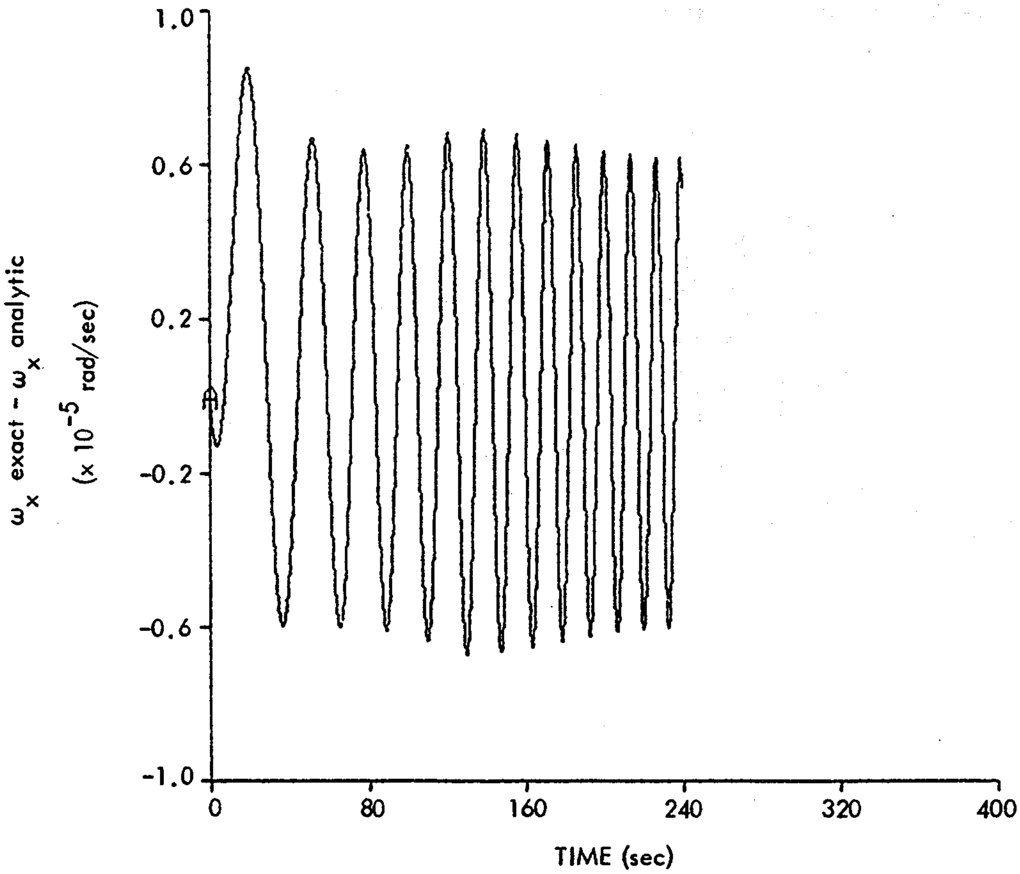

FIG. 2. Difference between Exact Numerical Solution (Case of Fig. 1) and Solution for Nearly Symmetric Body [equation (5)].

Bödewadt's solution was found using the relation

$$I_T = (I_x I_y)^{1/2} \tag{15}$$

to approximate the symmetric parameter.

When the result given by equation (5) is plotted, the accuracy is so high that the two plots, numerical and analytic, are indistinguishable, so this result is not shown. Fig. 2 shows the plot of ω_x exact minus ω_x analytic, which indicates a relative error of the order of one tenth of one percent.

Bödewadt's Solution for the Eulerian Angles

In [1], Leimanis presented Bödewadt's solution [3] for the angles of rotation of a rigid body subject to a time-independent, self-excitement with a body-fixed direction. The fundamental equation is

$$\dot{A} = AW \tag{16}$$

where W is the affinor of rotation

$$W = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \tag{17}$$

TABLE 1. Galileo Spacecraft Parameters

I_x	2985 kg m^2
I_y	2729 kg m^2
I_z	4183 kg m^2
M_x	−1.253 Nm
M_y	−1.494 Nm
M_z	13.5 Nm
ω_{x0}	0
ω_{y0}	0
ω_{z0}	0.33 rad/s
ω_{zf}	1.047 rad/s

and ω_x, ω_y, and ω_z are the body fixed angular velocity components. A is the tensor which transforms body fixed coordinates to the inertial coordinates, the solution of which gives the Eulerian angles. Bödewadt proposed a solution of equation (16) of the form

$$A = A(t_0)e^U \tag{18}$$

where

$$U = \begin{bmatrix} 0 & -\gamma & \beta \\ \gamma & 0 & -\alpha \\ -\beta & \alpha & 0 \end{bmatrix} \tag{19}$$

and

$$\alpha = \int_{t_0}^{t} \omega_x \, dt, \qquad \beta = \int_{t_0}^{t} \omega_y \, dt, \qquad \gamma = \int_{t_0}^{t} \omega_z \, dt \tag{20}$$

It is well-known that in such a case, the solution given by equation (18) is only valid when W and U commute. This is easily verified by substituting from equation (18) into equation (16).

The conditions for which W and U commute and equation (18) is correct are

$$\omega_x\beta = \omega_y\alpha \tag{21}$$

$$\omega_x\gamma = \omega_z\alpha \tag{22}$$

$$\omega_y\gamma = \omega_z\beta \tag{23}$$

Clearly when ω_x, ω_y, and ω_z are constant, conditions (21)–(23) are satisfied. These conditions can be stated in a number of interesting ways, for example,

$$\omega_x/\alpha = \omega_y/\beta = \omega_z/\gamma \quad \text{(for } \alpha, \beta, \gamma \text{ nonzero)} \tag{24}$$

or

$$\omega(t) \times \int_{t_0}^{t} \omega(\tau)\, d\tau = 0 \tag{25}$$

for all values of t_0, and t. By breaking the integral in equation (25) into parts, it is easy to see that

$$\boldsymbol{\omega}(t) \times \int_{t_0}^{t_1} \boldsymbol{\omega}(\tau)\, d\tau = 0 \tag{26}$$

Since equation (25) is also valid for $t = t_1$ then

$$\boldsymbol{\omega}(t) \times \boldsymbol{\omega}(t_1) = 0 \tag{27}$$

for all t and t_1.

Wertz [4] observes that equation (18) is a solution when

$$\boldsymbol{\omega}(t) = \omega(t)\mathbf{e} \tag{28}$$

where $\mathbf{e}$ is a constant vector. This is equivalent to equation (27). Thus, the direction of the spin vector, $\boldsymbol{\omega}$, must not change.

Now the special cases of Bödewadt can be reviewed.

Case 1: No Spin

In this case, for symmetric rigid bodies, Bödewadt shows that the solution of Euler's equations of motion, equations (1)–(3), is

$$\omega_x = (M_x/I_T)t + \omega_{x0} \tag{29}$$

$$\omega_y = (M_y/I_T)t + \omega_{y0} \tag{30}$$

$$\omega_z = 0 \tag{31}$$

when

$$I_x = I_y = I_T \tag{32}$$

$$\omega_{z0} = 0 \tag{33}$$

$$M_z = 0 \tag{34}$$

The parameters α, β, and γ, of equations (20) are

$$\alpha = M_x(2I_T)^{-1}(t^2 - t_0^2) + \omega_{x0}(t - t_0) \tag{35}$$

$$\beta = M_y(2I_T)^{-1}(t^2 - t_0^2) + \omega_{y0}(t - t_0) \tag{36}$$

$$\gamma = 0 \tag{37}$$

Substitution from equations (29)–(31) and (35)–(37) into equations (21)–(23) [or substitution from equations (29)–(31) into equation (27)] shows that the condition

$$M_x\omega_{y0} = M_y\omega_{x0} \tag{38}$$

must be satisfied for equation (18) to represent a solution, and not, in general, as supposed in [1].

Case 2: Constant Spin

In this case Bödewadt assumes that

$$I_x = I_y = I_T \tag{39}$$

$$\omega_z = \omega_{z0} \neq 0 \tag{40}$$

$$M_z = 0 \tag{41}$$

so that

$$\omega_x = e \cos \Omega t + f \sin \Omega t + g \tag{42}$$

$$\omega_y = -f \cos \Omega t + e \sin \Omega t + h \tag{43}$$

where

$$e = \omega_{x0} - g\,, \qquad f = -\omega_{y0} + h \tag{44}$$

$$g = -M_y(I_T\Omega)^{-1}, \qquad h = M_x(I_T\Omega)^{-1} \tag{45}$$

$$\lambda = (I_z - I_T)/I_T\,, \qquad \Omega = \omega_{z0}\lambda \tag{46}$$

By integrating equations (42) and (43) and using equations (21)–(23), it can be shown that equation (18) is a solution only when

$$\omega_x = \omega_{x0} = -M_y[(I_z - I_T)\omega_{z0}]^{-1} \tag{47}$$

$$\omega_y = \omega_{y0} = M_x[(I_z - I_T)\omega_{z0}]^{-1} \tag{48}$$

which is the case of constant angular velocity for all time; obviously equations (47) and (48) satisfy equation (27).

Case 3: Linearly Varying Spin

In this case, Bödewadt assumes that

$$I_x = I_y = I_T \tag{49}$$

$$M_x \neq 0\,, \qquad M_y \neq 0\,, \qquad M_z \neq 0 \tag{50}$$

$$\omega_z = (M_z/I_z)t + \omega_{z0} \tag{51}$$

The solution has already been discussed, and the value for ω_x can be found by applying equations (49)–(51) to equation (5). Once again by using equations (21)–(23) (the α, β, γ terms are given in [1] and [3]), or by employing equation (27), it can be shown that equation (18) is a solution for a very limited set of conditions, namely

$$\omega_x = \omega_y = 0 \tag{52}$$

but these conditions violate equations (49)–(51), since they imply that

$$M_x = M_y = 0 \tag{53}$$

Therefore, Case 3 cannot be solved by the method of equation (18) in general. The only related situation which can be solved this way is the trivial one in which equation (53) is used so that the only allowable torque is about the spin axis. To solve this problem one does not need to use equation (18), since the kinematic equations can be integrated directly:

$$\phi_x = \phi_{x0} \tag{54}$$

$$\phi_y = \phi_{y0} \tag{55}$$

$$\phi_z = (1/2)(M_z/I_z)t^2 + \omega_{z0}t + \phi_{z0} \tag{56}$$

where ϕ_x, ϕ_y, ϕ_z can be any set of Eulerian angles.

Case 4: Kinetic Symmetry

Bödewadt makes the following assumption:

$$I_x = I_y = I_z = I \tag{57}$$

so that

$$\omega_x = (M_x/I)t + \omega_{x0} \tag{58}$$

$$\omega_y = (M_y/I)t + \omega_{y0} \tag{59}$$

$$\omega_z = (M_z/I)t + \omega_{z0} \tag{60}$$

Integrating equations (58)–(60) to obtain α, β, and γ and using equations (21)–(23) indicates that

$$M_x\omega_{y0} = M_y\omega_{x0} \tag{61}$$

$$M_x\omega_{z0} = M_z\omega_{x0} \tag{62}$$

$$M_y\omega_{z0} = M_z\omega_{y0} \tag{63}$$

in order for the solution, equation (18), to be true. Equations (61)–(63) can be rearranged as

$$\omega_{x0}/M_x = \omega_{y0}/M_y = \omega_{z0}/M_z \tag{64}$$

but this particular form gives undefined results for cases where conditions (61)–(63) do not. Note that equations (61)–(63) and (64) are equivalent to $\boldsymbol{\omega}_0 \times \mathbf{M} = 0$ and consequently to $\boldsymbol{\omega} \times \mathbf{M} = 0$.

Solution for the Eulerian Angles

When equations (21)–(23) are satisfied, as for the specified conditions above, the solution for the Eulerian angles is facilitated by Bödewadt's equation,

$$e^U = I + U\nu^{-1}\sin\nu + U^2\nu^{-2}(1 - \cos\nu) \tag{65}$$

where

$$\nu^2 = \alpha^2 + \beta^2 + \gamma^2 \tag{66}$$

A can then be found from equation (18), and the individual Eulerian angles can be identified.

An Approximate Solution for the Eulerian Angles for a Near Symmetric Rigid Body Subject to Constant Moments

An approximate solution for the Eulerian angles has been found corresponding to Case 3 of Bödewadt [2]. The main restriction in the solution is that two of the Eulerian angles [ϕ_x and ϕ_y in equations (68)–(69)] must remain small so that, if ϕ is one of the two, the approximation

$$\sin\phi \approx \phi \tag{67}$$

holds. The form of the solution (for Type 3-1-2 Eulerian angles as defined in [4]) is

$$\phi_x(t) = \phi_{x0} \cos \mu\tau_1 + \phi_{y0} \sin \mu\tau_1 + u \operatorname{sgn} a[W_{ys}(\tau_1) + W_{xc}(\tau_1)] \tag{68}$$

$$\phi_y(t) = \phi_{y0} \cos \mu\tau_1 - \phi_{x0} \sin \mu\tau_1 + u \operatorname{sgn} a[W_{yc}(\tau_1) + W_{xs}(\tau_1)] \tag{69}$$

$$\phi_z(t) = M_z(2I_z)^{-1}t^2 + \omega_{z0}t + \phi_{z0} \tag{70}$$

where

$$\mu = (\lambda_1\lambda_2)^{-1/2} \tag{71}$$

and the notation of equation (5) is used. The W_{ys} and W_{xc} functions are

$$W_{ys}(\tau_1) = \int_0^{\tau_1} [\omega_y(\xi) \sin(\mu\tau_1 - \mu\xi)](b^2 + 2a\xi)^{-1/2} d\xi \tag{73}$$

$$W_{xc}(\tau_1) = \int_0^{\tau_1} [\omega_x(\xi) \cos(\mu\tau_1 - \mu\xi)](b^2 + 2a\xi)^{-1/2} d\xi \tag{72}$$

where ω_x and ω_y are given in equation (5). W_{yc} and W_{xs} are defined analogously. The details of writing these integrals in terms of Fresnel, sine and cosine integrals and elementary functions are given in [2].

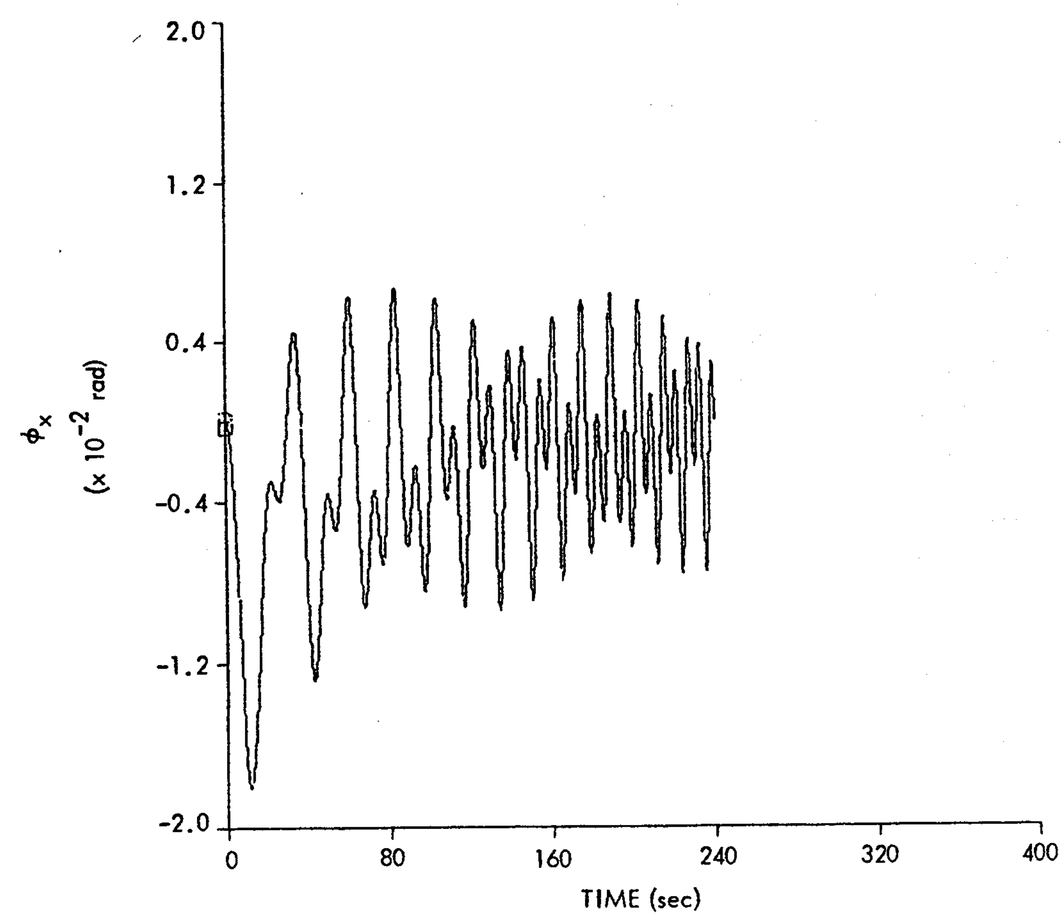

FIG. 3. Approximate Analytic Solution for the Eulerian Angle, ϕ_x

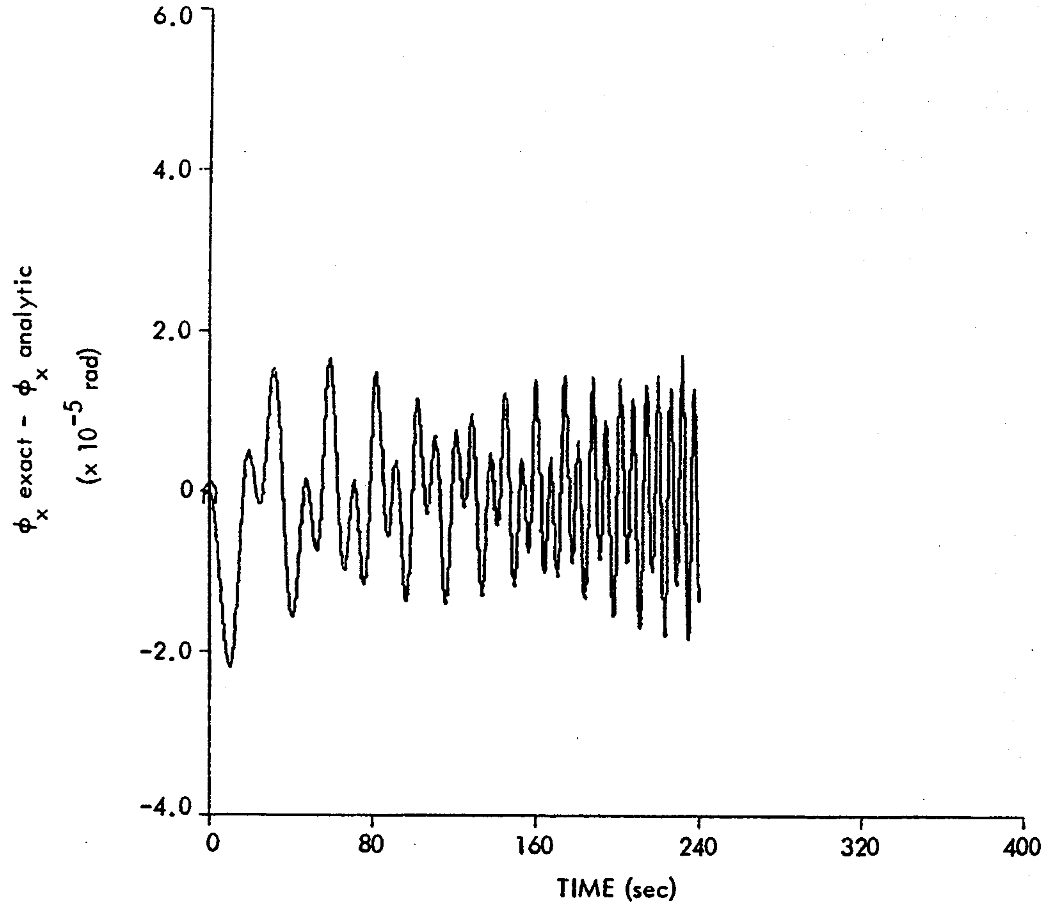

FIG. 4. Exact Minus Analytic Solution for the Eulerian Angle ϕ_x

Results

Figure 3 contains plots of both numerical and analytic solutions for $\phi_x(t)$ for the Galileo spin up maneuver using the same parameters used to obtain Figs. 1 and 2. The discrepancy between the plots is indiscernible, so Fig. 4 is presented to indicate the difference between the solutions. The relative error in the analytic solution is of the order of one half of one percent. The error in the $\phi_z(t)$ solution (not shown) is of the order of one hundredth of one percent. Clearly, the approximation given by equation (4) was instrumental in obtaining such accurate results for the nearly symmetric case.

The analytic solution, equations (68)–(70), is currently being used to study the motion of the Galileo spacecraft during spin up maneuvers. Of particular interest is the orientation of the angular momentum vector at the end of the maneuver (see [5]), which is found through equations (5) and (68)–(70). Since error analysis is being done, several cases must be simulated in order to find the variance of the final condition. If the analytic solution were not available, this type of study would be extremely time consuming and expensive.

Conclusions

While the Bödewadt solution of Euler's equations of motion are exact for the symmetric case, they cannot provide useful approximate solutions for the more typical situation of nearly symmetric bodies found in many spacecraft applications. An extremely accurate approximate analytic solution similar in form has been independently derived by the author.

The Bödewadt solution for the Eulerian angles is much more restricted than originally supposed. It is incorrect for the case of variable spin rate. However, a highly accurate approximate solution has been found for the case in which two of the Eulerian angles are small.

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