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TRANSVERSE VELOCITY SOLUTION FOR A SPINNING-UP RIGID BODY SUBJECT TO CONSTANT BODY-FIXED FORCES AND MOMENTS *

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A spinning, nearly-axisymmetric rigid body subject to constant body-fixed forces and moments about all three is considered. Because of constant axial torque along the spin axis, the spin rate increases linearly with time. By further assuming small deviation of spin axis (with respect to an inertially-fixed direction), an approximate closed-form analytical solution is obtained for transverse velocity. Numerical simulations confirm that the solutions are highly accurate when applied to typical motion of a spacecraft such as the Galileo spacecraft.

INTRODUCTION

In rigid body dynamics there is a rich history of analytical solutions, much of which is well represented in the treatise by Leimanis.[1] The work goes back to the classical analysis of the top and continues to modern works on spacecraft dynamics.[2–31] The early dynamicists had no access to computers and they devoted great effort to finding integrals of the motion and reducing the dynamics problem to "quadrature integrals." Kepler had a great interest in finding quadratures for simple geometric shapes and along these lines he discovered that the area swept out by a planet in its elliptical orbit about the Sun is the same at different points in the orbit when the time interval is the same. The idea of a quadrature integral is to find the square area under a given curve or function. Once a problem has been reduced to quadratures, it is possible to tabulate its value over the range of integration. Such tabulation could be

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accomplished by simple numerical integrations that can be made arbitrarily accurate. Mathematical handbooks are replete with examples of famous quadrature integrals such as Jacobian Elliptic functions, the Fresnel integrals, Bessel functions, the error function, etc.

One of the advantages of a closed-form analytical solution is to help scientists and engineers to perform parametric studies. While it is easy to numerically solve the equations of motion governing a rigid body subject to moments and forces, it is not easy to determine uncertainty in geometric parameters, mass properties and related parameters that affect the solution and the final conditions. Ref. 32 is an example of such study.

Among the numerous benefits of analytical closed-form solutions is verifying the numerical simulations and results. Suppose that we are studying a new dynamical system. We run our program on a computer and it generate reams of data. Most of the time we need to understand system dynamics to recognize incorrect numerical results. (Here we recall the old computer proverb “Garbage in, garbage out.”) By having analytical solutions we can check our results and also have better insight into the problem.

In this paper, we use the results of Tsiotras and Longuski[27] and Gick[31] to find the analytical velocity solution of a spinning-up rigid-body. The spin rate increases linearly with time and the body-fixed forces and moments are assumed constant. We present approximate closed-form solution for transverse velocities. The results are valid for axisymmetric, nearly-axisymmetric, and under certain conditions, for asymmetric rigid bodies.

Euler’s Equations of Motion

The motion of a self-excited rigid body in inertial reference frame can be described by Euler’s equations of motion which can be written as

$$\dot{\omega}_x(t) = M_x/I_x - [(I_z - I_y)/I_x]\,\omega_y\omega_z \tag{1}$$

$$\dot{\omega}_y(t) = M_y/I_y - [(I_x - I_z)/I_y]\,\omega_z\omega_x \tag{2}$$

$$\dot{\omega}_z(t) = M_z/I_z - [(I_y - I_x)/I_z]\,\omega_x\omega_y \tag{3}$$

where ω_x, ω_y, and ω_z are components of the absolute angular velocity of the rigid body in the body-fixed reference frame, M_x, M_y, and M_z are constant body-fixed moments and I_x, I_y, and I_z are principal moments of inertia around the x, y, and z axes of the body-fixed reference frame, respectively. For axisymmetric, nearly-axisymmetric rigid bodies or asymmetric rigid bodies where the product $\omega_x\omega_y$ is small enough, Eqs.(1-3) can be simplified as:

$$\dot{\omega}_x(t) = M_x/I_x - [(I_z - I_y)/I_x]\,\omega_y\omega_z \tag{4}$$

$$\dot{\omega}_y(t) = M_y/I_y - [(I_x - I_z)/I_y]\,\omega_z\omega_x \tag{5}$$

$$\dot{\omega}_z(t) \approx M_z/I_z \tag{6}$$

By integrating Eq.(6) and assuming that axial moment, M_z, is constant, we obtain

$$\omega_z(t) \approx (M_z/I_z)t + \omega_{z0}, \ \omega_{z0} \triangleq \omega_z(0) \tag{7}$$

which is, of course, exact for axisymmetric rigid bodies.

The Kinematic Equations

By using a Type I: 3-1-2 Euler angle sequence[33] which relates the orientation of body-fixed reference frame to inertial reference frame, the kinematic equations can be written as follows:

$$\dot{\phi}_x = \omega_x \ cos \ \phi_y + \omega_z \ sin \ \phi_y \tag{8}$$

$$\dot{\phi}_y = \omega_y - (\omega_z \ cos \ \phi_y - \omega_x \ sin \ \phi_y) \ tan \ \phi_x \tag{9}$$

$$\dot{\phi}_z = (\omega_z \ cos \ \phi_y - \omega_x \ sin \ \phi_y) \ sec \ \phi_x \tag{10}$$

where ϕ_x, ϕ_y and ϕ_z are the Eulerian angles. With the assumption that ϕ_x and ϕ_y are small and $\phi_y\omega_x$ is small compared to ω_z, Eqs.(8-10) can be simplified as:

$$\dot{\phi}_x = \omega_x + \omega_z \ \phi_y \tag{11}$$

$$\dot{\phi}_y = \omega_y - \phi_x\omega_z \tag{12}$$

$$\dot{\phi}_z = \omega_z \tag{13}$$

After substituting Eq.(13) into Eq.(6) and integrating, we obtain

$$\phi_z = \frac{1}{2}\frac{M_z}{I_z}t^2 + \omega_{z0}t + \phi_{z0}, \ \phi_{z0} \triangleq \phi_z(0) \tag{14}$$

The Inertial Acceleration Equation

In the presence of constant body-fixed forces f_x, f_y, and f_z, the rigid body will accelerate with respect to the inertial reference frame. The following equation relates the acceleration in the body-reference frame with respect to the inertial reference frame

$$\left\{ \begin{array}{c} \dot{v}_X(t) \\ \dot{v}_Y(t) \\ \dot{v}_Z(t) \end{array} \right\} = [A] \left\{ \begin{array}{c} f_x/m \\ f_y/m \\ f_z/m \end{array} \right\} \tag{15}$$

where [A], is the direction cosine matrix

$$[A] = \left[\begin{array}{ccc} c\phi_z c\phi_y - s\phi_z s\phi_x s\phi_y & -s\phi_z c\phi_x & c\phi_z s\phi_y + s\phi_z s\phi_x c\phi_y \\ s\phi_z c\phi_y + c\phi_z s\phi_x s\phi_y & c\phi_z c\phi_x & s\phi_z s\phi_y - c\phi_z s\phi_x c\phi_y \\ -c\phi_x s\phi_y & s\phi_x & c\phi_x c\phi_y \end{array} \right] \tag{16}$$

When ϕ_x and ϕ_y are small, the direction cosine matrix can be simplified as

$$[A] = \left[\begin{array}{ccc} c\phi_z & -s\phi_z & \phi_y c\phi_z + \phi_x s\phi_z \\ s\phi_z & c\phi_z & \phi_y s\phi_z - \phi_x c\phi_z \\ -\phi_y & \phi_x & 1 \end{array} \right] \tag{17}$$

By introducing the complex functions

$$\phi = \phi_x(t) + i\ \phi_y(t) \tag{18}$$

$$v(t) = v_X(t) + i\ v_Y(t) \tag{19}$$

$$f = f_x + i\ f_y \tag{20}$$

and using the first two rows of Eq.(15), the transverse acceleration can be written in the following compact form

$$\dot{v}(t) = e^{i\phi_z(t)}\ \left[\frac{f}{m} - \frac{if_z}{m}\ \phi(t)\right] \tag{21}$$

Before integrating Eq.(21) and finding the transverse velocity solution, we need to know the closed-form solutions for $\phi_z(t)$ and $\phi(t)$ which are given in Ref. 31. Because the closed-form solutions for the Euler angles are bases of the transverse velocity solution, in the next section we provide a brief review of those results.

Closed-Form Analytical Solution for the Eulerian Angles

By substituting a new variable τ,

$$\tau(t) \triangleq \omega_z = (M_z/I_z)\ t + \omega_{z0},\ \ \tau(0) \triangleq \omega_{z0} \tag{22}$$

into Eq.(13) and integrating with respect to τ, $\phi_z(\tau)$ can be obtained as

$$\phi_z(\tau) = \frac{\lambda}{2}(\tau^2 - \tau_0^2) + \phi_{z0} \tag{23}$$

where λ is defined as

$$\lambda \triangleq \frac{I_z}{M_z} \tag{24}$$

It can be shown that Eqs.(11-12) can be combined and written in the following complex form

$$\phi'(\tau) + i\ \lambda\tau\phi(\tau) = \lambda\omega(\tau) \tag{25}$$

where the complex function $\omega(\tau)$ is defined as

$$\omega(\tau) = \omega_x(\tau) + i\ \omega_y(\tau) \tag{26}$$

Equation(25) is a first order non-homogeneous differential equation with a variable coefficient. It can be shown that the solution is:[27]

$$\phi(\tau) = \phi(\tau_0)\ e^{-i\lambda(\tau^2-\tau_0^2)/2} + \lambda\ e^{(-i\lambda\tau^2/2)}\ I_\phi(\tau_0, \tau; \lambda, \rho) \tag{27}$$

where

$$I_\phi(\tau_0, \tau; \lambda, \rho) \triangleq \int_{\tau_0}^{\tau} e^{(i\lambda u^2/2)}\omega(u)\ du \tag{28}$$

$$I_\phi(\tau_0, \tau; \lambda, \rho) = k_1 I_{\phi 1}(\tau_0, \tau; \lambda, \rho) + k_2 I_{\phi 2}(\tau_0, \tau; \kappa, \rho) \tag{29}$$

$$k_1 \triangleq (\sqrt{k_x} + \sqrt{k_y})/2k, \qquad k_2 \triangleq (\sqrt{k_x} - \sqrt{k_y})/2k \tag{30}$$

$$k_x \triangleq (I_z - I_y)/I_x, \qquad k_y \triangleq (I_z - I_x)/I_y \tag{31}$$

$$k \triangleq \sqrt{k_x\ k_y} \tag{32}$$

where the k's represent the mass properties of the rigid body and $I_{\phi 2}$ provides the contribution from an asymmetric body.

$$I_{\phi 1}(\tau_0, \tau; \lambda, \rho) \triangleq \int_{\tau_0}^{\tau} e^{(i\lambda u^2/2)} \Omega(u)\ du \tag{33}$$

$$I_{\phi 2}(\tau_0, \tau; \kappa, \rho) \triangleq \int_{\tau_0}^{\tau} e^{(i\lambda u^2/2)} \bar{\Omega}(u)\ du \tag{34}$$

We see that the integrals $I_{\phi 1}$ and $I_{\phi 2}$ have $\Omega(u)$ in the integrand, which represents the solution for the Euler's equations of motion:

$$\Omega(u) = \Omega_x(u) + i\ \Omega_y(u) \tag{35}$$

$$\Omega_x(u) = \omega_x(u)\sqrt{k_y}, \qquad \Omega_y(u) = \omega_y(u)\sqrt{k_x} \tag{36}$$

$$\begin{aligned} I_{\phi 1}(\tau_0, \tau; \lambda, \rho) = {} & [\Omega(\tau_0) e^{(-i\rho\tau_0^2/2)} - F\ \bar{I}_{u0}(\tau_0; \rho)]\ \bar{I}_{u0}(\tau_0, \tau; -\mu) \\ & + F J_{u0}(\tau_0, \tau; \mu, \rho) \end{aligned} \tag{37}$$

$$\begin{aligned} I_{\phi 2}(\tau_0, \tau; \kappa, \rho) = {} & [\bar{\Omega}(\tau_0) e^{(i\rho\tau_0^2/2)} - \bar{F}\ I_{u0}(\tau_0; \rho)]\ I_{u0}(\tau_0, \tau; \kappa) \\ & + \bar{F}\ \bar{J}_{u0}(\tau_0, \tau; -\kappa, \rho) \end{aligned} \tag{38}$$

$$\rho \triangleq k\lambda, \qquad \mu \triangleq \lambda + \rho, \qquad \kappa \triangleq \lambda - \rho \tag{39}$$

Here we observe that $I_{\phi 1}$ and $I_{\phi 2}$ depend ultimately on the Fresnel integral:

$$I_{u0}(\tau_0, \tau; \lambda) \triangleq \int_{\tau_0}^{\tau} e^{(i\lambda u^2/2)}\ du \tag{40}$$

$$F = F_x + iF_y \tag{41}$$

where F represents the transverse body-fixed torque:

$$F_x \triangleq (M_x/I_x)(I_z/M_z)\sqrt{k_y} \tag{42}$$

$$F_y \triangleq (M_y/I_y)(I_z/M_z)\sqrt{k_x} \tag{43}$$

and where J_{u0} is an integral of the Fresnel integral:

$$J_{u0}(\tau_0, \tau; \mu, \rho) \triangleq \int_{\tau_0}^{\tau} e^{(i\mu u^2/2)} \; \bar{I}_{u0}(u; \rho) \; du \tag{44}$$

which we analyze later. Now by having the closed-form solutions for the Eulerian angles, we will find the transverse velocity solution in the next section.

TRANSVERSE VELOCITY SOLUTION

Before integrating Eq.(21), we need to replace the variable t with τ by multiplying both sides of Eq.(21) by $dt/d\tau$ as follows

$$\frac{dv}{d\tau} = \frac{\lambda}{m} \; e^{i\phi_z(\tau)}[\; f - if_z \; \phi(\tau)] \tag{45}$$

after integration of both sides, we get

$$v(\tau) = v(\tau_0) + \frac{\lambda f}{m} T_{11}(\tau_0, \tau; \lambda) - \frac{i\lambda f_z}{m} T_{11}(\tau_0, \tau; \lambda) \tag{46}$$

where T_{11} and T_{12} are defined as

$$T_{11}(\tau_0, \tau; \lambda) \triangleq \int_{\tau_0}^{\tau} e^{i\phi_z(u)} \; du \tag{47}$$

$$T_{12}(\tau_0, \tau; \lambda) \triangleq \int_{\tau_0}^{\tau} e^{i\phi_z(u)} \phi(u) du \tag{48}$$

By substituting Eq.(23) into Eq.(47), T_{11} can be written in the following form.

$$T_{11}(\tau_0, \tau; \lambda) = e^{i(\phi_{z0} - \lambda\tau_0^2/2)} \; I_{u0}(\tau_0, \tau; \lambda) \tag{49}$$

where

$$I_{u0}(u; \mu) = \sqrt{\frac{\pi}{|\mu|}} \; sgn(u) \widetilde{E} \left(\sqrt{\frac{\pi}{|\mu|}} \; u \right) \tag{50}$$

$$\widetilde{E}(x) = \begin{cases} E(x) & when \; \mu < 0. \\ \bar{E}(x) & when \; \mu \geq 0. \end{cases} \tag{51}$$

and

$$E(x) \triangleq \int_0^x e^{(-i\pi u^2/2)} du \tag{52}$$

is the complex Fresnel integral. The $sgn(.)$ symbol in Eq.(50) represents the signum function which is $sgn(x) = 1$ for $x \geq 0$ and $sgn(x) = -1$ for $x < 0$.

Substituting Eq.(23) and Eq.(27) into Eq.(48) (after some algebra) provides

$$T_{12}(\tau_0, \tau; \lambda) = \phi(\tau_0)e^{i\phi_{z0}}(\tau - \tau_0) + \lambda\ e^{i(\phi_{z0} - \lambda\tau_0^2/2)} T_{21}(\tau_0, \tau; \lambda, \rho) \tag{53}$$

where

$$T_{21}(\tau_0, \tau; \lambda, \rho) \triangleq \int_{\tau_0}^{\tau} I_\phi(\tau_0, u; \lambda, \rho) du \tag{54}$$

Substituting Eq.(29) into Eq.(54), yields

$$T_{21}(\tau_0, \tau; \lambda, \rho) = k_1 T_{31}(\tau_0, \tau; \lambda, \rho) + k_2 T_{32}(\tau_0, \tau; \kappa, \rho) \tag{55}$$

where T_{31} and T_{32} are defined as

$$T_{31}(\tau_0, \tau; \lambda, \rho) \triangleq \int_{\tau_0}^{\tau} I_{\phi 1}(\tau_0, u; \lambda, \rho) du \tag{56}$$

$$T_{32}(\tau_0, \tau; \kappa, \rho) \triangleq \int_{\tau_0}^{\tau} I_{\phi 2}(\tau_0, u; \kappa, \rho) du \tag{57}$$

By integration by parts, we can show that T_{31} and T_{32} can be determined as

$$\begin{aligned} T_{31}(\tau_0, \tau; \lambda, \rho) = {} & \tau I_{\phi 1}(\tau_0, \tau; \lambda, \rho) - \frac{F}{i\mu}\left[e^{i\mu\tau^2/2}\bar{I}_{u0}(\tau; \rho) - e^{i\mu\tau_0^2/2}\bar{I}_{u0}(\tau_0; \rho)\right] \\ & - (i\mu)^{-1}\left[\Omega_0 e^{-i\rho\tau_0^2/2} - F\ \bar{I}_{u0}(\tau_0; \rho)\right]\left(e^{i\mu\tau^2/2} - e^{i\mu\tau_0^2/2}\right) \\ & + \frac{F}{i\mu} I_{u0}(\tau_0, \tau; \lambda) \end{aligned} \tag{58}$$

$$\begin{aligned} T_{32}(\tau_0, \tau; \kappa, \rho) = {} & \tau I_{\phi 2}(\tau_0, \tau; \kappa, \rho) - \frac{F}{i\kappa}\left[e^{i\kappa\tau^2/2} I_{u0}(\tau; \rho) - e^{i\kappa\tau_0^2/2} I_{u0}(\tau_0; \rho)\right] \\ & - (i\kappa)^{-1}\left[\bar{\Omega}_0 e^{i\rho\tau_0^2/2} - \bar{F}\ I_{u0}(\tau_0; \rho)\right]\left(e^{i\kappa\tau^2/2} - e^{i\kappa\tau_0^2/2}\right) \\ & + \frac{\bar{F}}{i\kappa} I_{u0}(\tau_0, \tau; \lambda) \end{aligned} \tag{59}$$

With T_{31} and T_{32}, the analytical closed-form transverse velocity solution is completed.

SIMULATION AND NUMERICAL RESULTS

We compare our analytical solution with the exact solution. By "exact solution," we mean a highly accurate numerical integration of Eqs. (1-3, 8-10, 15-16). Because of the approximation of $J_{u0}(\tau_0, \tau; \mu, \rho)$ function via two piecewise continuous functions (see appendix A) for small and large arguments, we present the results for two cases: The "low spin rate" and the "high spin rate." The low spin rate includes spin rates from 0–2.24 rpm and the high spin rate is from for spin rates higher than 2.24 rpm which in our case is 2.24–10 rpm. We note that our approximation solution for J_{u0} is very accurate for all spin rates–it does not break down at intermediate spin rates. The following mass properties and body-fixed forces and moments (inspired from the

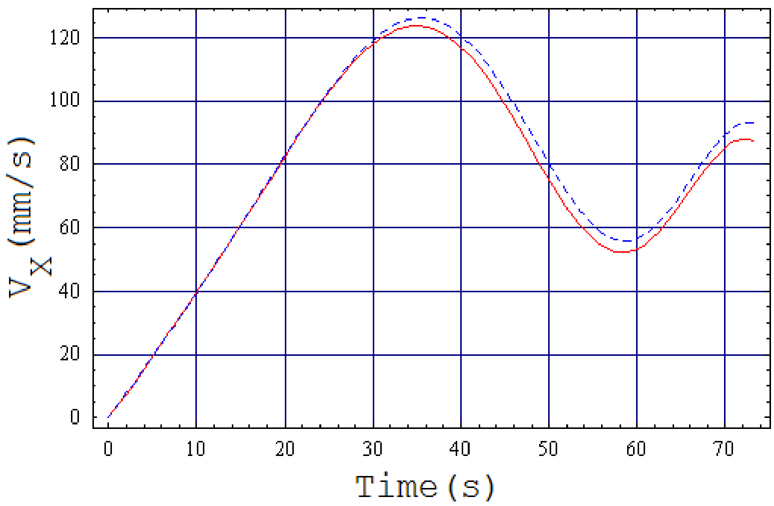

Figure 1: Exact and Analytical Solutions for Inertial Velocity v_X at Low Spin Rate

case of the Galileo spacecraft) are used during the simulation with setting all the initial conditions to zero.

$$m = 2000\ kg, \qquad I_x = 2985, \qquad I_y = 2729, \qquad I_z = 4183\ kg.m^2 \tag{60}$$

$$f_x = 7.66, \qquad f_y = -6.42, \qquad f_z = 10.0\ N \tag{61}$$

$$M_x = -1.253, \qquad M_y = -1.494, \qquad M_z = 13.5\ N.m \tag{62}$$

Figure 1 shows the x component of the transverse velocity for low spin rate in inertial frame. The solid line and dashed lines represent the exact and analytical solutions, respectively. Because of the low spin rate (corresponding to low gyroscopic rigidity), there is a noticeable difference between the exact and numerical solutions. The associated error between the exact and analytical solutions is shown in Fig. 2. It can be seen that the order of magnitude of the relative error is about 10^{-2} mm/s. In Fig. 3, we show the x component of the transverse velocity for high spin rate. Again the solid line represents the exact solution and the dashed line represents the analytical solution which are indistinguishable in this case. Figure 4 shows the associated error between the exact and analytical solutions. The order of magnitude of the relative error is about 10^{-3} mm/s.

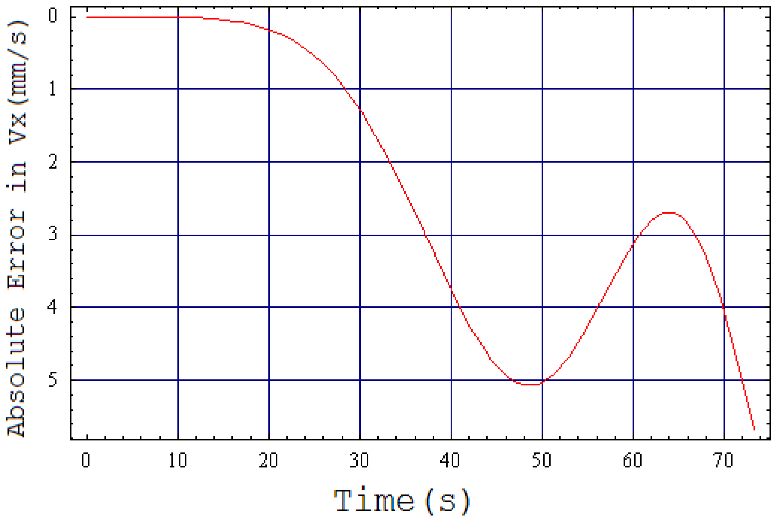

Figure 2: Exact Minus Analytical Solution of Inertial Velocity v_X at Low Spin Rate

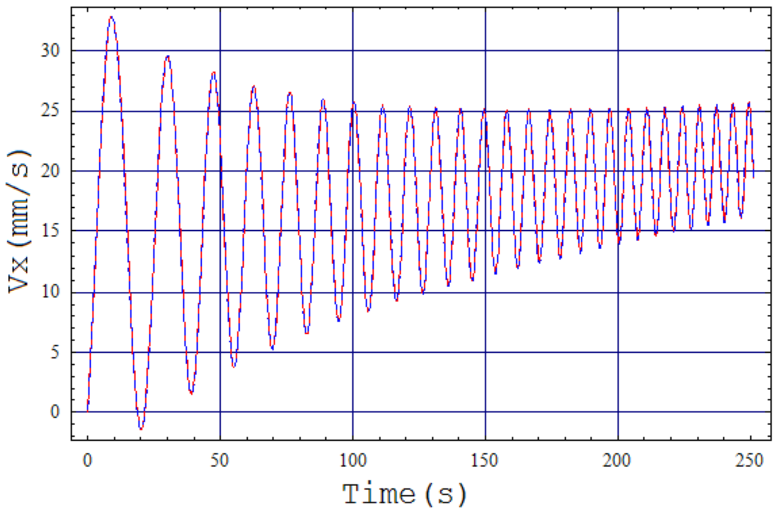

Figure 3: Exact and Analytical Solutions for Inertial Velocity v_X at High Spin Rate

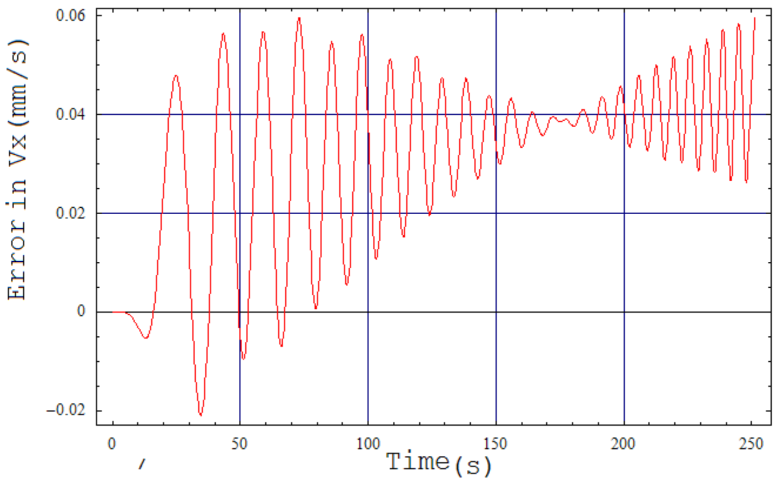

Figure 4: Exact Minus Analytical Solution of Inertial Velocity v_X at High Spin Rate

CONCLUSION

A complete, closed-form, approximate analytical solution has been found for the transverse velocity of a spinning rigid body subject to constant forces and torques about all three body axes. We demonstrate that this solution is highly accurate when compared to the exact solution. Our analysis applies to axisymmetric, nearly-axisymmetric and (under special conditions) asymmetric rigid bodies. This behavior of the rigid body is fundamentally based on the Fresnel integral and related integrals. Applications of this analytical theory may include spacecraft on-board computations, probabilistic error modeling for mission-planning, and development of the new control concepts for spacecraft maneuvers. The solution follows the classical works of Rosser and Davis from the 1940's and 1950's which seek to develop a mathematical theory of rocket flight; it complements the contributions of numerous contemporary authors in the literature.

NOTATION

A = transformation matrix relating body and inertial frames
c = cosine
F = rescaled transverse body-fixed moments, $1/s^2$
f = body-fixed force, N
I = moment of inertia, kg-m^2
M = body-fixed moment, N-m
m = rigid body mass, kg

ϕ	= Euler angle, rad
Ω	= rescaled transverse angular velocities, rad/s
ω	= angular velocity, rad/s
τ	= spin rate, rad/s
Subscripts	
X,Y,Z	= components in inertial frame
x,y,z	= components in body-fixed frame

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APPENDIX

A. Approximating J_{u0} function

In the definition of $J_{u0}(\tau_0, u; \mu, \rho)$, Eq.(44), it was shown[34] that J_{u0} can be approximated as

$$J_{u0}(\tau_0, u; \mu, \rho) = \begin{cases} J_{u0s}(u; \mu, \rho), & if\ u \leq \tau_s = \sqrt{8/|\rho|} \\ J_{u0s}(\tau_s; \mu, \rho) + J_{u0l}(\tau_s, u; \mu, \rho), & otherwise. \end{cases} \tag{A-1}$$

where, for $u \leq \tau_s$

$$\begin{aligned} J_{u0s}(u; \mu, \rho) \triangleq & \sqrt{\frac{\pi}{|\rho|}} \sum_{n=0}^{11} (a_n + i\ b_n) \left(\frac{|\rho|}{8}\right)^{\left(n+\frac{1}{2}\right)} \\ & \times \int_0^u e^{(i\lambda\xi^2/2)} \xi^{2n+1} d\xi \end{aligned} \tag{A-2}$$

and for $u \geq \tau_s$ we have

$$\begin{aligned} J_{u0l}(\tau_s, u; \mu, \rho) \triangleq & \sqrt{\frac{\pi}{|\rho|}} \frac{(1-i)}{2} \int_{\tau_s}^u e^{(i\mu\xi^2/2)} d\xi \\ & + \sqrt{\frac{\pi}{|\rho|}} \sum_{n=0}^{11} (c_n + i\ d_n) \left(\frac{8}{|\rho|}\right)^{\left(n+\frac{1}{2}\right)} \int_{\tau_s}^u \frac{e^{(i\lambda\xi^2/2)}}{\xi^{2n+1}}\ d\xi \end{aligned} \tag{A-3}$$

B. $I_u(u; \lambda, n)$ function and its integral

$$I_u(\tau_0, \tau; \lambda, n) \triangleq \int_{\tau_0}^{\tau} e^{(i\lambda u^2/2)} u^n\ du \tag{B-1}$$

$$\begin{aligned} \int_0^\tau I_u(\xi; \lambda, n) d\xi = & \frac{-i}{\lambda} I_u(\tau; \lambda, n-1) \\ & + \frac{i(n-1)}{\lambda} \int_0^\tau I_u(\xi; \lambda, n-2) d\xi \qquad (n = 0, \ldots, 11) \end{aligned} \tag{B-2}$$

$$\int_0^\tau I_{u0}(\xi; \lambda) d\xi = \sqrt{\frac{\pi}{|\lambda|}} \int_0^\tau sgn(\xi) \widetilde{E}(\sqrt{|\lambda|/\pi}\ \xi) d\xi \tag{B-3}$$

$$\begin{aligned} \int_0^\tau I_u(\xi; \lambda, 1) d\xi &= \int_0^\tau \frac{-i}{\lambda} \left[e^{(i\lambda\xi^2/2)} - 1\right] d\xi \\ &= \frac{-i}{\lambda} [\tau - I_{u0}(\tau; \lambda)] \end{aligned} \tag{B-4}$$

Table A-1:
BOERSMA'S NUMERICAL VALUES OF COEFFICIENTS FOR THE $J_{uo}(\tau;\mu,\rho)$ APPROXIMATION [34]

i	a_i	b_i	c_i	d_i
0	1.595769140	-0.000000033	0.000000000	0.199471140
1	-0.000001702	4.255387524	-0.024933975	0.000000023
2	-6.808568854	-0.000092810	0.000003936	-0.009351341
3	-0.000576361	-7.780020400	0.005770956	0.000023006
4	6.920691902	-0.009520895	0.000689892	0.004851466
5	-0.016898657	5.075161298	-0.009497136	0.001903218
6	-3.050485660	-0.138341947	0.011948809	-0.017122914
7	-0.075752419	-1.363729124	-0.006748873	0.029064067
8	0.850663781	-0.403349276	0.000246420	-0.027928955
9	-0.025639041	0.702222016	0.002102967	0.016497308
10	-0.150230960	-0.216195929	-0.001217930	-0.005598515
11	0.034404779	0.019547031	0.000233939	0.000838386

C. $I_d(u;\lambda,n)$ function and its integral

$$I_d(\tau_0,\tau;\lambda,n) \triangleq \int_{\tau_0}^{\tau} \frac{e^{(i\lambda u^2/2)}}{u^n}\, du \qquad \text{(C-1)}$$

$$\begin{aligned}\int_{\tau_s}^{\tau} I_d(\xi;\lambda,n)d\xi = \frac{1}{n-1}\,[I_d(\tau_s,\tau;\lambda,n-1) \\ + i\lambda \int_{\tau_s}^{\tau} I_d(\xi;\lambda,n-2)d\xi\ ] \qquad (n=0,\ldots,11)\end{aligned} \qquad \text{(C-2)}$$

$$\int_{\tau_s}^{\tau} I_{d0}(\xi;\lambda)d\xi = I_{u0}(\infty;\lambda)(\tau-\tau_s) - \sqrt{\frac{\pi}{|\lambda|}} \int_{\tau_0}^{\tau} sgn(\xi)\widetilde{E}(\sqrt{|\lambda|/\pi}\ \xi)d\xi \qquad \text{(C-3)}$$

$$\int_{\tau_s}^{\tau} I_d(\xi;\lambda,1)d\xi = \tau I_d(\tau;\lambda,1) - \tau_s I_d(\tau_s;\lambda,1) + I_{u0}(\tau_s,\tau;\lambda) \qquad \text{(C-4)}$$