Axial Velocity Solution for a Spinning-up Rigid Body Subject to Constant Body-Fixed Forces and Moments

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An approximate closed-form analytical solution is derived for the axial velocity of a spinning, axisymmetric or nearly-axisymmetric rigid body subject to constant body-fixed forces and moments about three axes. The analytical solution is approximate due to the assumption that two of the Euler angles are small. Numerical simulations shows that the solutions are highly accurate when applied to typical motion of a spacecraft such as the Galileo.

I. Introduction

Mathematicians and dynamicists have been working on the problem of the rigid body motion for over two centuries. There is a rich history of analytical solutions on rigid body, rocket and spacecraft problems[1]-.[33] Among those problems, the three-dimensional translational motion of a rigid body subject to constant body-fixed forces and moments are not an easy one when the translational and rotational motions are coupled. This coupling could be happened when the total external forces are dependent of rotational motion variables or the external moments are dependent of translational motion variables. The velocity problem of a rigid body when the rigid body is subjected to forces and moments which are orientation dependent is crucial in space vehicles and is the scope of this paper.

We use the results of Tsiotras and Longuski[29] and Gick[32] to find the analytical axial velocity solution of a spinning-up rigid-body. The spin rate increases linearly with time and the body-fixed forces and moments are assumed constant. In this paper, we provide the corresponding solution for the axial velocity. The result is exact for axisymmetric, and approximate for nearly-axisymmetric, and under certain conditions, for asymmetric rigid bodies.

II. Euler's Equations of Motion and The Kinematic Equations

Consider a spinning rigid body in an inertial frame with constant body-fixed forces and moments as shown in Fig. 1. Euler's equations of motion can be written as

$$\dot{\omega}_x(t) = M_x/I_x - [(I_z - I_y)/I_x]\,\omega_y\omega_z \tag{1}$$

$$\dot{\omega}_y(t) = M_y/I_y - [(I_x - I_z)/I_y]\,\omega_z\omega_x \tag{2}$$

$$\dot{\omega}_z(t) = M_z/I_z - [(I_y - I_x)/I_z]\,\omega_x\omega_y \tag{3}$$

where ω_x, ω_y, and ω_z are components of the absolute angular velocity of the rigid body in the body-fixed reference frame, M_x, M_y, and M_z are body-fixed moments and I_x, I_y, and I_z are principal moments of inertia about the x, y, and z axes of the body-fixed reference frame, respectively. We assume throughout that the

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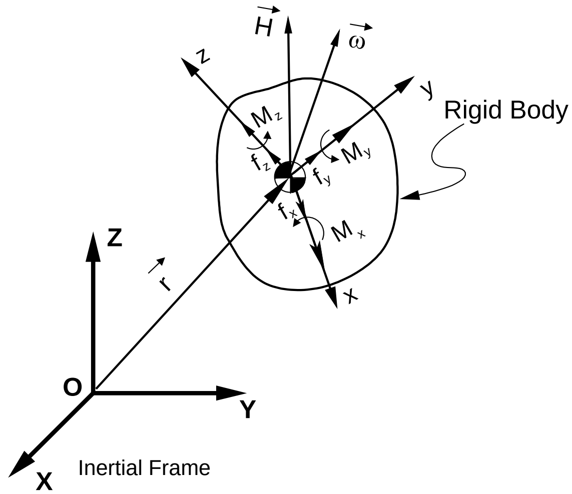


Figure 1. A spinning rigid body in an inertial frame.

body-fixed moments are constant. For axisymmetric, nearly-axisymmetric rigid bodies or asymmetric rigid bodies where the product $\omega_x \ \omega_y$ is small enough, Eqs.(1-3) can be simplified as:

$$\dot{\omega}_x(t) = M_x/I_x - [(I_z - I_y)/I_x]\, \omega_y \omega_z \tag{4}$$

$$\dot{\omega}_y(t) = M_y/I_y - [(I_x - I_z)/I_y]\, \omega_z \omega_x \tag{5}$$

$$\dot{\omega}_z(t) \approx M_z/I_z \tag{6}$$

By integrating Eq.(6) and assuming that axial moment, M_z, is constant, we obtain

$$\omega_z(t) \approx (M_z/I_z)t + \omega_{z0}, \ \ \omega_{z0} \triangleq \omega_z(0) \tag{7}$$

which is, of course, exact for axisymmetric rigid bodies.

We use a Type I: 3-1-2 Euler angle sequence[34] which relates the orientation of body-fixed reference frame to inertial reference frame, the kinematic equations can be written as follows:

$$\dot{\phi}_x = \omega_x \ cos \ \phi_y + \omega_z \ sin \ \phi_y \tag{8}$$

$$\dot{\phi}_y = \omega_y - (\omega_z \ cos \ \phi_y - \omega_x \ sin \ \phi_y) \ tan \ \phi_x \tag{9}$$

$$\dot{\phi}_z = (\omega_z \ cos \ \phi_y - \omega_x \ sin \ \phi_y) \ sec \ \phi_x \tag{10}$$

where ϕ_x, ϕ_y and ϕ_z are the Eulerian angles. With the assumption that ϕ_x and ϕ_y are small and $\phi_y \omega_x$ is small compared to ω_z, Eqs.(8-10) can be simplified as:

$$\dot{\phi}_x = \omega_x + \omega_z \ \phi_y \tag{11}$$

$$\dot{\phi}_y = \omega_y - \phi_x \omega_z \tag{12}$$

$$\dot{\phi}_z = \omega_z \tag{13}$$

After substituting Eq.(13) into Eq.(6) and integrating, we obtain

$$\phi_z = \frac{1}{2} \frac{M_z}{I_z} t^2 + \omega_{z0} t + \phi_{z0}, \ \ \phi_{z0} \triangleq \phi_z(0) \tag{14}$$

III. Closed-Form Analytical Solution for the Angular Velocities

By using Tsiotras and Longuski[29] results, the closed-form solutions for scaled angular velocities, Ω_x and Ω_y, can be written as

$$\Omega(\tau) = \Omega(\tau_0)e^{i\rho(\tau^2-\tau_0^2)/2} + Fe^{i\rho\tau^2/2}\bar{I}_{u0}(\tau_0, \tau; \rho) \tag{15}$$

where the overbar denotes complex conjugate and

$$\Omega \triangleq \Omega_x + i\Omega_y \tag{16}$$

$$\Omega_x \triangleq \omega_x\sqrt{k_y} \quad , \qquad \Omega_y \triangleq \omega_y\sqrt{k_x} \tag{17}$$

$$k_x \triangleq \frac{I_z - I_y}{I_x} \quad , \qquad k_y \triangleq \frac{I_z - I_x}{I_y} \tag{18}$$

In the Eq.(17), ω_x and ω_y are the angular velocities in the body-fixed frame and F in the Eq.(15) is defined as

$$F \triangleq F_x + iF_y \tag{19}$$

where

$$F_x \triangleq (M_x/I_x)(I_z/M_z)\sqrt{k_y} \tag{20}$$

$$F_y \triangleq (M_y/I_y)(I_z/M_z)\sqrt{k_x} \tag{21}$$

The function $I_{u0}(\tau_0, \tau; \rho)$ in Eq.(15) is defined as

$$I_{u0}(\tau_0, \tau; \lambda) \triangleq \int_{\tau_0}^{\tau} e^{(i\lambda u^2/2)}\, du = I_{u0}(\tau; \lambda) - I_{u0}(\tau_0; \lambda) \tag{22}$$

where

$$\rho \triangleq k\,(I_z/M_z) \tag{23}$$

$$k \triangleq \sqrt{k_x\; k_y} \tag{24}$$

$$\tau(t) \triangleq \omega_z(t) = (M_z/I_z)\; t + \omega_{z0}; \qquad \omega_{z0} = \tau_0 \tag{25}$$

IV. Closed-Form Analytical Solution for the Eulerian Angles

With this assumption that ϕ_x, ϕ_y, and $\phi_y\omega_x$ are small, Tsiotras and Longuski[29] showed that for the Euler angle sequence 3-1-2 (ϕz, ϕx, ϕy), the closed-form solution is

$$\phi(\tau) = \phi(\tau_0)\; e^{-i\lambda(\tau^2-\tau_0^2)/2} + \lambda\; e^{(-i\lambda\tau^2/2)}\; I_\phi(\tau_0, \tau; \lambda, \rho) \tag{26}$$

where

$$\phi = \phi_x(t) + i\;\phi_y(t) \tag{27}$$

$$I_\phi(\tau_0, \tau; \lambda, \rho) \triangleq \int_{\tau_0}^{\tau} e^{(i\lambda u^2/2)}\omega(u)\; du \tag{28}$$

$$I_\phi(\tau_0, \tau; \lambda, \rho) = k_1 I_{\phi 1}(\tau_0, \tau; \lambda, \rho) + k_2 I_{\phi 2}(\tau_0, \tau; \kappa, \rho) \tag{29}$$

$$k_1 \triangleq (\sqrt{k_x} + \sqrt{k_y})/2k, \qquad k_2 \triangleq (\sqrt{k_x} - \sqrt{k_y})/2k \tag{30}$$

$$\rho \triangleq k\lambda, \qquad \mu \triangleq \lambda + \rho = \lambda(1+k), \qquad \kappa \triangleq \lambda - \rho = \lambda(1-k) \tag{31}$$

where the k's represent the mass properties of the rigid body and $I_{\phi 2}$ provides the contribution from an asymmetric body.

$$I_{\phi 1}(\tau_0, \tau; \lambda, \rho) \triangleq \int_{\tau_0}^{\tau} e^{(i\lambda u^2/2)} \Omega(u)\ du \tag{32}$$

$$I_{\phi 2}(\tau_0, \tau; \kappa, \rho) \triangleq \int_{\tau_0}^{\tau} e^{(i\lambda u^2/2)} \bar{\Omega}(u)\ du \tag{33}$$

$$\begin{aligned} I_{\phi 1}(\tau_0, \tau; \lambda, \rho) = {} & [\Omega(\tau_0) e^{(-i\rho\tau_0^2/2)} - F\ \bar{I}_{u0}(\tau_0; \rho)]\ \bar{I}_{u0}(\tau_0, \tau; -\mu) \\ & + F J_{u0}(\tau_0, \tau; \mu, \rho) \end{aligned} \tag{34}$$

$$\begin{aligned} I_{\phi 2}(\tau_0, \tau; \kappa, \rho) = {} & [\bar{\Omega}(\tau_0) e^{(i\rho\tau_0^2/2)} - \bar{F}\ I_{u0}(\tau_0; \rho)]\ I_{u0}(\tau_0, \tau; \kappa) \\ & + \bar{F}\ \bar{J}_{u0}(\tau_0, \tau; -\kappa, \rho) \end{aligned} \tag{35}$$

$$J_{u0}(\tau_0, \tau; \mu, \rho) \triangleq \int_{\tau_0}^{\tau} e^{(i\mu u^2/2)}\ \bar{I}_{u0}(u; \rho)\ du = J_{u0}(\tau; \mu, \rho) - J_{u0}(\tau_0; \mu, \rho) \tag{36}$$

Appendix A shows $J_{u0}(\tau_0, \tau; \mu, \rho)$ approximation.

V. The Inertial Acceleration Equation

Consider a rigid body which is subjected to constant body-fixed forces f_x, f_y, and f_z(as shown in Fig. 1). The following equation relates the acceleration in the body-fixed frame with respect to the inertial reference frame

$$\begin{Bmatrix} \dot{v}_X(t) \\ \dot{v}_Y(t) \\ \dot{v}_Z(t) \end{Bmatrix} = [A] \begin{Bmatrix} f_x/m \\ f_y/m \\ f_z/m \end{Bmatrix} \tag{37}$$

where [A] is the direction cosine matrix and for a 3-1-2 Euler sequence is given by the following equation.[34]

$$[A] = \begin{bmatrix} c\phi_z c\phi_y - s\phi_z s\phi_x s\phi_y & -s\phi_z c\phi_x & c\phi_z s\phi_y + s\phi_z s\phi_x c\phi_y \\ s\phi_z c\phi_y + c\phi_z s\phi_x s\phi_y & c\phi_z c\phi_x & s\phi_z s\phi_y - c\phi_z s\phi_x c\phi_y \\ -c\phi_x s\phi_y & s\phi_x & c\phi_x c\phi_y \end{bmatrix} \tag{38}$$

When ϕ_x and ϕ_y are small, the direction cosine matrix can be simplified as

$$[A] = \begin{bmatrix} c\phi_z & -s\phi_z & \phi_y c\phi_z + \phi_x s\phi_z \\ s\phi_z & c\phi_z & \phi_y s\phi_z - \phi_x c\phi_z \\ -\phi_y & \phi_x & 1 \end{bmatrix} \tag{39}$$

Using the last row of Eq.(37), the axial acceleration can be written in the following form

$$\dot{v_Z}(t) = f_z/m + (i/2m) \left[\bar{f}\phi(t) - f\bar{\phi}(t)\right] \tag{40}$$

In the next section we provide a closed-form solution for the axial velocity by integrating Eq.(40).

VI. Closed-Form Analytical Solution for the Axial Velocity

Before integrating Eq.(40), we will find it useful to replace the variable t with τ by multiplying both sides of Eq.(40) by $dt/d\tau$ and using Eq.(25), we obtain

$$\frac{dv_Z}{d\tau} = \frac{\lambda f_z}{m} - \frac{\lambda}{m} Im[\bar{f}\phi(\tau)] \tag{41}$$

Where Im[.] stands for imaginary part. After integration of both sides of Eq.(41) with respect to τ, we get

$$v_Z(\tau) = v_Z(\tau_0) + \frac{\lambda f_z}{m}(\tau - \tau_0) - \frac{\lambda}{m} Im[A_{v11}(\tau_0, \tau; \lambda)] \tag{42}$$

where A_{v11} is defined as

$$A_{v11}(\tau_0, \tau; \lambda) \triangleq \int_{\tau_0}^{\tau} \bar{f}\phi(u)\, du \tag{43}$$

By substituting Eq.(26) into Eq.(43), it can be shown that Eq.(43) simplify to

$$A_{v11}(\tau_0, \tau; \lambda) = \bar{f}\phi(\tau_0)e^{(i\lambda\tau_0^2/2)}\bar{I}_{u0}(\tau_0, \tau; \lambda) + \lambda\bar{f}A_{v21}(\tau_0, \tau; \lambda, \rho) \tag{44}$$

where

$$A_{v21}(\tau_0, \tau; \lambda, \rho) \triangleq \int_{\tau_0}^{\tau} e^{(-i\lambda u^2/2)} I_{\phi}(\tau_0, u; \lambda, \rho) du \tag{45}$$

A_{v21} can be simplified after substituting Eqs.(29), (34) and (35) into Eq.(45) as follows

$$A_{v21}(\tau_0, \tau; \lambda, \rho) = k_1 A_{v31}(\tau_0, \tau; \lambda, \rho) + k_2 A_{v32}(\tau_0, \tau; \kappa, \rho) \tag{46}$$

$$A_{v31}(\tau_0, \tau; \lambda, \rho) \triangleq \int_{\tau_0}^{\tau} e^{(-i\lambda u^2/2)} I_{\phi 1}(\tau_0, u; \lambda, \rho) du \tag{47}$$

$$A_{v32}(\tau_0, \tau; \kappa, \rho) \triangleq \int_{\tau_0}^{\tau} e^{(-i\lambda u^2/2)} I_{\phi 2}(\tau_0, u; \kappa, \rho) du \tag{48}$$

A_{v31} and A_{v32} can be written as

$$A_{v31}(\tau_0, \tau; \lambda, \rho) = A_{v41}(\tau_0, \tau; \lambda, \rho) + A_{v42}(\tau_0, \tau; \lambda, \rho) \tag{49}$$

$$A_{v32}(\tau_0, \tau; \kappa, \rho) = A_{v43}(\tau_0, \tau; \kappa, \rho) + A_{v44}(\tau_0, \tau; \kappa, \rho) \tag{50}$$

where

$$A_{v41}(\tau_0, \tau; \lambda, \mu) \triangleq \left[\Omega(\tau_0)e^{(-i\rho\tau_0^2/2)} - F\bar{I}_{u0}(\tau_0; \rho)\right] \int_{\tau_0}^{\tau} e^{(-i\lambda u^2/2)} I_{u0}(\tau_0, u; \mu) du \tag{51}$$

$$A_{v42}(\tau_0, \tau; \lambda, \mu) \triangleq \int_{\tau_0}^{\tau} F e^{(-i\lambda u^2/2)} J_{u0}(\tau_0, u; \mu, \rho) du \tag{52}$$

$$A_{v43}(\tau_0, \tau; \lambda, \kappa) \triangleq \left[\bar{\Omega}(\tau_0)e^{(i\rho\tau_0^2/2)} - \bar{F}I_{u0}(\tau_0; \rho)\right] \int_{\tau_0}^{\tau} e^{(-i\lambda u^2/2)} I_{u0}(\tau_0, u; \kappa) du \tag{53}$$

$$A_{v44}(\tau_0, \tau; \lambda, \kappa) \triangleq \int_{\tau_0}^{\tau} \bar{F} e^{(-i\lambda u^2/2)} \bar{J}_{u0}(\tau_0, u; -\kappa, \rho) du \tag{54}$$

Using Eqs.(22) and (36) and expanding the integrals on the right hand side of Eqs.(51) and (52), A_{v41} and A_{v42} can be rewritten as

$$A_{v41}(\tau_0, \tau; \lambda, \mu) = \left[\Omega(\tau_0)e^{(-i\rho\tau_0^2/2)} - F\bar{I}_{u0}(\tau_0; \rho)\right] \left[\bar{J}_{u0}(\tau_0, \tau; \lambda, \mu) - I_{u0}(\tau_0; \mu)\bar{I}_{u0}(\tau_0, \tau; \lambda)\right] \tag{55}$$

$$A_{v42}(\tau_0, \tau; \lambda, \mu) = A_{v51}(\tau; \mu, \rho) - FJ_{u0}(\tau_0; \mu, \rho)\bar{I}_{u0}(\tau_0, \tau; \lambda) \tag{56}$$

where

$$A_{v51}(\tau_0, \tau; \lambda, \mu) \triangleq \int_{\tau_0}^{\tau} F e^{(-i\lambda u^2/2)} J_{u0}(u; \mu, \rho) du \tag{57}$$

Similar to A_{v41} and A_{v42}, A_{v43} and A_{v44} can be rewritten as

$$A_{v43}(\tau_0, \tau; \lambda, \kappa) = \left[\bar{\Omega}(\tau_0)e^{(i\rho\tau_0^2/2)} - \bar{F}I_{u0}(\tau_0; \rho)\right] \left[\bar{J}_{u0}(\tau_0, \tau; \lambda, \kappa) - I_{u0}(\tau_0; \kappa)\bar{I}_{u0}(\tau_0, \tau; \lambda)\right] \tag{58}$$

$$A_{v44}(\tau_0, \tau; \lambda, \kappa) = A_{v52}(\tau; \kappa, \rho) - \bar{F}\bar{J}_{u0}(\tau_0; -\kappa, \rho)\bar{I}_{u0}(\tau_0, \tau; \lambda) \tag{59}$$

where

$$A_{v52}(\tau_0, \tau; \lambda, \kappa) \triangleq \int_{\tau_0}^{\tau} \bar{F}e^{(-i\lambda u^2/2)}\bar{J}_{u0}(u; -\kappa, \rho)du \tag{60}$$

After substituting Eq.(A-1) into Eq.(57), we have

$$A_{v51}(\tau_s, \tau; \lambda, \mu) = A_{v61}(\tau; \mu, \rho) + A_{v62}(\tau_s, \tau; \mu, \rho) + FJ_{u0s}(\tau_s; \mu, \rho)\bar{I}_{u0}(\tau_0, \tau; \lambda) \tag{61}$$

where

$$A_{v61}(\tau; \lambda, \mu) \triangleq \int_{0}^{\tau} Fe^{(-i\lambda u^2/2)}J_{u0s}(u; \mu, \rho)du, \qquad (\tau \leq \tau_s) \tag{62}$$

$$A_{v62}(\tau_s, \tau; \lambda, \mu) \triangleq \int_{\tau_s}^{\tau} Fe^{(-i\lambda u^2/2)}J_{u0l}(u; \mu, \rho)du, \qquad (\tau > \tau_s) \tag{63}$$

Similar to A_{v51}, A_{v52} can be written as

$$A_{v52}(\tau_s, \tau; \lambda, \kappa) = A_{v63}(\tau; \lambda, \kappa) + A_{v64}(\tau_s, \tau; \lambda, \kappa) + \bar{F}\bar{J}_{u0s}(\tau_s; -\kappa, \rho)\bar{I}_{u0}(\tau_0, \tau; \lambda) \tag{64}$$

where

$$A_{v63}(\tau; \lambda, \kappa) \triangleq \int_{0}^{\tau} \bar{F}e^{(-i\lambda u^2/2)}\bar{J}_{u0s}(u; -\kappa, \rho)du, \qquad (\tau \leq \tau_s) \tag{65}$$

$$A_{v64}(\tau_s, \tau; \lambda, \kappa) \triangleq \int_{\tau_s}^{\tau} \bar{F}e^{(-i\lambda u^2/2)}\bar{J}_{u0l}(u; -\kappa, \rho)du, \qquad (\tau > \tau_s) \tag{66}$$

The functions A_{v61} and A_{v62} can be determined by substituting Eqs.(A-2) and (A-3) into Eqs.(62) and (63), respectively as follows

$$A_{v61}(\tau; \lambda, \mu) = \sqrt{\frac{\pi}{|\rho|}} \sum_{n=0}^{11} (a_n + i\ b_n) \left(\frac{|\rho|}{8}\right)^{(n+\frac{1}{2})} \bar{J}_u(\tau; \lambda, 2n+1) \tag{67}$$

$$J_u(\tau; \lambda, n) \triangleq \int_{0}^{\tau} e^{(i\lambda u^2/2)}\bar{I}_u(u; \lambda, n)du \qquad (n = 0, \ldots, 11) \tag{68}$$

and

$$A_{v62}(\tau_s, \tau; \lambda, \mu) = A_{v71}(\tau_s, \tau; \lambda, \mu) + A_{v72}(\tau_s, \tau; \lambda, \rho) \tag{69}$$

where A_{v71} and A_{v72} are defined and simplified as

$$\begin{aligned} A_{v71}(\tau_s, \tau; \lambda, \mu) &\triangleq F\sqrt{\frac{\pi}{|\rho|}}\ \frac{(1-i)}{2} \int_{\tau_s}^{\tau} e^{(-i\lambda u^2/2)}I_{u0}(\tau_s, u; \mu)du \\ &= F\sqrt{\frac{\pi}{|\rho|}}\ \frac{(1-i)}{2} \left[\bar{J}_{u0}(\tau_s, \tau; \lambda, \mu) - I_{u0}(\tau_s; \mu)\bar{I}_{u0}(\tau_s, \tau; \lambda)\right] \end{aligned} \tag{70}$$

$$\begin{aligned} A_{v72}(\tau_s, \tau; \lambda, \rho) &\triangleq F\sqrt{\frac{\pi}{|\rho|}} \sum_{n=0}^{11} (c_n + i\ d_n) \left(\frac{8}{|\rho|}\right)^{(n+\frac{1}{2})} \int_{\tau_s}^{\tau} e^{(-i\lambda u^2/2)}I_d(\tau_s, u; \lambda, 2n+1)du \\ &= F\sqrt{\frac{\pi}{|\rho|}} \sum_{n=0}^{11} (c_n + i\ d_n) \left(\frac{8}{|\rho|}\right)^{(n+\frac{1}{2})} \times \\ &\quad [I_d(\tau_s; \lambda, 2n+1)\bar{I}_{u0}(\tau_s, \tau; \lambda) - J_d(\tau_s, \tau; \lambda, 2n+1)] \end{aligned} \tag{71}$$

$$J_d(\tau_s, \tau; \lambda, n) \triangleq \int_{\tau_s}^{\tau} e^{(-i\lambda u^2/2)}I_d(u; \lambda, n)du \qquad (n = 0, \ldots, 11) \tag{72}$$

See appendices C, D, and E for more details on $J_u(\tau; \lambda, n)$, $I_d(\tau_s, \tau; \lambda, n)$, and $J_d(\tau_s, \tau; \lambda, n)$ functions, respectively.

It can be shown that A_{v63} and A_{v64} can be determined similar to A_{v61} and A_{v62} as

$$A_{v63}(\tau;\lambda,\kappa) = \bar{F}\sqrt{\frac{\pi}{|\rho|}}\,\sum_{n=0}^{11}(a_n - i\ b_n)\left(\frac{|\rho|}{8}\right)^{\left(n+\frac{1}{2}\right)}\ \bar{J}_u(\tau;\lambda,2n+1) \tag{73}$$

$$A_{v64}(\tau_s,\tau;\lambda,\kappa) = A_{v73}(\tau_s,\tau;\lambda,\kappa) + A_{v74}(\tau_s,\tau;\lambda,\rho) \tag{74}$$

$$\begin{aligned} A_{v73}(\tau_s,\tau;\lambda,\kappa) &\triangleq \bar{F}\sqrt{\frac{\pi}{|\rho|}}\,\frac{(1+i)}{2}\int_{\tau_s}^{\tau} e^{(-i\lambda u^2/2)}\bar{I}_{u0}(\tau_s,u;-\kappa)du \\ &= \bar{F}\sqrt{\frac{\pi}{|\rho|}}\,\frac{(1+i)}{2}\left[\bar{J}_{u0}(\tau_s,\tau;\lambda,\kappa) - I_{u0}(\tau_s;\kappa)\bar{I}_{u0}(\tau_s,\tau;\lambda)\right] \end{aligned} \tag{75}$$

$$\begin{aligned} A_{v74}(\tau_s,\tau;\lambda,\rho) &\triangleq \bar{F}\sqrt{\frac{\pi}{|\rho|}}\,\sum_{n=0}^{11}(c_n - i\ d_n)\left(\frac{8}{|\rho|}\right)^{\left(n+\frac{1}{2}\right)}\int_{\tau_s}^{\tau} e^{(-i\lambda u^2/2)}I_d(\tau_s,u;\lambda,2n+1)du \\ &= \bar{F}\sqrt{\frac{\pi}{|\rho|}}\,\sum_{n=0}^{11}(c_n - i\ d_n)\left(\frac{8}{|\rho|}\right)^{\left(n+\frac{1}{2}\right)}\times \\ &\quad [I_d(\tau_s;\lambda,2n+1)\bar{I}_{u0}(\tau_s,\tau;\lambda) - J_d(\tau_s,\tau;\lambda,2n+1)] \end{aligned} \tag{76}$$

By finding the functions A_{v71}, A_{v72}, A_{v73}, and A_{v74}, the closed-form analytical solution for the axial velocity problem is completed.

VII. Simulation and Numerical Results

In this section we show our analytical and the exact solutions. By "exact solution," we mean a highly accurate numerical integration of Euler's equations of motion, Eqs.(1-3) the kinematic equations, Eqs.(8-10), and the inertial acceleration equations, Eqs.(37-38). Also, the results were presented for the 'low spin rate" and the 'high spin rate" because of the approximation of $J_{u0}(\tau_0,\tau;\mu,\rho)$ function via two piecewise continuous functions (see appendix A). The low spin rate includes spin rates from 0–2.36 rpm and the high spin rate is from for spin rates higher than 2.36 rpm which in our case is 2.36–10 rpm. The following mass properties and body-fixed forces and moments (inspired from the case of the Galileo spacecraft) are used during the simulation with all the initial conditions set to zero. We use the symbolic language manipulation software package MATHEMATICA®[35] in this simulation to generate exact and analytical solutions.

$$m = 2000\ kg, \qquad I_x = 2985, \qquad I_y = 2729, \qquad I_z = 4183\ kg.m^2 \tag{77}$$

$$f_x = 7.66, \qquad f_y = -6.42, \qquad f_z = 10.0\ N \tag{78}$$

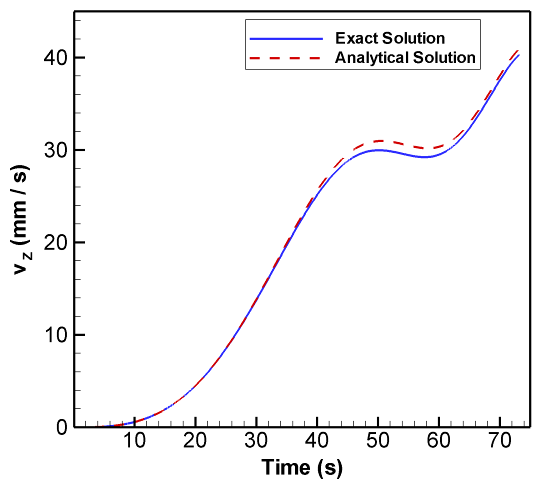
$$M_x = -1.253, \qquad M_y = -1.494, \qquad M_z = 13.5\ N.m \tag{79}$$

Figures 2(a) and 2(b) show the Z component of the axial velocity for low and high spin rates in inertial frame. The solid lines and dashed lines represent the exact and analytical solutions, respectively. As we expected, due to low gyroscopic rigidity, there is a noticeable difference between the exact and numerical solutions.

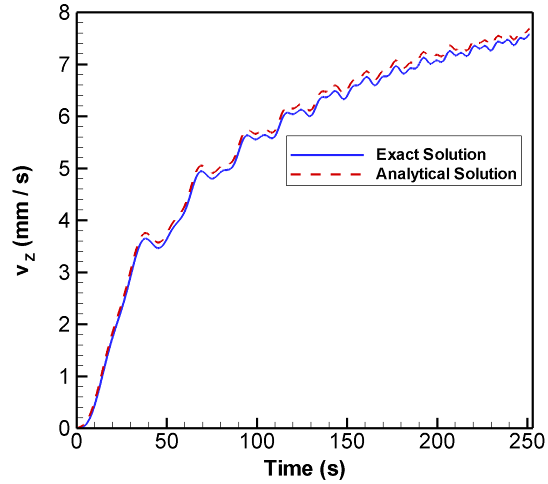
The associated error between the exact and analytical solutions of axial velocity are shown for the low and high spin rates in Figures 2(c) and 2(d), respectively. It can be seen that the order of magnitude of the axial velocity relative errors for low and high spin rates are about 10^{-2}.

VIII. Conclusion

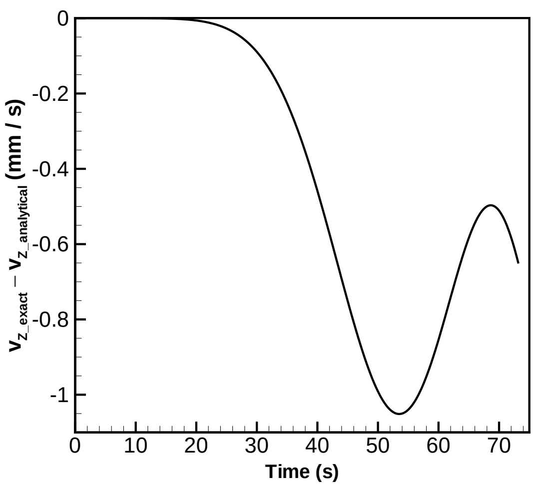
A complete, closed-form, approximate analytical solution has been found for the axial velocity of a spinning rigid body subject to constant forces and torques about all three body axes. We demonstrate that this solution is highly accurate when compared to the exact solution. Our analysis applies to axisymmetric,



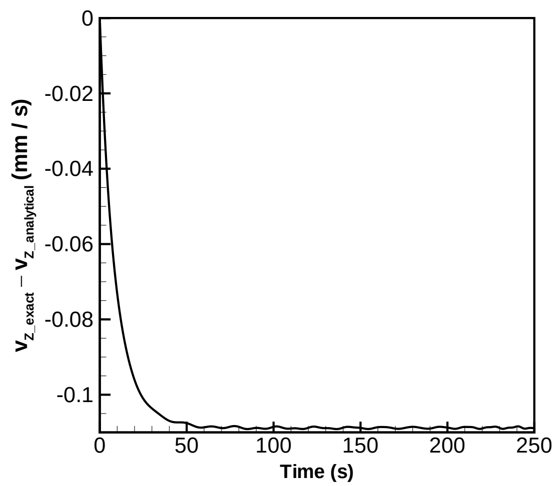
(a) Exact and analytical solutions for inertial axial velocity v_Z at low spin rate.



(b) Exact and analytical solution of inertial axial velocity v_Z at high spin rate.



(c) Exact minus analytical solution of inertial axial velocity v_Z at low spin rate.



(d) Exact minus analytical solution of inertial axial velocity v_Z at high spin rate.

Figure 2. The Z component of the axial velocity for low and high spin rates and the associated error between the exact and analytical solutions.

nearly-axisymmetric and (under special conditions) asymmetric rigid bodies. This behavior of the rigid body is fundamentally based on the Fresnel integral and related integrals. Applications of this analytical theory may include spacecraft on-board computations, probabilistic error modeling for mission-planning, development of new control concepts for spacecraft maneuvers, and maneuver analysis. This work complements the contributions of numerous authors in the literature.

APPENDIX A: $J_{u0}(\tau_0, u; \mu, \rho)$ Function

In the definition of $J_{u0}(\tau_0, u; \mu, \rho)$, Eq.(36), it was shown by Boersma[36] that J_{u0} can be approximated as

$$J_{u0}(\tau_0, \tau; \mu, \rho) = \begin{cases} J_{u0s}(\tau; \mu, \rho), & if\ \tau \leq \tau_s = \sqrt{8/|\rho|} \\ J_{u0s}(\tau_s; \mu, \rho) + J_{u0l}(\tau_s, \tau; \mu, \rho), & otherwise. \end{cases} \tag{A-1}$$

where, for $\tau \leq \tau_s$

$$J_{u0s}(u;\mu,\rho) \triangleq \sqrt{\frac{\pi}{|\rho|}} \sum_{n=0}^{11} (a_n + i\ b_n) \left(\frac{|\rho|}{8}\right)^{\left(n+\frac{1}{2}\right)} I_u(\tau;\lambda,2n+1) \tag{A-2}$$

and for $\tau > \tau_s$ we have

$$\begin{aligned} J_{u0l}(\tau_s,\tau;\mu,\rho) \triangleq & \sqrt{\frac{\pi}{|\rho|}} \frac{(1-i)}{2} \int_{\tau_s}^{u} e^{(i\mu\xi^2/2)} d\xi \\ & + \sqrt{\frac{\pi}{|\rho|}} \sum_{n=0}^{11} (c_n + i\ d_n) \left(\frac{8}{|\rho|}\right)^{\left(n+\frac{1}{2}\right)} I_d(\tau_s,\tau;\lambda,2n+1) \end{aligned} \tag{A-3}$$

See appendices B and D for more details on the $I_u(\tau;\lambda,n)$ and $I_d(\tau_s,\tau;\lambda,n)$ functions.

Table A-1. Boersma's numerical values of coefficients for the $J_{uo}(\tau;\mu,\rho)$ approximation[36]

i	a_i	b_i	c_i	d_i
0	1.595769140	-0.000000033	0.000000000	0.199471140
1	-0.000001702	4.255387524	-0.024933975	0.000000023
2	-6.808568854	-0.000092810	0.000003936	-0.009351341
3	-0.000576361	-7.780020400	0.005770956	0.000023006
4	6.920691902	-0.009520895	0.000689892	0.004851466
5	-0.016898657	5.075161298	-0.009497136	0.001903218
6	-3.050485660	-0.138341947	0.011948809	-0.017122914
7	-0.075752419	-1.363729124	-0.006748873	0.029064067
8	0.850663781	-0.403349276	0.000246420	-0.027928955
9	-0.025639041	0.702222016	0.002102967	0.016497308
10	-0.150230960	-0.216195929	-0.001217930	-0.005598515
11	0.034404779	0.019547031	0.000233939	0.000838386

APPENDIX B: $I_u(u;\lambda,n)$ Function

I_{un} function is defined as

$$I_u(\tau_0,\tau;\lambda,n) \triangleq \int_{\tau_0}^{\tau} e^{(i\lambda u^2/2)} u^n\ du \tag{B-1}$$

Using integration by parts, we can show that

$$I_u(\tau;\lambda,n) = \frac{-i\tau^{n-1}}{\lambda} e^{(i\lambda\tau^2/2)} + \frac{i(n-1)}{\lambda} I_u(\tau;\lambda,n-2) \qquad (n \geq 2) \tag{B-2}$$

$$I_u(\tau;\lambda,1) = \frac{-i}{\lambda}\left[e^{(i\lambda\tau^2/2)} - 1\right] \tag{B-3}$$

$$I_{u0}(\tau;\lambda) = \sqrt{\frac{\pi}{|\lambda|}}\ sgn(\tau)\widetilde{E}\left(\sqrt{|\lambda|/\pi}\ \tau\right) \tag{B-4}$$

where

$$\widetilde{E}\left(\sqrt{|\lambda|/\pi}\ \tau\right) = \begin{cases} E\left(\sqrt{|\lambda|/\pi}\ \tau\right) & when\ \lambda < 0 \\ \bar{E}\left(\sqrt{|\lambda|/\pi}\ \tau\right) & when\ \lambda > 0 \end{cases} \tag{B-5}$$

and

$$E(x) \triangleq \int_0^x e^{(-i\pi u^2/2)} du \tag{B-6}$$

is the complex Fresnel integral. The $sgn(.)$ symbol in Eq.(B-4) represents the signum function which is $sgn(x) = 1$ for $x \geq 0$ and $sgn(x) = -1$ for $x < 0$.

$$I_u(\tau; 0, n) = \frac{\tau^{n+1}}{n+1} \tag{B-7}$$

$$I_{u0}(\tau; 0) = \frac{\tau^2}{2} \tag{B-8}$$

It can be shown that

$$\int_0^\tau I_u(\xi; \lambda, n) d\xi = \frac{-i}{\lambda} I_u(\tau; \lambda, n-1) + \frac{i(n-1)}{\lambda} \int_0^\tau I_u(\xi; \lambda, n-2) d\xi \qquad (n \geq 2) \tag{B-9}$$

$$\int_0^\tau I_{u0}(\xi; \lambda) d\xi = \sqrt{\frac{\pi}{|\lambda|}} \int_0^\tau sgn(\xi) \widetilde{E}(\sqrt{|\lambda| / \pi}\ \xi) d\xi \tag{B-10}$$

$$\int_0^\tau I_u(\xi; \lambda, 1) d\xi = \int_0^\tau \frac{-i}{\lambda} \left[e^{(i\lambda\xi^2/2)} - 1 \right] d\xi = \frac{-i}{\lambda} [\tau - I_{u0}(\tau; \lambda)] \tag{B-11}$$

APPENDIX C: $J_u(u; \lambda, n)$ Function

We define J_u function as

$$J_u(\tau_0, \tau; \mu, \rho, n) \triangleq \int_{\tau_0}^{\tau} e^{(i\mu u^2/2)} I_u(u; \rho, n)\ du = J_u(\tau; \mu, \rho, n) - J_u(\tau_0; \mu, \rho, n) \tag{C-1}$$

For a special case, when μ and ρ are replaced with λ, the function defined in C-1 reduces to $J_u(\tau; \lambda, n)$. A recursive formula can be found by using integration by parts as

$$J_u(\tau_0, \tau; \mu, \rho, n) = \frac{i}{\rho} I_u(\tau_0, \tau; \rho - \mu, n-1) - i\frac{n-1}{\rho} J_u(\tau_0, \tau; \mu, \rho, n-2) \qquad (n \geq 2) \tag{C-2}$$

and

$$J_u(\tau_0, \tau; \mu, \rho, 1) = \frac{i}{\rho} [I_{u0}(\tau_0, \tau; \lambda) - I_{u0}(\tau_0, \tau; \mu)] \tag{C-3}$$

APPENDIX D: $I_d(u; \lambda, n)$ Function

I_{dn} function is defined as

$$I_d(\tau_0, \tau; \lambda, n) \triangleq \int_{\tau_0}^{\tau} \frac{e^{(i\lambda u^2/2)}}{u^n}\ du \tag{D-1}$$

By using integration by parts, it can be shown that

$$I_d(\tau; \lambda, n) = \begin{cases} \frac{e^{(i\lambda\tau^2/2)}}{(n-1)\tau^{(n-1)}} + \frac{i\lambda}{n-1} I_d(\tau; \lambda, n-2), & (n \geq 2; \lambda \neq 0) \\ \frac{1}{(n-1)\tau^{(n-1)}}, & (n \geq 2; \lambda = 0) \end{cases} \tag{D-2}$$

$$I_{d0}(\tau; \lambda) = \begin{cases} I_{u0}(\infty; \lambda) - I_{u0}(\tau; \lambda), & (\lambda \neq 0) \\ -\tau, & (\lambda = 0) \end{cases} \tag{D-3}$$

$$I_d(\tau; \lambda, 1) = \frac{1}{2} E_i \left(\frac{|\lambda| \tau^2}{2} \right) \tag{D-4}$$

where $E_i(\tau)$ is the Exponential Integral function and is defined as

$$E_i(\tau) \triangleq \int_{\tau}^{\infty} \frac{e^{i\xi}}{\xi}\ d\xi \tag{D-5}$$

We can show that

$$\int_{\tau_s}^{\tau} I_d(\xi;\lambda,n)d\xi = \frac{1}{n-1}\ [I_d(\tau_s,\tau;\lambda,n-1) + i\lambda\int_{\tau_s}^{\tau} I_d(\xi;\lambda,n-2)d\xi\ ] \qquad (n\geq 2) \tag{D-6}$$

and

$$\int_{\tau_s}^{\tau} I_{d0}(\xi;\lambda)d\xi = I_{u0}(\infty;\lambda)(\tau-\tau_s) - \sqrt{\frac{\pi}{|\lambda|}}\int_{\tau_0}^{\tau} sgn(\xi)\widetilde{E}(\sqrt{|\lambda|/\pi}\ \xi)d\xi \tag{D-7}$$

$$\int_{\tau_s}^{\tau} I_d(\xi;\lambda,1)d\xi = \tau I_d(\tau;\lambda,1) - \tau_s I_d(\tau_s;\lambda,1) + I_{u0}(\tau_s,\tau;\lambda) \tag{D-8}$$

APPENDIX E: $J_d(u;\lambda,n)$ Function

We defined J_{dn} as

$$J_d(\tau_s,\tau;\mu,\rho,n) \triangleq \int_{\tau_s}^{\tau} e^{(-i\mu u^2/2)} I_d(u;\rho,n)\ du \tag{E-1}$$

By using integration by parts, we can find the following recursive formula

$$\begin{aligned} J_d(\tau_0,\tau;\mu,\rho,n) = \frac{1}{(n-1)(n-2)}\left[\frac{1}{\tau_s^{(n-2)}} - \frac{1}{\tau^{(n-2)}}\right] + \\ \frac{i\lambda}{n-1} J_d(\tau_s,\tau;\mu,\rho,n-2) \qquad (n\geq 2) \end{aligned} \tag{E-2}$$

and

$$J_d(\tau_0,\tau;\mu,\rho,0) = I_{u0}(\infty;\lambda)\bar{I}_{u0}(\tau_s,\tau;\lambda) - \bar{J}_{u0}(\tau_s,\tau;\mu,\rho) \tag{E-3}$$

In general, determining $J_d(\tau_s,\tau;\mu,\rho,1)$ is not easy and in addition, what we need is $J_d(\tau_s,\tau;\lambda,1)$. In the following, we introduce an asymptotic expansion for the function $J_d(\tau_s,\tau;\lambda,1)$. Recall the $J_d(\tau_s,\tau;\lambda,1)$ definition which is

$$J_d(\tau_s,\tau;\lambda,1) \triangleq \int_{\tau_s}^{\tau} e^{(-i\lambda u^2/2)} I_d(u;\lambda,1)\ du \tag{E-4}$$

By substituting Eq.(D-4) into Eq.(E-4), we obtain

$$J_d(\tau_s,\tau;\lambda,1) = \frac{1}{2}\int_{\tau_s}^{\tau} e^{(-i\lambda u^2/2)} E_i\left(\frac{\lambda u^2}{2}\right)\ du \tag{E-5}$$

Using integration by part technique, we can show that $E_i\left(\frac{\lambda u^2}{2}\right)$ has an asymptotic expansion which is

$$E_i\left(\frac{\lambda u^2}{2}\right) \sim \sum_{n=0}^{\infty} \frac{-n!\ e^{(-i\lambda u^2/2)}}{(i\lambda u^2/2)^{(n+1)}} \tag{E-6}$$

It can be shown that after substituting Eq.(E-6)into Eq.(E-4) and integration, $J_d(\tau_s,\tau;\mu,\rho,1)$ can be written as

$$J_d(\tau_s,\tau;\mu,\rho,1) \sim \sum_{n=0}^{\infty} \frac{2^n n!}{(2n+1)(i\lambda)^{(n+1)}}\left[\frac{1}{\tau^{(2n+1)}} - \frac{1}{\tau_s^{(2n+1)}}\right] \tag{E-7}$$

The expansion (E-7) is very accurate and it can be shown that its convergence rate is fast. By only considering three terms of the series (E-7), we can show that the order of magnitude of the relative error is about 10^{-6}.

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