Reduced Parameterization for Optimization of Low-Thrust Gravity-Assist Trajectories: Case Studies

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Low-thrust trajectories can be modeled as a series of impulsive (DV) maneuvers connected by conic arcs. We study new ways of parameterizing the DV vectors with a reduced number of variables and constraints. When optimizing low-thrust gravity-assist trajectories, the DV magnitudes can be parameterized with switch on/off times of the engine; the steering angles can be parameterized with coefficients of a Chebyshev series. We present numerical results for several missions, including: Earth-Jupiter rendezvous, Earth-Mars-Vesta flyby, Earth-Mercury rendezvous and a seven-synodic-period Earth-Mars roundtrip mission. In most of these cases, we found significant improvements in convergence speed (with acceptable accuracy) with the new formulations.

Nomenclature

k_N = Number of On/Off-Nodes
k_θ = Degree of a Chebyshev series for modeling the clock angle θ
k_ψ = Degree of a Chebyshev series for modeling the clock angle ψ
m = Number of nonlinear constraints
N = Number of segments on a low-thrust trajectory
n = Number of optimization variables
D**V** = Impulsive maneuver vector, km/s
ΔV = Magnitude of a maneuver, km/s
θ = Clock angle of a maneuver, deg
ψ = Cone angle of a maneuver, deg

I. Introduction

INTERPLANETARY missions via electric propulsion (EP) have been proposed in the literature for many years,[1-6] with destinations that include Mercury,[7-9] Mars,[10-11] Jupiter,[12-16] the Outer Planets,[17-21] and the Heliospheric boundary of the solar system.[22] In 1998, NASA launched Deep Space 1[23] which successfully demonstrated the first use of ion propulsion in an interplanetary mission. With a specific impulse of about ten times that of a chemical rocket,[24] Deep Space 1's (solar-powered) propulsion system provides a large saving in propellant mass (and thus mission costs). Following Deep Space 1, four missions are expected to further demonstrate the effectiveness of electric propulsion. SMART-1[25] was launched in 2003 and is orbiting around the Moon while Hayabusa[26] was launched in that same year as a sample return mission from an asteroid. The DAWN[27] mission is planned to be launched to Ceres and Vesta in 2007, and BepiColombo[9] is scheduled for launch in 2013 to visit Mercury.

With its high efficiency, electric propulsion (compared with chemical propulsion) can help increase the payload mass and reduce the trip time of a mission. However, the design of trajectories for electric propulsion missions is often more challenging than missions using chemical propulsion. Acceleration of a spacecraft propelled by EP is usually very small and the thrust duration can be a significant portion of the total time-of-flight (TOF). For low-thrust missions, the control parameter is the spacecraft's thrust vector as a function of time, which is infinite dimensional.

In optimizing low-thrust trajectories, we vary the thrust vector (as a function of time) to find a solution with minimum propellant spent (or with maximum final mass). Typically, there are two approaches to solve a low-thrust trajectory optimization problem: indirect or direct.

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Indirect methods[28-30] are based on calculus of variations, which often lead to a two-point boundary-value problem. Problems solved by indirect methods are usually very sensitive to their initial guesses, especially for trajectories with multiple gravity-assists. Indirect methods also require an initial guess for the Lagrange multipliers (which may not have any physical meaning) to satisfy the terminal constraints.

On the other hand, direct methods parameterize the problem to a set of finite variables, subject to the constraints of the mission (e.g. maximum allowable thrust). We can use nonlinear programming techniques to solve the direct problem. Comparatively, it is often less sensitive to the initial guesses of the variables. A variety of the direct method techniques have been studied in the literature.[31-34] In 1999, Sims and Flanagan[35] proposed a direct method for optimizing low-thrust trajectories. Several papers[10,11,15-17,20,21] have been written based on this model on the design and optimization of low-thrust, gravity-assist (LTGA) trajectories to various destinations in the Solar System.

The size of the problem can vary quite a lot with the direct optimization method. For example, in the Sims and Flanagan model, the number of variables, n, and the number of constraints, m, can range from $n = 100$ and $m = 50$ for an Earth-Jupiter direct mission, to $n = 2000$ and $m = 800$ for a 45-year Earth-Mars Cycler.[10] Often, the computational run time increases with the problem size.

In this paper, we study new ways to parameterize low-thrust, gravity-assist trajectories with a reduced number of variables and constraints. Our goal is to save computation time in the trajectory design process. Even if we have to sacrifice some accuracy, a reduced parameterization of the problem can be very valuable for mission designers, especially for those studying a broad design space[15,16] (e.g. launch period, TOF, hardware parameters of low-thrust missions to Jupiter).

We follow the work of Ref. 36 and present optimized results using the new formulations. We compare the speed, accuracy, and robustness of the new parameterizations with the original method.

II. Trajectory Model

Figure 1 illustrates the trajectory model proposed by Sims and Flanagan.[35] Trajectory is divided into legs which begin and end with a planet or a control node (a control node can be any point in space). Low-thrust arcs on each leg are modeled as sequences of impulsive maneuvers (D**V**), connected by conic arcs. The D**V** at each segment should not exceed a maximum magnitude, ΔV_{max}, where ΔV_{max} is the velocity change accumulated by the spacecraft when it is operated at full thrust during that segment. At each leg, trajectory is propagated (with a two-body model) forward and backward to a matchpoint (usually halfway through a leg). The forward- and backward-propagated half-legs should meet at the matchpoint in order to have a feasible trajectory. We assume planetary flybys happen instantaneously.

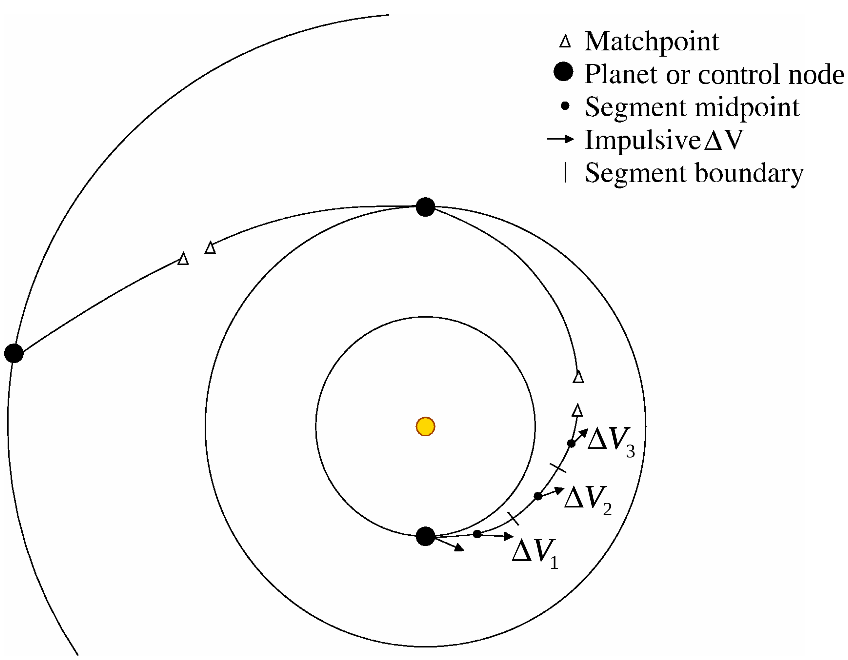

Figure 1: Trajectory model (after Sims and Flanagan[35]).

III. Optimization Problem and Software

Our LTGA trajectory optimization program, GALLOP[17,37] (which stands for Gravity-Assist Low-thrust Local Optimization Program), is written based on the Sims and Flanagan model. The objective of GALLOP is to either maximize the final mass or minimize the initial mass of the spacecraft; subjected to the constraints on the D**V** magnitude (where $\Delta V \leq \Delta V_{max}$), the matchpoint (where the position, velocity, and mass of the spacecraft should be continuous), and the flyby altitude (where it should above a minimum height at a flyby body). Additional constraints can be added such as total time-of-flight (TOF), $\mathbf{V}_\infty$ magnitude, and encounter dates of a body. Variables in GALLOP includes (but not limited to): D**V** on each segment, Julian dates at each planetary encounter, states of the spacecraft at each control node (which includes position, velocity, mass, and time), and the encounter conditions at each body (such as the $\mathbf{V}_\infty$ magnitude, flyby altitude, and B-plane angle).

GALLOP uses a nonlinear programming software called NPOPT, which is a wrapper for a sparse nonlinear optimizer SNOPT,[38,39] to solve the constrained nonlinear optimization problem. SNOPT implements sequential quadratic programming (SQP) to find a *locally* optimal solution. We can specify some parameters for NPOPT which determines when to end an optimization run. These parameters include the major iteration limit, the major feasibility tolerance, ε_{FEA}, and the major optimality tolerance, ε_{OPT}. (For a detailed explanation of these parameters, please see Ref. 39.) The major feasibility tolerance specifies how tightly the constraints should be satisfied, while the major optimality tolerance specifies closeness to the optimality conditions. Unless otherwise stated, we set both ε_{FEA} and ε_{OPT} to 10^{-6} (the default in NPOPT).

IV. Parameterizations of the DV

The D**V** on each segment can be written in many Coordinate systems,[40] such as Spherical Coordinates and Cartesian Coordinates. In this paper, we assume the D**V** is expressed in Spherical Coordinates. We present four ways to parameterize the D**V**.[36] Although we use GALLOP, these parameterizations may also be applied to other low-thrust (with or without gravity-assist) trajectory optimization programs.

A. N-Vector (Original) Formulation

As the "Original" formulation, N-Vector uses the D**V** coordinates on each segment, namely ΔV_i, θ_i, ψ_i, as optimization variables, where i runs from 1 to N. For a trajectory of N segments, there are 3N variables on the D**V** (not including other variables such as $\mathbf{V}_\infty$) and N nonlinear constraints on ΔV_i (where $\Delta V_i \leq \Delta V_{max,i}$).

B. Node (On/Off-Node) Formulation

The Node formulation replaces the N D**V** magnitudes into a set of On/Off-Nodes. If we assume a low-thrust trajectory follows a sequence of Maximum-Thrust (MT) and Null-Thrust (NT) arcs, then an On-Node defines the switching point from NT to MT while at an Off-Node the spacecraft switches from thrusting to coasting. As an example, in Fig. 2 for an Earth-Jupiter rendezvous mission, we can assume the trajectory follows an MT-NT-MT (or thrust-coast-thrust) scheme. An Off-Node and an On-Node can be placed at the 34th and the 54th segment, respectively.

We can impose linear constraints on the node epochs such that the order of the On/Off-Nodes remains unchanged. In the case where an On-Node merges with an Off-Node, the two corresponding MT arcs combines to become one and the NT arc vanishes. For trajectories with multiple thrust pulses, the number of nodes should be carefully selected such that it has a fair representation of the thrust profile.

There are 8 optimization variables (position, velocity, mass, and time of the spacecraft) which corresponds to each On/Off-Node. For a trajectory with k_N nodes, there are in total $8k_N + 2N$ optimization variables on the D**V**. The constraints on the **ΔV** magnitudes are also removed since the ΔV are either at maximum or zero (i.e. it cannot violate the constraint of $\Delta V \leq \Delta V_{max}$). As compared with N-Vector formulation, however, there are constraints on the additional matchpoint(s) between each On-Node and Off-Node, since each extra On/Off-Node adds a leg (and hence a matchpoint) to the trajectory optimization problem.

C. Chebyshev Formulation

In the Chebyshev formulation, for each planet-to-planet leg, we model the D**V** angles (θ and ψ) as Chebyshev series:[41]

$$c_0 T_0(u) + c_1 T_1(u) + \ldots + c_k T_k(u) \tag{1}$$

where $T_k(u)$ is the Chebyshev polynomial of degree k; and u, as the independent variable of the Chebyshev polynomial, is the leg-duration normalized to span from -1 to 1. Figure 3 shows a schematic of the modeling.

The parameters associated with the D**V** angles are the coefficients on the Chebyshev series, $\{c_0, c_1, \ldots, c_k\}$. If the degree of Chebyshev series on θ and ψ are k_θ and k_ψ, respectively, then the number of variables on the D**V** is $N + k_\theta + k_\psi + 2$. The degree of the Chebyshev series should be chosen carefully such that it can follow the shape of the D**V** angles. In general, the higher the degree, the more robust the modeling is. However, increasing the degree may not always improve the modeling accuracy, especially when the angles jump through different quadrants. We should pick a coordinate frame (e.g. inertial frame, a spacecraft rotating frame like V-C-N or r-θ-h) for the D**V** angles such that it can be modeled easily by a Chebyshev series.

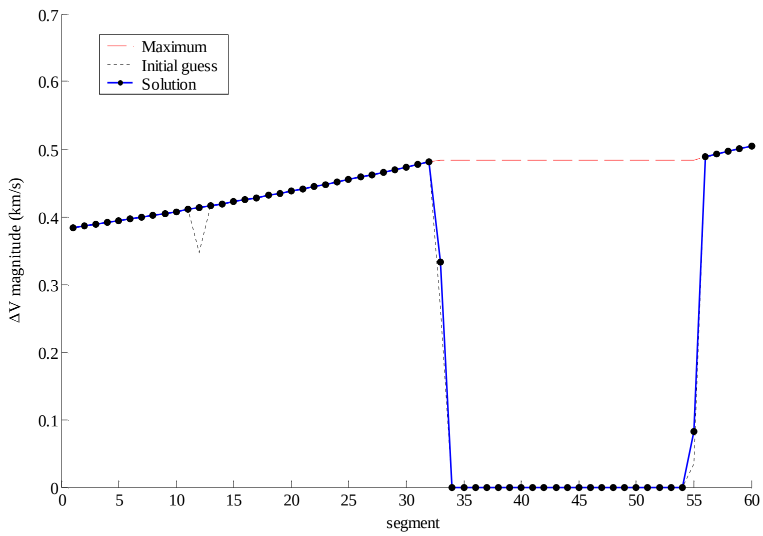

Figure 2: ΔV magnitude of an Earth-Jupiter rendezvous mission.

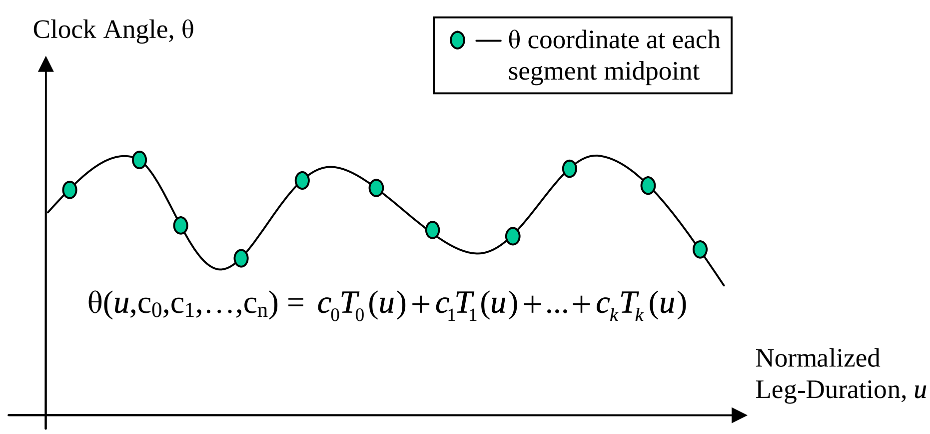

Figure 3: DV** angle written as Chebyshev series.**

D. Node + Chebyshev (N.C.) Formulation

The Node + Chebyshev (N.C.) formulation is the combination of the Node formulation and the Chebyshev formulation. It uses On/Off-Nodes to parameterize the D**V** magnitude, while the D**V** angles are parameterized with coefficients of Chebyshev series. With the On/Off-Nodes, a trajectory using N.C. is divided into a set of thrusting and coasting arcs. Unlike the Chebyshev formulation, for each individual thrusting arc, there is one, for each angle θ and ψ, Chebyshev series modeling the D**V** angles. For coasting arcs, no variable on the D**V** (magnitude and directions) is required. Using the Node + Chebyshev formulation, the number of variables on the D**V** becomes $8k_N + k_\theta + k_\psi + 2$ and the nonlinear constraints on the D**V** magnitudes are removed.

V. Numerical Results

We present results optimized by the four formulations in this section. Unless otherwise stated, our objective is to maximize the final mass of the spacecraft and the D**V** are expressed in Spherical Coordinates. All tests are performed on a Sun Blade 1000 Workstation.

A. Earth-Jupiter Rendezvous

In Ref. 36, we presented an Earth-Jupiter trade study using the N-Vector, Node, and Node + Chebyshev formulations (i.e. without the Chebyshev formulation). We provide the results for the Chebyshev formulation here for completeness. Table 1 gives the parameters of a nuclear electric propulsion (NEP) spacecraft for this mission.

Figure 4 shows the trade-off curves of a time-of-flight (TOF) trade study for the four formulations. The study begins with a TOF of 2330 days and ends with a TOF of 2150 days, with a step size of 20 days. The solution from the previous step is used as the initial guess for the next step.

Among the four formulations, results found by three of them agree closely, except for the Chebyshev formulation with TOF from 2230 to 2330 days. The difference is not due to the modeling error in Chebyshev formulation, but from the fact that it has converged to another locally optimal solution family (i.e. a solution with a different number of revs around the Sun). Ignoring the points with TOF from 2230 to 2330 days, the final masses found by the three new formulations agree within 20 kg (or 0.1% of the final mass) with those found by N-Vector.

We also note that with TOF from 2150 to 2330 days, the N-Vector formulation converges to some suboptimal solutions, with a lower final mass than the other formulations. In our experience, the N-Vector formulation sometimes switches to a suboptimal point when the D**V** are written in Spherical Coordinates, due to the fact that the D**V** angles in the coasting segments may be perturbed during the iterations of the optimizer. On the other hand, the Node formulation and the Node + Chebyshev formulation do not diverge to a suboptimal solution in this test case.

Table 2 shows the overall performance of the four formulations, including the total run time. Run times for individual steps are provided in Figure 5. The Node formulation takes an exceptionally long time to converge in some of the runs (TOF = 2270, 2210, 2150 days). Overall, the Node + Chebyshev formulation has the best performance among the four, which is 10 times faster than the N-Vector formulation.

Table 1: Spacecraft Parameters for an Earth-Jupiter Rendezvous Mission

Parameter	Values
Power Available to the Thrusters	95 kW
Specific Impulse	6,000 s
Overall Efficiency	70 %
Thrust	2.26 N
Mass Flow Rate	38.4 mg/s
Initial Mass at zero V_∞	20,000 kg

Table 2: Comparing Four Formulations on an E-J Trade Study

Formulation	Number of Optimization Variables	Number of Nonlinear Constraints	Total Run Time[b] in the Trade Study, sec
N-Vector	187	67	160
Node	129	21	321
Chebyshev[a]	93	67	241
Node + Chebyshev[a]	63	21	15

[a] Degree of Chebyshev Series (for both θ and ψ): 10 on Leg 1 and 5 on Leg 3.
[b] On a Sun Blade 1000 Workstation.

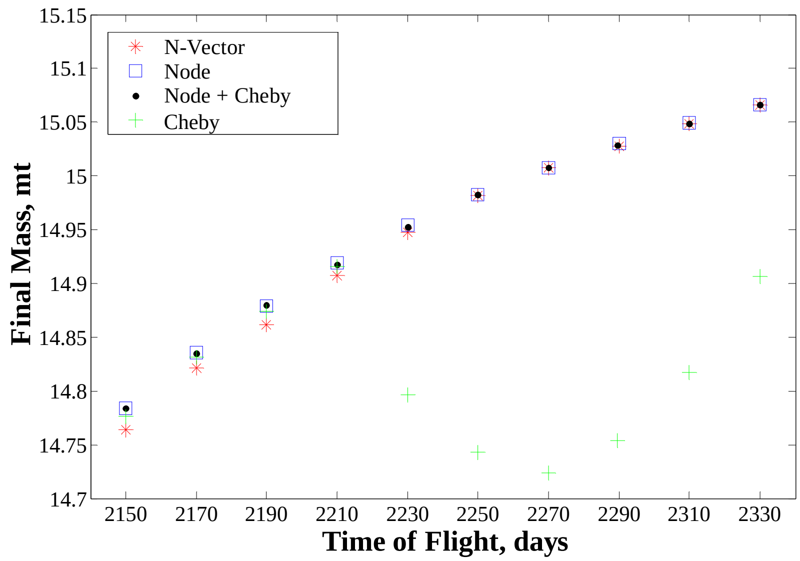

Figure 4: TOF trade study of an E-J rendezvous mission (starting from 2330 days, ending at 2150 days).

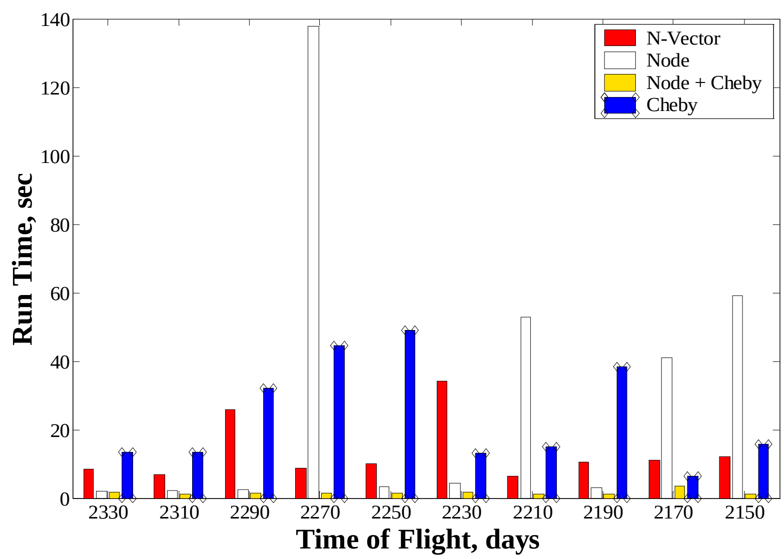

Figure 5: Run times (on a Sun Blade 1000 Workstation) for an E-J rendezvous mission trade study (starting from 2330 days, ending at 2150 days).

B. Earth-Mars-Vesta Flyby

We can assess the accuracy of a formulation by comparing results with the literature. We study an Earth-Mars-Vesta flyby mission found in Ref. 35. The spacecraft is propelled by a solar-powered ion thruster, similar to that used in Deep Space 1.[24] We compare results of our new formulations to those from CLSEP, a direct optimization tool based on the trajectory model in Fig. 1, and to those from an indirect optimization tool called SEPTOP.[5] Table 3 summarizes the mission specification and Fig. 6 shows the trajectory plot.

We assume a coast-thrust-coast sequence on the Earth-Mars leg and a thrust-coast sequence on the leg from Mars to Vesta. For the D**V** directions, a fourth degree Chebyshev series is used to parameterize the angles. The D**V** angles are expressed in the r-θ-h frame (a spacecraft rotating frame). There are 26 and 36 segments on the leg from Earth to Mars and on the leg from Mars to Vesta respectively.

Table 3: Parameters of an Earth-Mars-Vesta Flyby Mission

Characteristics	**Values**
Power Available to the Spacecraft at 1AU	10 kW
Earth Launch Date	Oct 4, 2009
Mars Flyby Date	May 2, 2010
Vesta Flyby Date	Jan 27, 2011
Launch V_∞[a]	2.8 km/s
Initial Mass	545 kg
Lower Bound on Mars Flyby Altitude	200 km

[a] The spacecraft is launched by Delta 7326.

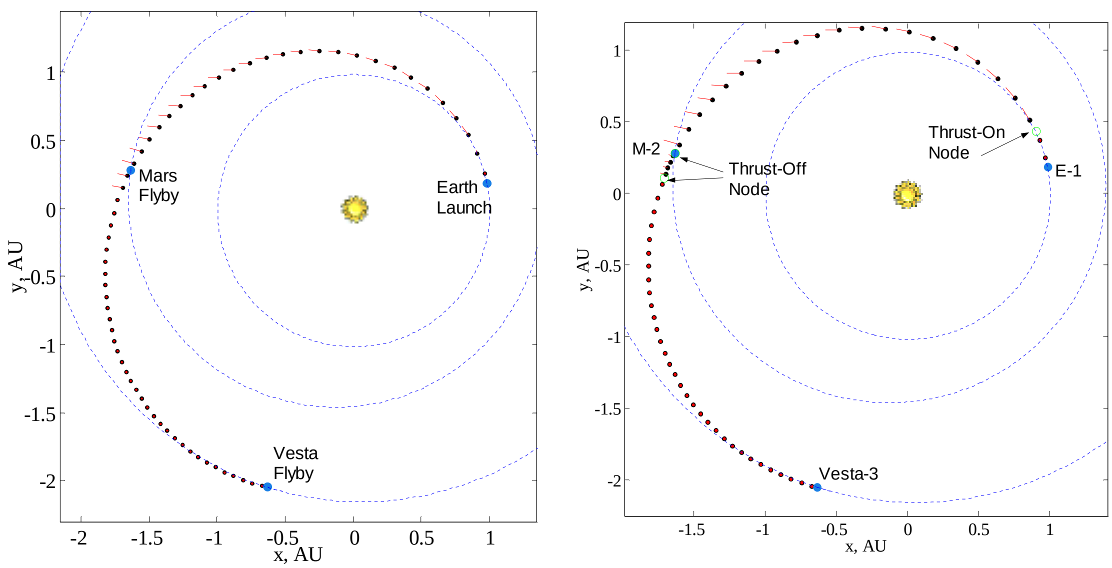

Figure 6: Trajectory plot of an Earth-Mars-Vesta flyby mission optimized by N-Vector formulation (left) and Node (right) formulation.

Table 4: Accuracy for the Earth-Mars-Vesta Flyby Test Case

Launch V_∞	**Earth Launch Date**	**Mars Flyby Date**	**Vesta Arrival Date**	**Final Mass, kg**					
				CLSEP[a]	**SEPTOP[a]**	**GALLOP N-Vector**	**GALLOP Node**	**GALLOP Cheby.**	**GALLOP Node + Cheby.**
Fixed	Fixed	Fixed	Fixed	493.73	493.74	493.727	493.741	493.726	493.740
Free	Free	Fixed	Fixed	503.46	503.39	503.439	503.398	503.438	503.397
Free	Free	Free	Fixed	504.43	504.36	504.423	504.368	504.423	504.368
Free	Free	Free	Free	535.24	535.23	535.652	535.708	535.652	535.708

[a] Results of CLSEP and SEPTOP are from Ref. 35.

Table 5: Speed of Convergence on the Earth-Mars-Vesta Flyby Test Case

Launch V_∞	Earth Launch Date	Mars Flyby Date	Vesta Arrival Date	Run Time, sec GALLOP N-Vector	GALLOP Node	GALLOP Cheby.	GALLOP Node + Cheby.
Fixed	Fixed	Fixed	Fixed	16.1[a]	6.0[a]	9.0[a]	2.9[a]
Free	Free	Fixed	Fixed	3.0	5.9	3.0	1.5
Free	Free	Free	Fixed	4.8	2.7	5.3	1.3
Free	Free	Free	Free	19.1	10.5	4.0	36.5

[a] Initial Guesses are from a shape-based tool called STOUR-LTGA.[42]

We begin with a fixed (or frozen) launch V_∞ and encounter dates, then subsequently release these variables in a series of runs; Tables 4 and 5 summarize the results. As can be seen in the tables, the final masses found by the four formulations agree within a half kg with those found by CLSEP and SEPTOP. The differences could be caused by an updated ephemeris (JPL-de405) used by GALLOP. For the convergence speed, using the new formulations (Node, Chebyshev, and Node + Chebyshev) provides only some improvements over the N-Vector formulation. Among the four formulations, Node + Chebyshev is the fastest except for the run when the Vesta arrival date is free.

In addition, we perform a launch period study by varying the launch date from July 28, 2009 through Nov 20, 2009, with a step size of 5 days. The starting guess is from the last run in Table 4, where all the encounter dates are free and the Earth launch date is on Sept. 23, 2009. Solutions are found by adding or subtracting launch dates from the converged solution of the immediately preceding step. Figure 7 shows the trade-off curves for the four formulations. We notice that the final masses of the four formulations agree closely for the most part. For launch range from 0 to 50 days since July 25, 2009, the N-Vector formulation finds suboptimal solutions in which the final mass is about 2 to 6 kg (about 1% of the final mass) less than the other formulations. These suboptimal points found by N-Vector can be improved if we perturb the Vesta arrival dates to the optimal values (found by the other formulations) and by doing so the final masses of the four formulations agree within 0.5 kg. The run times of the four formulations are shown in Table 6. We note that the run time can be cut by half using the Node formulation or the Chebyshev formulation, while the Node + Chebyshev formulation saves 75% of time compared with the N-Vector formulation.

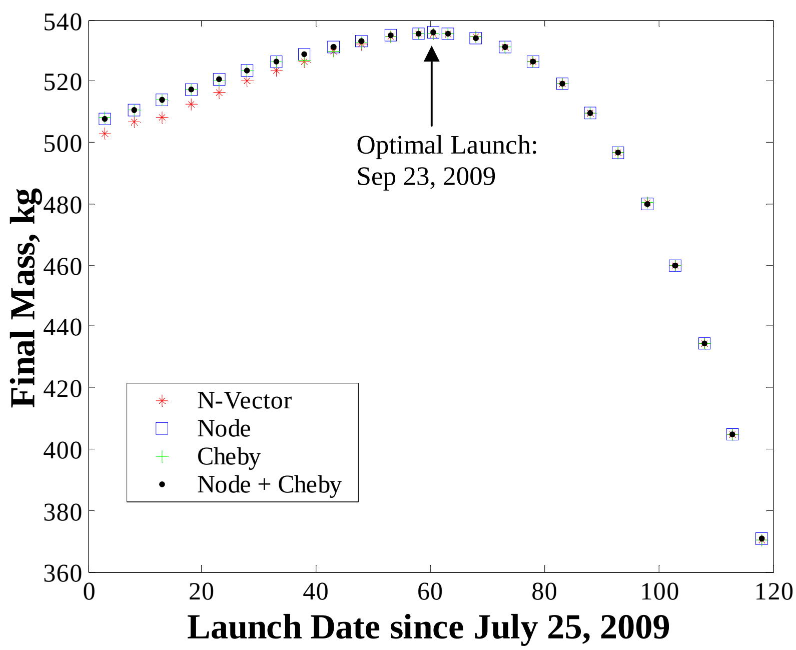

Figure 7: Launch period study of an Earth-Mars-Vesta flyby mission.

Table 6: Convergence Speed of a Launch Period Study for an Earth-Mars-Vesta Flyby Mission

Run Time, sec	N-Vector	Node	Chebyshev	Node + Chebyshev
Average	2.6	1.1	1.4	0.6
Maximum	5.1	2.1	7.3	0.8
Total	62.5	27.1	34.1	15.3

C. Earth-Mars Roundtrip Mission

An Earth-Mars roundtrip mission found in Ref. 11 is considered as a test case here. The mission uses nuclear electric propulsion (NEP) to transfer back and forth between Earth and Mars over 15 years (seven Earth-Mars synodic periods). Table 7 gives the key parameters for this mission.

Figure 8 shows the trajectory for the first cycle of this mission. The spacecraft launches at E-1 and arrives Mars in about one year. The transfer vehicle stays at Mars' orbit for 30 days (for the exploration on the surface of Mars) and begins its inbound trip back to the Earth. We assume a net mass drop of about 45 mt (metric tons) at Mars to account for the propellant expenditure of the crew taxi vehicle. At the arrival back at Earth, the transfer vehicle stays for 30 days before it continues its outbound trip to Mars again. The net mass gain at Earth is about 120 mt to account for injected payload from the surface, which includes propellant and consumables on the transfer vehicle. We assume a V_∞ of zero (i.e. rendezvous) on each Earth and Mars encounter.

The optimization of this problem includes 15 planetary encounters and 371 segments in total, which comprises over 1,000 variables and 400 nonlinear constraints using the N-Vector formulation (see Table 8). We assume a Thrust-Coast-Thrust scheme for each leg. The degree of the Chebyshev series ranges from 3 to 6 for parameterizing the D**V** angles and the angles are expressed in the inertial frame for legs from Earth to Mars and in the V-C-N frame for legs from Mars to Earth.

Table 7: Parameters for an Earth-Mars Roundtrip Mission

Parameter	Values
Power Available to the Thrusters	11.6 MW
Specific Impulse	10,204 s
Overall Efficiency	64.5 %
Thrust	150 N
Initial Mass at E-1[a]	346,636 kg
Final Mass at E-15[b]	214,525 kg

[a] Initial guess value (i.e. not yet optimized).
[b] Frozen during optimization.

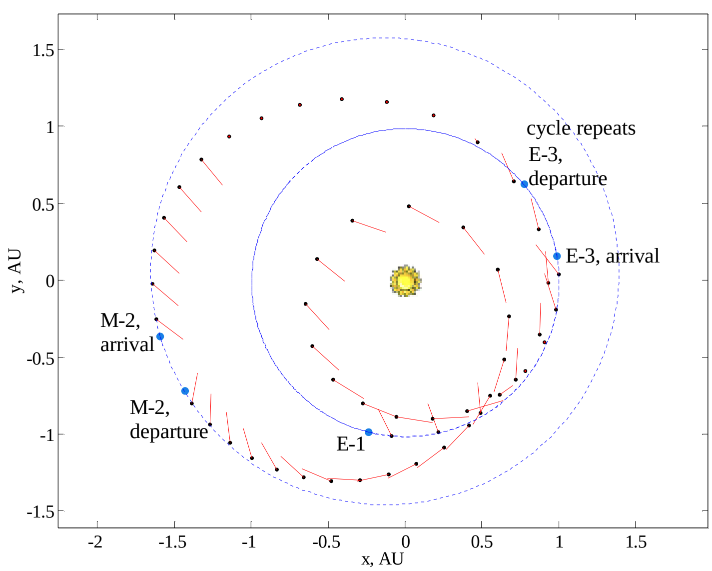

Figure 8: First cycle of an Earth-Mars round trip mission.

Table 8: Comparing Four Formulations on the Earth-Mars Round Trip Mission

Characteristics	N-Vector	Node	Chebyshev	Node + Chebyshev
Number of Variables	1198	1067	624	641
Number of Nonlinear Constraints	469	312	469	280

The starting guess of this problem is from the concatenated solutions found in Ref. 11. That is, we patch the solutions which are optimized within a single synodic period to a series of seven roundtrip mission between Earth and Mars. Our objective is to minimize the initial mass on E-1, which corresponds to the wet mass of the transfer vehicle for its first trip to Mars.

Table 9 summarizes the results of this large scale optimization using four different formulations. The N-Vector formulations took over 15 hours to complete the run and ends up with an initial mass slighter higher (by 180 kg) than the value of the initial guess (where the value of the initial guess may be infeasible to the problem). The Node formulation took over 20 hours to find a solution slightly better than that of N-Vector (by 713 kg or 0.2%). On the other hand, both the Chebyshev formulation and the Node + Chebyshev formulation fail to converge with the default values (10^{-6}) for their major optimality and feasibility tolerances.

If we loosen the tolerances to 10^{-4}, the Node + Chebyshev formulation finds a solution with a greater initial mass (by 3,694 kg or 1%) than that found by N-Vector (with tolerances of 10^{-6}). However, the savings in run time using the Node + Chebyshev formulation is quite remarkable: it takes less than 2 hours while N-Vector takes over 15 hours (12 hours for tolerances equal to 10^{-4}) to converge. For this large scale problem, it seems difficult for the Node + Chebyshev formulation to converge within the default tolerances.

For preliminary trade studies, often an accuracy of 1% is sufficient. If we need more precision on a solution, we can apply the Node + Chebyshev formulation to find an intermediate solution, which can then be used as an initial guess for the N-Vector formulation to further improve the accuracy. The total computational time for this two-step process could be shorter than using the N-Vector alone.

Table 9: Optimization of the Earth-Mars Round Trip Mission using Four Formulations

Formulation	Exit Condition of NPOPT	Run Time, hour	Initial Mass at the end of the run, kg	Additional Notes
N-Vector[a]	Optimal Solution Found[c]	15.5	346,816	
Node[a]	Optimal Solution Found[c]	20.1	346,102	
Chebyshev[a]	Infeasible Problem[d]	0.6	354,613	Final Feasibility ~ 10^{-2}
Node + Chebyshev[a]	Infeasible Problem[d]	0.7	368,264	Final Feasibility ~ 10^{-5}
N-Vector[b]	Optimal Solution Found[c]	12.2	346,816[e]	Final Feasibility ~ 10^{-5}
Node + Chebyshev[b]	Optimal Solution Found[c]	1.7	350,510	Final Feasibility ~ 10^{-10}

[a] Major feasibility tolerance and major optimality tolerance are both set to 10^{-6} (the default for NPOPT).
[b] Major feasibility tolerance and major optimality tolerance are both set to 10^{-4}.
[c] *Optimal Solution Found* means the feasibility and optimality at the end of the optimization run both satisfy their tolerances.
[d] *Infeasible Problem* means the optimizer fails to find a solution which satisfy the constraints within the feasibility tolerance and so the objective function (initial mass) reported is not meaningful.
[e] The initial mass found by setting the tolerances to 10^{-4} is 0.15 kg higher than that of the tolerances equal to 10^{-6} (for N-Vector).

D. Earth-Mercury Rendezvous

We study an Earth-Mercury mission where the spacecraft makes many revs around the Sun. The spacecraft uses solar electric propulsion similar to the propulsion system used in Deep Space 1.[24] Table 10 gives the parameters of this mission and Fig. 9 shows the trajectory plot. In Fig. 9, for the Node + Chebyshev formulation, we note that the number of segments on coasting arcs is reduced to 2 (the minimum for each leg in GALLOP), so that redundant variables on coasting segments are reduced to minimal during optimization. All variables shown in Table 10 are kept frozen during optimization.

The spacecraft makes 12 revs around the Sun during its 6.3 years interplanetary flight to Mercury. The D**V** of this trajectory is changing quite rapidly, both in magnitude and directions (see Fig. 10 for the D**V** angles). There are 15 on/off pulses in total and we only use 13 pairs of On/Off-Nodes to model the D**V** magnitude (since some of the pulses are too short). For the D**V** angles, with the Chebyshev formulation, we study cases with the degree of Chebyshev series from 4 to 9. With Node + Chebyshev, we use a 2nd to 4th degree Chebyshev series to model the angles on each thrusting arc (13 in total).

We use 242 segments in this problem. The N-Vector formulation has more than 700 variables and 200 nonlinear constraints, which can be considered as a mid-size optimization problem (as compared with the Earth-Mars roundtrip mission). The initial guess is created by moving the Mercury arrival date of an optimal solution later by 0.1 days.

Table 10: Parameters of an Earth-Mercury Rendezvous Mission

Characteristics	Values
Power Available to the Spacecraft at 1AU	10 kW
Earth Launch Date	March 21, 2007
Mercury Arrival Date	Aug. 22, 2013
Launch V_∞	0 km/s
Initial Mass	660 kg

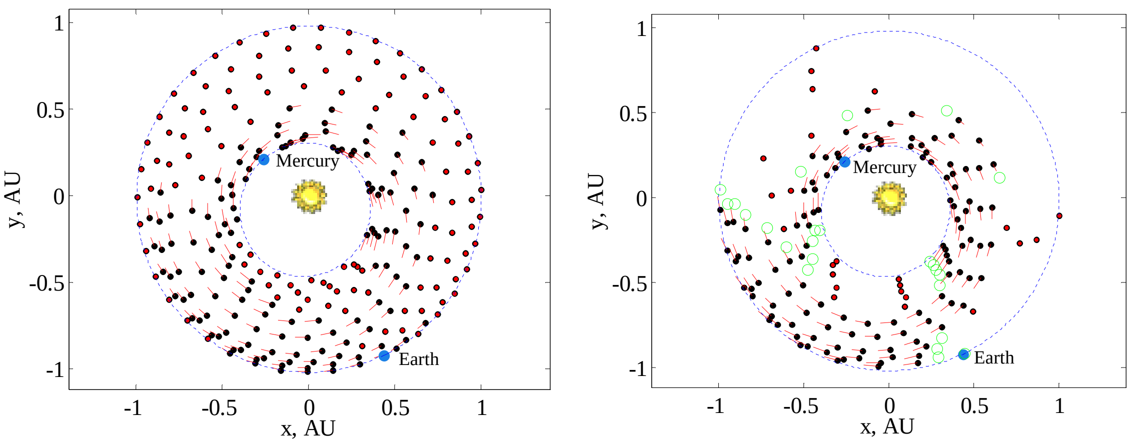

Figure 9: Trajectory plot of an Earth-Mercury rendezvous mission optimized by N-Vector formulation (left) and Node + Chebyshev formulation (right).

Table 11 summarizes the results of this test case using different formulations. We note that N-Vector has the shortest convergence time among the other formulations. Node formulation fails to converge for over 3 hours, while Chebyshev formulation (with various degrees) finds a solution 2% (7 kg) less than that found by N-Vector. With the Node + Chebyshev formulation, we have to loosen the major optimality tolerance to 10^{-3} in order to find a solution which has fair accuracy comparing with N-Vector (0.03% or 0.11 kg difference). However, unlike the previous test cases (where N.C. is usually fastest), Node + Chebyshev takes almost twice as long as the N-Vector to converge.

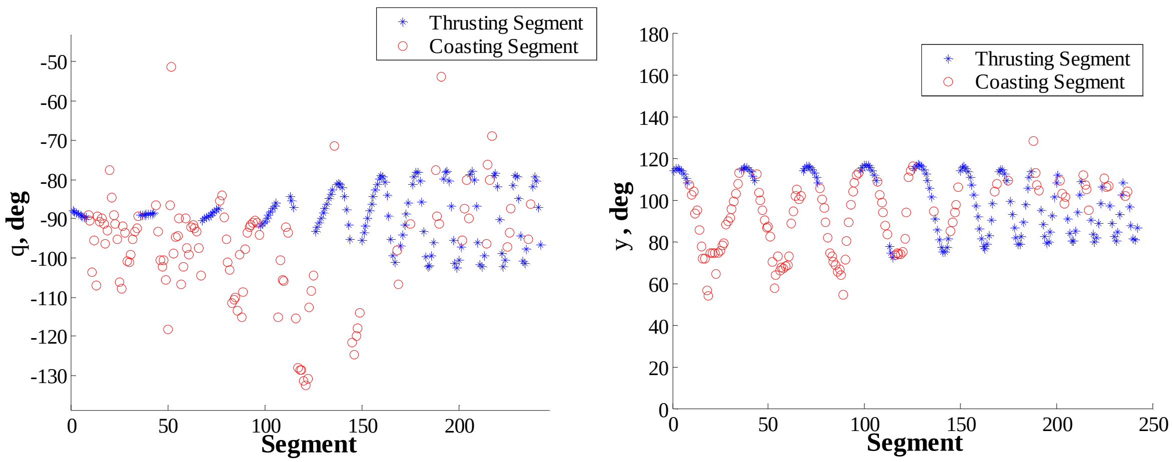

Figure 10: The clock angle θ (left) and the cone angle ψ (right) of the DV, in the r-θ-h frame, for an Earth-Mercury rendezvous mission, optimized by N-Vector formulation.

Table 11: Optimization of an Earth-Mercury Rendezvous Mission using Different Formulations

Formulation	***n***	***m***	**Exit Condition of NPOPT**	**Run Time, min**	**Final Mass at the end of the run, kg**
N-Vector[a]	733	249	Optimal Solution Found[c]	4.5	371.73
Node[a]	547	175	User stopped the run	> 180	Not Available
Chebyshev ($k_\theta = 4$, $k_\psi = 5$)[a]	260	249	Optimal Solution Found[c]	6.6	364.25
Chebyshev ($k_\theta = 7$, $k_\psi = 7$)[a]	265	249	Optimal Solution Found[c]	13	364.99
Chebyshev ($k_\theta = 9$, $k_\psi = 9$)[a]	269	249	Optimal Solution Found[c]	16	365.24
Node + Chebyshev[a]	390	175	MILE[d]	69	371.75
Node + Chebyshev[b]	390	175	Optimal Solution Found[c]	8	371.62

[a] Major feasibility tolerance and major optimality tolerance are both set to 10^{-6} (the default for NPOPT).
[b] Major feasibility tolerance is set to 10^{-6} and major optimality tolerance is set to 10^{-3}.
[c] *Optimal Solution Found* means the feasibility and optimality at the end of the optimization run both satisfy their tolerances.
[d] *Major Iteration Limit* (5000) *Exceeded*, with a feasibility < 10^{-6} and an optimality ~ 10^{-4} at the end of the run.

Furthermore, we perform a launch period study for this mission using N-Vector and Node + Chebyshev. We vary the launch date from April 14, 2007 to July 18, 2007, with a step size of 5 days (20 runs in total). Solution from the immediate preceding step is used as an initial guess for the next step. Figure 11 shows the trade-off curves of this launch period study. We note that Node + Chebyshev does not quite follow the shape of N-Vector for launch date from 60 to 100 since April 9, 2007. Nevertheless, the maximum deviation in final mass between the two formulations is only 0.62 kg, which is less than 0.2% of the final mass. Table 12 compares the run times of the formulations. We note from Table 11 that Node + Chebyshev has to loosen its optimality tolerance, so we do the same for this launch period study. The Node + Chebyshev formulation takes 3 hours less than N-Vector to complete the launch period study, even if we set the tolerances for both formulations to be the same.

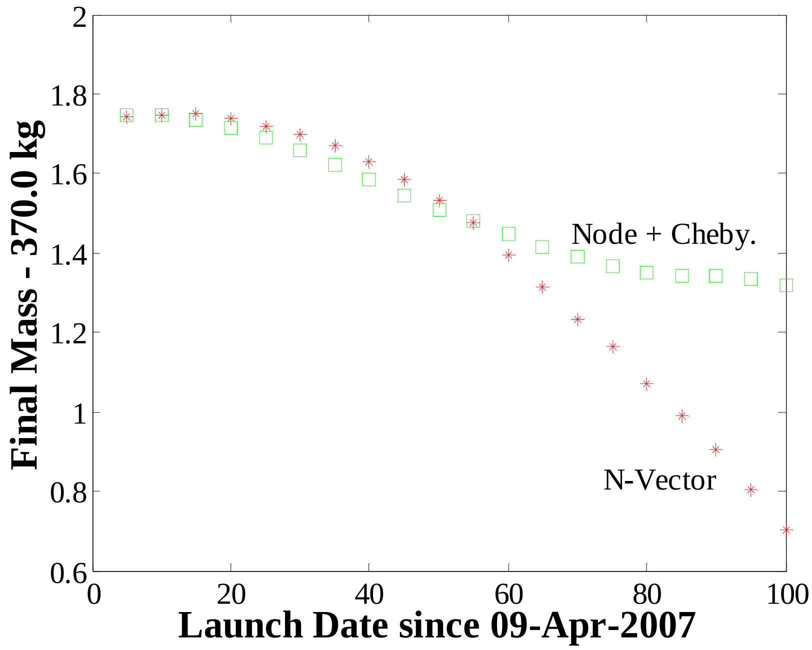

Figure 11: Launch period study of an Earth-Mercury rendezvous mission. *(Solutions of N-Vector are found by setting the major feasibility and optimality tolerances to 10^{-6}, while solutions of Node + Chebyshev are found by setting the major feasibility tolerance to 10^{-6} and the major optimality tolerance to 10^{-3}.)*

Table 12: Convergence Speed of a Launch Period Study for an Earth-Mercury Rendezvous Mission

Run Time	N-Vector[a]	N-Vector[b]	Node + Chebyshev[b]
Average	9.8 min	11 min	21 sec
Maximum	15.9 min	176 min	4.6 min
Total	197 min	219 min	7.1 min

[a] Major feasibility tolerance and major optimality tolerance are both set to 10^{-6} (the default for NPOPT).
[b] Major feasibility tolerance is set to 10^{-6} and major optimality tolerance is set to 10^{-3}.

VI. Discussion

We make the following observations and comments based on the research we have done:

1. N-Vector is most stable (converges in most cases with the default NPOPT parameters) among the four formulations, but it may not be the fastest approach. It can be served as a good standard method (or workhorse) for cases where we only need to perform a single (or a few) run. The N-Vector formulation should be the first method to use when solving a new problem.
2. Node + Chebyshev can save hours of computer operation time for mission designer who look at broad ranges of design space.
3. For cases we have studied, the N-Vector formulation can converge to suboptimal solutions in a parametric study while Node + Chebyshev stays on the same family of optimal solutions.
4. When solving a difficult problem (e.g. when $n > 500$ and $m > 250$), we might need to loosen the tolerances in the optimizer when we are using the new formulations (Node, Chebyshev, and Node + Chebyshev).
5. The performance of the four formulations can be very different than the results we have shown in this paper when solving a new type of problem. It is difficult to predict the behavior of the optimizer when tackling problems that we have never attempted.
6. As future work, we would like to study other parametric functions (e.g. trigonometric functions) to model the D**V** angles (other than Chebyshev polynomials).
7. The accuracy of the Node + Chebyshev can be further improved if we optimize the solution found by Node + Chebyshev using the N-Vector formulation. We can use Node + Chebyshev to find an intermediate solution, then use N-Vector to find a better solution. Yet, the effectiveness of this process is still unknown.
8. There is no definite answer of which formulation is the best; it depends on how accurate and how fast we need for a certain case. It is helpful for the user to have an array of methods for tackling different types of problems. We can make an analogy of our formulations to different methods in numerical integration:[43] Runge-Kutta (RK4) is a standard method for solving general problems but the Adams method can be more efficient than RK4 at stringent tolerances; N-Vector is the standard formulation for solving a new problem but Node + Chebyshev is usually faster in parametric runs.

VII. Conclusion

We study new ways in parameterizing the D**V** when optimizing low-thrust, gravity assist trajectories. We compare the performance of four formulations: N-Vector, Node, Chebyshev, and Node + Chebyshev. For the cases we studied, Node + Chebyshev formulation is the fastest, with the exceptions of N-Vector being faster on the Earth-Mars-Vesta flyby mission (on one of the runs) and the Earth-Mercury rendezvous mission (on one of the runs). For an Earth-Jupiter rendezvous mission and an Earth-Mars-Vesta flyby mission (small scale problems), the Node + Chebyshev formulation can saves 75% to 90% of the run time with an accuracy of 0.1% compared with N-Vector. For a seven-synodic-period Earth-Mars roundtrip mission (a large scale problem), Node + Chebyshev saves over 10 hours of run time with an error of 1% in the objective function; and for the launch period study of an Earth-Mercury rendezvous mission (a mid scale problem), the convergence speed of Node + Chebyshev is 3 hours faster than N-Vector, with an accuracy of 0.2%. The Node + Chebyshev formulation is particularly useful in parametric studies (e.g. flight time, launch period, hardware parameters) where we can save hours of computational time with fairly accurate results.

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