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Analytic Solution for the Velocity of a Rigid Body During Spinning-Up Maneuvers

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When attitude maneuvers are performed with thrusters, the ensuing spacecraft motion may involve both rotational and translational behavior. Using a complex formulation, analytic solutions are derived for the translational velocity of a near-symmetric rigid body subject to constant body-fixed torques and forces. For the case of an axisymmetric body, the solution of Euler's equations of motion is exact. Due to a small angle assumption, the analytic solutions for the Eulerian angles and the inertial velocities are approximate, but these apply to a wide variety of practical problems. Numerical solutions for a typical problem involving constant body-fixed torques and forces provide an indication of the accuracy.

Introduction

The thrusting, spinning rocket problem (see Fig. 1 and [1]) represents a new challenge in the analysis of rigid body motion. This problem was unknown to early dynamicists such as Euler, Lagrange, Poinsot and others. Classical analysis of rigid body motion has provided closed-form solutions for torque-free motion and the motion of a top. The thrusting, spinning rocket problem is the next logical step in the analytic solution of rigid body behavior. This practical modern day problem is more difficult to solve analytically than the known classical problems, where integrals of the motion exist. It involves forced motion where forces and torques operate along all three body axes, and is governed by highly nonlinear differential equations.

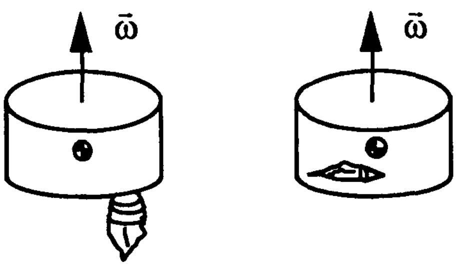

Fig. 1 Thrusting / Spinning-up maneuver [1].

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Analytic solutions for this type of problem extend our knowledge of the fundamental behavior of a rigid body subject to body-fixed forces and torques. Analytic models can be of great help in obtaining a qualitative understanding of the complex dynamic behavior; even a simple heuristic analytic result may provide a quick and relatively accurate solution for maneuver analysis. Analytic solutions for the attitude motion of a rigid body have been obtained recently for the constant body-fixed torque problem by Longuski [2,3] and Tsiotras and Longuski [4]. An analytic solution for the transverse velocity components accumulated during the spin-up maneuver (corresponding to the right figure of Fig. 1) is obtained by Hintz and Longuski [5] in terms of a simple Fresnel integral. Integration of the transverse equations is significantly facilitated by neglecting the axial force. In Klumpe and Longuski [6] an attempt is made to include the axial force (corresponding to the combination of both figures in Fig. 1) and after cumbersome analysis, a solution is found for the secular terms.

In this paper we briefly review the solution of Tsiotras and Longuski [4] and use it for the foundation of the analytic solution for the velocity problem. The resulting solution provides the translational velocity components of the thrusting, spinning rocket problem in a much more complete form. It incorporates the previously known results, but in addition includes periodic behavior that was not addressed in earlier investigations.

Euler's Equations of Motion

The motion of a rigid body with respect to the center of mass is governed by Euler's equations of motion

$$M_x = I_x \dot{\omega}_x + (I_z - I_y)\omega_y\omega_z \quad (1a)$$

$$M_y = I_y \dot{\omega}_y + (I_x - I_z)\omega_z\omega_x \quad (1b)$$

$$M_z = I_z \dot{\omega}_z + (I_y - I_x)\omega_x\omega_y \quad (1c)$$

where M_x, M_y and M_z are torque components, ω_x, ω_y and ω_z are the angular velocity components, and I_x, I_y and I_z are the principal moments of inertia. As usual, a dot represents differentiation with respect to time. We will assume that $I_z > I_x > I_y$ and the body is spinning about its z axis. No explicit analytic solutions of this system of nonlinear differential equations are known to exist for arbitrary functions of the external torques M_x, M_y and M_z. In fact, no exact solutions are known, without the need of some simplifying assumptions, even for the case of M_x, M_y and M_z being constant. Assuming that only M_z is constant and that the last term of equation (1c) is small (either because of near symmetry or because of the product of $\omega_x \omega_y$ being small) we obtain

$$\omega_z \approx (M_z / I_z)\, t + \omega_{z0}, \qquad \omega_{z0} \equiv \omega_z(0). \tag{2}$$

Of course, equation (2) is the exact solution for ω_z, for an axisymmetric rigid body. This approximation has been very useful in previous developments [2-4]. This approximation is very accurate also for the case of a spin-stabilized spacecraft, when both ω_x and ω_y tend to remain small, even when no symmetry assumption can be made. The approximation in the solution of equation (1c) allows us to decouple the third order system of nonlinear differential equations (1). Therefore, assuming the validity of equation (2), we can merely concentrate on equations (1a) and (1b), which now become a set of two coupled, but linear time-varying differential equations.

Analytic Solution for the Angular Velocities

Although we assume that $I_x = I_y$ in order to uncouple equation (1c) from equations (1a) and (1b), we will retain the distinction between I_x and I_y in the latter two equations. Therefore in essence, we have replaced the system of equations (1) by the following system

$$M_x = I_x \dot{\omega}_x + (I_z - I_y)\omega_y \omega_z \tag{3a}$$

$$M_y = I_y \dot{\omega}_y + (I_x - I_z)\omega_z \omega_x \tag{3b}$$

$$M_z = I_z \dot{\omega}_z. \tag{3c}$$

Defining the new independent variable

$$\tau(t) \equiv \omega_z(t) = (M_z / I_z)\, t + \omega_{z0}, \quad \tau(0) = \omega_{z0} = \tau_0 \tag{4}$$

and the transformation of the dependent variables

$$\Omega_x = \omega_x \sqrt{k_y}, \qquad \Omega_y = \omega_y \sqrt{k_x} \tag{5a}$$

$$k_x \equiv (I_z - I_y)/I_x, \qquad k_y \equiv (I_z - I_x)/I_y \tag{5b}$$

$$k \equiv \sqrt{k_x\, k_y} \tag{5c}$$

allows us to combine equations (3a) and (3b) into the following linear first order scalar, but complex differential equation with time-varying coefficient

$$\Omega' - i\rho\tau\Omega = F \tag{6}$$

where

$$\Omega \equiv \Omega_x + i\Omega_y, \quad \rho \equiv k(I_z / M_z), \quad F \equiv F_x + iF_y \tag{7}$$

$$F_x \equiv (M_x / I_x)(I_z / M_z)\sqrt{k_y} \tag{8a}$$

$$F_y \equiv (M_y / I_y)(I_z / M_z)\sqrt{k_x}. \tag{8b}$$

Note that the prime in (6) denotes differentiation with respect to the new independent variable τ. As shown in [4] the solution for the transverse angular velocities can be written immediately as follows:

$$\Omega(\tau) = \Omega_0 \exp\left(i\,\rho\tau^2/2\right) + F\exp\left(i\,\rho\tau^2/2\right) I_0(\tau_0, \tau; \rho), \quad \Omega_0 \equiv \Omega(\tau_0). \tag{9}$$

The first term in the above expression is the solution due to the initial conditions, also called the homogeneous solution. The second term describes the forced response due to the forcing function F, also called the nonhomogeneous solution. The only difficulty that arises in the computation of the solution for the transverse angular velocities ω_x and ω_y comes from the integral of the nonhomogeneous solution. We are therefore, merely interested in computing the integral appearing in (9) where

$$I_0(\tau_0, \tau; \rho) \equiv \int_{\tau_0}^{\tau} \exp\left(-i\,\rho u^2/2\right) du = I_0(\tau; \rho) - I_0(\tau_0; \rho) \tag{10}$$

and

$$I_0(x; \rho) \equiv \int_0^x \exp\left(-i\,\rho u^2/2\right) du. \tag{11}$$

Integrals of the form

$$I_n(x; \rho) \equiv \int_0^x \exp\left(-i\,\rho u^2/2\right) u^n\, du \quad n = 0, 1, 2, \ldots, m \tag{12}$$

can be easily evaluated by means of the recurrence formula

$$I_n(x;\rho) = i\frac{x^{n-1}}{\rho}\exp\left(-i\,\rho x^2/2\right) - i\frac{n-1}{\rho}\int_0^x \exp\left(-i\,\rho u^2/2\right) u^{n-2}\,du \qquad (13)$$

$$n = 2,3,4,\ldots$$

To obtain all the integrals from (12) we must find the first two terms of the sequence. This can be done as follows:

$$I_0(x;\rho) \equiv \int_0^x \exp\left(-i\,\rho u^2/2\right) du = \sqrt{\pi/|\rho|}\; s(x)\; E\left(\sqrt{|\rho|/\pi}\,x\right) \qquad (14)$$

$$I_1(x;\rho) \equiv \int_0^x \exp\left(-i\,\rho u^2/2\right) u\,du = (i/\rho)\left[\exp\left(-i\,\rho x^2/2\right) - 1\right] \qquad (15)$$

where s(x) is defined as the signum function, given by

$$s(x) \equiv \mathrm{sgn}(x) = \begin{cases} +1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases} \qquad (16)$$

and E(x) represents the complex Fresnel integral function of the first kind defined by

$$E(x) \equiv \int_0^x \exp\left(-i\,\pi u^2/2\right) du. \qquad (17)$$

We have assumed in equation (14) that ρ is positive. In the event that ρ is negative we merely take the complex conjugate of equation (14), that is,

$$I_0^*(x;\rho) = I_0(x;-\rho) \qquad (18)$$

where the asterisk denotes complex conjugation. Both cases can be handled at the same time by defining the function

$$\bar{E}\left(\sqrt{|\rho|/\pi}\,x\right) \equiv \begin{cases} E\left(\sqrt{|\rho|/\pi}\,x\right) & \rho > 0 \\ E^*\left(\sqrt{|\rho|/\pi}\,x\right) & \rho < 0. \end{cases} \qquad (19)$$

Then we find that, for both cases, the integral $I_0(x;\rho)$ can be evaluated by

$$I_0(x;\rho) = \sqrt{\pi/|\rho|}\; s(x)\; \bar{E}\left(\sqrt{|\rho|/\pi}\,x\right). \qquad (20)$$

Analytic Solution for the Eulerian Angles

Using a 3-1-2 Euler angle sequence to describe the orientation of the body-fixed reference frame, with respect to an inertially fixed reference frame, the following kinematic equations hold:

$$\begin{aligned} \dot{\varphi}_x &= \omega_x \cos\varphi_y + \omega_z \sin\varphi_y \\ \dot{\varphi}_y &= \omega_y - (\omega_z \cos\varphi_y - \omega_x \sin\varphi_y)\tan\varphi_x \\ \dot{\varphi}_z &= (\omega_z \cos\varphi_y - \omega_x \sin\varphi_y)\sec\varphi_x. \end{aligned} \qquad (21)$$

A small angle approximation for φ_x and φ_y and a further assumption that the product $\varphi_y\omega_x$ is small compared to ω_z, allows us to solve directly for φ_z and reduces equations (21) to equations (22).

$$\begin{aligned} \dot{\varphi}_x &= \omega_x + \varphi_y \omega_z \\ \dot{\varphi}_y &= \omega_y - \varphi_x \omega_z \\ \dot{\varphi}_z &= \omega_z \end{aligned} \qquad (22)$$

Introducing the new independent variable τ as was done in (4) the solution for $\varphi_z(\tau)$ is given by

$$\varphi_z(\tau) = \lambda\left(\tau^2 - \tau_0^2\right)/2 + \varphi_{z0}, \quad \varphi_{z0} \equiv \varphi_z(\tau_0) \qquad (23)$$

where λ is defined by

$$\lambda \equiv \rho / k = I_z / M_z. \qquad (24)$$

Using the solution of ω_z from equation (2), we can combine the first two equations of (22) into the following single complex equation

$$\varphi' + i\,\lambda\tau\varphi = \lambda\omega \qquad (25)$$

where the complex variables $\varphi(\tau)$ and $\omega(\tau)$ are defined as

$$\varphi(\tau) \equiv \varphi_x(\tau) + i\,\varphi_y(\tau), \quad \omega(\tau) \equiv \omega_x(\tau) + i\,\omega_y(\tau). \qquad (26)$$

The differential equation (25) has the solution

$$\varphi(\tau) = \varphi_0 \exp\left(-i\,\lambda\tau^2/2\right) + \lambda\exp\left(-i\,\lambda\tau^2/2\right) I_\varphi(\tau_0,\tau;\lambda,\rho) \qquad (27)$$

where $\varphi_0 \equiv \varphi(\tau_0)$ is the initial condition for the transverse angles φ_x and φ_y in the new independent variable. The nonhomogeneous solution involves the integral $I_\varphi(\tau_0,\tau;\lambda,\rho)$ which is defined as follows:

$$I_\varphi(\tau_0,\tau;\lambda,\rho) \equiv \int_{\tau_0}^{\tau} \exp\left(i\,\lambda u^2/2\right)\omega(u)\,du. \qquad (28)$$

Therefore, in order to solve for the Eulerian angles, we need to evaluate the integral $I_\varphi(\tau_0,\tau;\lambda,\rho)$. The solution to (28) involves an expression for $\omega(\tau)$ instead of $\Omega(\tau)$ which has been already found in equation (9). However, it is easy to see that $\omega(\tau)$ can be expressed in terms of the already known solution of $\Omega(\tau)$ by using the relationships in equations (5a), (7) and (26) to obtain

$$\omega(\tau) = k_1\,\Omega(\tau) + k_2\,\Omega^*(\tau). \qquad (29)$$

The method for the evaluating $I_\varphi(\tau_0,\tau;\lambda,\rho)$ is outlined as follows. We rewrite $I_\varphi(\tau_0,\tau;\lambda,\rho)$ as the sum of two independent integrals

$$I_\varphi(\tau_0,\tau;\lambda,\rho) = k_1 I_{\varphi_1}(\tau_0,\tau;\lambda,\rho) + k_2 I_{\varphi_2}(\tau_0,\tau;\lambda,\rho) \qquad (30)$$

where k_1 and k_2 correspond to the symmetric and nonsymmetric portions of the solution respectively. The terms k_1 and k_2 are defined as follows:

$$k_1 = \left(\sqrt{k_x} + \sqrt{k_y}\right)/2k, \quad k_2 = \left(\sqrt{k_x} - \sqrt{k_y}\right)/2k. \qquad (31)$$

Therefore in order to evaluate $I_\varphi(\tau_0,\tau;\lambda,\rho)$ we need to consider the two independent integrals of equation (30) defined as

$$I_{\varphi_1}(\tau_0,\tau;\lambda,\rho) \equiv \int_{\tau_0}^{\tau} \exp\left(i\,\lambda u^2/2\right)\Omega(u)\,du \qquad (32)$$

$$I_{\varphi_2}(\tau_0,\tau;\lambda,\rho) \equiv \int_{\tau_0}^{\tau} \exp\left(i\,\lambda u^2/2\right)\Omega^*(u)\,du. \qquad (33)$$

Substituting equation (9) into (32) and (33), and carrying out the algebra, we obtain the compact forms of the integrals $I_{\varphi_1}(\tau_0,\tau;\lambda,\rho)$ and $I_{\varphi_2}(\tau_0,\tau;\lambda,\rho)$ given below:

$$\begin{aligned} I_{\varphi_1}(\tau_0,\tau;\lambda,\rho) &= \left[\Omega_0 - F\,I_0(\tau_0;\rho)\right] I_0^*(\tau_0,\tau;\mu) \\ &\quad + F\,J_0(\tau_0,\tau;\mu,\rho) \end{aligned} \qquad (34)$$

$$\begin{aligned} I_{\varphi_2}(\tau_0,\tau;\lambda,\rho) &= \left[\Omega_0^* - F^* I_0^*(\tau_0;\rho)\right] I_0^*(\tau_0,\tau;\kappa) \\ &\quad + F^* J_0^*(\tau_0,\tau;-\kappa,\rho) \end{aligned} \qquad (35)$$

where

$$\mu \equiv \lambda + \rho = \lambda(1+k), \qquad \kappa \equiv \lambda - \rho = \lambda(1-k). \qquad (36)$$

The first two integrals in equations (34) and (35) are of the form $I_0(x;\rho)$ as defined in equation (11). The third integrals in these equations, $J_0(\tau_0,\tau;\mu,\rho)$ and $J_0^*(\tau_0,\tau;-\kappa,\rho)$, respectively, are defined as

$$J_0(\tau_0,\tau;\mu,\rho) = \sqrt{\pi/|\rho|}\int_{\tau_0}^{\tau} s(u)\exp\left(i\,|\mu| u^2/2\right)\overline{E}\left(\sqrt{|\rho|/\pi}\,u\right)du \qquad (37)$$

$$J_0^*(\tau_0,\tau;-\kappa,\rho) = \sqrt{\pi/|\rho|}\int_{\tau_0}^{\tau} s(u)\exp\left(i\,|\kappa| u^2/2\right)\overline{E}^*\left(\sqrt{|\rho|/\pi}\,u\right)du. \qquad (38)$$

We have again assumed that in equations (37) and (38) that ρ, μ and κ are positive. We see in (39) that for ρ and μ negative we merely take the complex conjugate of equations (37) and (38), that is,

$$J_0^*(\tau_0,\tau;\mu,\rho) = J_0(\tau_0,\tau;-\mu,-\rho) \qquad (39)$$

It is noted for the reader's convenience that when ρ is positive, μ, κ, and λ are also positive, while the case of ρ negative corresponds to negative values for μ, κ, and λ. The integrals (37) and (38) are specific cases of the more general form of integral given below (see [7-9])

$$J_n(\tau_0,\tau;\mu,\rho) \equiv \int_{\tau_0}^{\tau} \exp\left(i\,\mu u^2/2\right) I_n(u;\rho)\,du \qquad n = 0,1,2,\ldots,m. \qquad (40)$$

From the recurrence formula for $I_n(x;\rho)$ given in equation (13), the following recurrence formula for $J_n(\tau_0,\tau;\mu,\rho)$ can be easily verified.

$$\begin{aligned} J_n(\tau_0,\tau;\mu,\rho) &= \frac{i}{\rho} I_{n-1}(\tau_0,\tau;-\lambda) \\ &\quad - i\frac{n-1}{\rho} J_{n-2}(\tau_0,\tau;\mu,\rho) \qquad n = 2,3,4,\ldots \end{aligned} \qquad (41)$$

The use of the above recurrence formula, allows us to reduce the evaluation of the integral (40) to the evaluation of the first two unknown terms of the sequence, given by

$$J_0(\tau_0,\tau;\mu,\rho) \equiv \int_{\tau_0}^{\tau} \exp\left(i\,\mu\,u^2/2\right) I_0(u;\rho)\,du \tag{42}$$

$$J_1(\tau_0,\tau;\mu,\rho) \equiv \int_{\tau_0}^{\tau} \exp\left(i\,\mu\,u^2/2\right) I_1(u;\rho)\,du. \tag{43}$$

Using equation (20), the first integral (42) takes the form

$$J_0(\tau_0,\tau;\mu,\rho) = \sqrt{\pi/|\rho|}\int_{\tau_0}^{\tau} s(u)\exp\left(i\,\mu\,u^2/2\right)\overline{E}\left(\sqrt{|\rho|/\pi}\,u\right)du \tag{44}$$

The evaluation of this integral will be discussed in the next section. The second integral (43) can easily be computed using equation (15)

$$\begin{aligned} J_1(\tau_0,\tau;\mu,\rho) &= (i/\rho)\int_{\tau_0}^{\tau}\exp\left(i\,\lambda\,u^2/2\right)du \\ &\quad -(i/\rho)\int_{\tau_0}^{\tau}\exp\left(i\,\mu\,u^2/2\right)du \\ &= (i/\rho)\left[I_0^*(\tau_0,\tau;\lambda) - I_0^*(\tau_0,\tau;\mu)\right] \end{aligned} \tag{45}$$

Thus we have completed the analytic solution for the attitude motion and we can proceed with the velocity solution.

Analytic Solution for the Inertial Velocities

When body-fixed forces $\left(f_x, f_y, f_z\right)$ are present, the rigid body accelerates according to

$$\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} = [A]\begin{bmatrix} f_x/m \\ f_y/m \\ f_z/m \end{bmatrix} \tag{46}$$

where a_x, a_y, a_z are acceleration components in inertial space and matrix A is given in (47).

$$[A] = \begin{bmatrix} c\varphi_z c\varphi_y - s\varphi_z s\varphi_x s\varphi_y & -s\varphi_z c\varphi_x & c\varphi_z s\varphi_y + s\varphi_z s\varphi_x c\varphi_y \\ s\varphi_z c\varphi_y + c\varphi_z s\varphi_x s\varphi_y & c\varphi_z c\varphi_x & s\varphi_z s\varphi_y - c\varphi_z s\varphi_x c\varphi_y \\ -c\varphi_x s\varphi_y & s\varphi_x & c\varphi_x c\varphi_y \end{bmatrix} \tag{47}$$

When φ_x and φ_y are small, (47) becomes

$$[A] = \begin{bmatrix} c\varphi_z & -s\varphi_z & \varphi_y c\varphi_z + \varphi_x s\varphi_z \\ s\varphi_z & c\varphi_z & \varphi_y s\varphi_z - \varphi_x c\varphi_z \\ -\varphi_y & \varphi_x & 1 \end{bmatrix}. \tag{48}$$

Introducing the complex variables

$$a \equiv a_x + i a_y, \quad v \equiv v_x + i v_y, \quad f \equiv f_x + i f_y \tag{49}$$

we can combine the transverse portion of equation (48) into the following complex equation for the transverse velocity in inertial space.

$$\begin{aligned} v(\tau) &= v_0 + (f/m)\,\lambda\int_{\tau_0}^{\tau}\exp[i\,\varphi_z(u)]\,du \\ &\quad - i\,(f_z/m)\,\lambda\int_{\tau_0}^{\tau}\varphi(u)\exp[i\,\varphi_z(u)]\,du \end{aligned} \tag{50}$$

with the initial condition defined as

$$v_0 \equiv v(\tau_0), \qquad v_0 = v_{x0} + i\,v_{y0} \tag{51}$$

where, recalling (23) and (27), we have

$$\begin{aligned} \varphi(\tau) &= \varphi_0\exp\left(-i\,\lambda\tau^2/2\right) \\ &\quad + \lambda\exp\left(-i\,\lambda\tau^2/2\right) I_\varphi(\tau_0,\tau;\lambda,\rho) \\ \varphi_z(\tau) &= \lambda\left(\tau^2 - \tau_0^2\right)/2 + \varphi_{z0}. \end{aligned}$$

The first integral in (50) is due to the transverse forces acting on the body. The solution to this integral is easily solved for in terms of Fresnel integral functions by substituting the expression for $\varphi_z(\tau)$ given above. The resulting solution is given as follows:

$$\int_{\tau_0}^{\tau}\exp[i\,\varphi_z(u)]\,du = \exp(i\,\varphi_{z0})\exp\left(-i\,\lambda\tau_0^2/2\right) I_0^*(\tau_0,\tau;\lambda) \tag{52}$$

recalling that the solution for $I_0(x;\rho)$ was given in (20). The solution to the second integral in (50), resulting from axial forces acting on the body, is more involved and the details are outlined below. Substituting the expressions for $\varphi(\tau)$ and $\varphi_z(\tau)$, the integral becomes

$$\int_{\tau_0}^{\tau} \varphi(u) \exp[i\varphi_z(u)]\, du = \exp(i\varphi_{z0}) \exp\left(-i\lambda\tau_0^2/2\right) \times \left[\varphi_0(\tau-\tau_0) + \lambda \int_{\tau_0}^{\tau} I_\varphi(\tau_0, u; \lambda, \rho)\, du\right]. \tag{53}$$

Now, in order to solve for the transverse inertial velocity, we need to evaluate the integral of the integral $I_\varphi(\tau_0, \tau; \lambda, \rho)$:

$$\int_{\tau_0}^{\tau} I_\varphi(\tau_0, u; \lambda, \rho)\, du = k_1 \int_{\tau_0}^{\tau} I_{\varphi_1}(\tau_0, u; \lambda, \rho)\, du + k_2 \int_{\tau_0}^{\tau} I_{\varphi_2}(\tau_0, u; \lambda, \rho)\, du. \tag{54}$$

Recalling the compact forms of the two integrals $I_{\varphi_1}(\tau_0, \tau; \lambda, \rho)$ and $I_{\varphi_2}(\tau_0, \tau; \lambda, \rho)$ given in (34) and (35), we need to consider the integration of the terms:

$$I_{\varphi_1}(\tau_0, \tau; \lambda, \rho) = \left[\Omega_0 - F I_0(\tau_0; \rho)\right] I_0^*(\tau_0, \tau; \mu) + F J_0(\tau_0, \tau; \mu, \rho)$$

$$I_{\varphi_2}(\tau_0, \tau; \lambda, \rho) = \left[\Omega_0^* - F^* I_0^*(\tau_0; \rho)\right] I_0^*(\tau_0, \tau; \kappa) + F^* J_0^*(\tau_0, \tau; -\kappa, \rho).$$

The first integral we consider is the integral of $I_0^*(\tau_0, \tau; \mu)$. Rewriting $I_0^*(\tau_0, \tau; \mu)$, in terms of $I_0(x; \rho)$, as given in (20), we have

$$I_0^*(\tau_0, \tau; \mu) = I_0^*(\tau; \mu) - I_0^*(\tau_0; \mu) = \sqrt{\pi/|\mu|}\left[s(\tau)\,\overline{E}^*\left(\sqrt{|\mu|/\pi}\,\tau\right) - s(\tau_0)\,\overline{E}^*\left(\sqrt{|\mu|/\pi}\,\tau_0\right)\right]. \tag{55}$$

The integration of (55) is reduced to the integral of the Fresnel integral function $\overline{E}^*(x)$ given below.

$$\int_0^x \overline{E}^*(u)\, du = x\,\overline{E}^*(x) + (i/\pi)\left[\exp\left(i\pi x^2/2\right) - 1\right] \tag{56}$$

Similarly, the integral $I_0^*(\tau_0, \tau; \kappa)$ from the $I_{\varphi_2}(\tau_0, \tau; \lambda, \rho)$ expression is

$$I_0^*(\tau_0, \tau; \kappa) = I_0^*(\tau; \kappa) - I_0^*(\tau_0; \kappa) \quad \sqrt{\pi/|\kappa|}\left[s(\tau)\overline{E}^*\left(\sqrt{|\kappa|/\pi}\,\tau\right) - s(\tau_0)\overline{E}^*\left(\sqrt{|\kappa|/\pi}\,\tau_0\right)\right] \tag{57}$$

The evaluation of the Fresnel integral as well as the evaluation and integration of $J_0(\tau_0, \tau; \mu, \rho)$ are presented in the following three subsections.

Evaluation of the Fresnel Integral

The Fresnel integral is difficult to approximate over a large range of its argument. An excellent approximation based on the τ-method of Lanczos [10], given by Boersma [11], is suitable for this purpose and is reviewed here. There exists a complex Fresnel integral function of the second kind defined by

$$E_2(x) \equiv \left(1/\sqrt{2\pi}\right)\int_0^x \left[\exp(-i\,u)/\sqrt{u}\right] du \tag{58}$$

where $E_2(x)$ is related to $E(x)$ by

$$E(x) = E_2\left(\pi x^2/2\right). \tag{59}$$

According to this method, two approximations are used, the first, valid for small arguments, is given as

$$E_2(x) = \exp(-ix) \sum_{n=0}^{11} (a_n + i b_n)(x/4)^{n+1/2} \qquad 0 \le x \le 4 \tag{60}$$

and the second, valid for large arguments, is

$$E_2(x) = (1-i)/2 + \exp(-ix) \sum_{n=0}^{11} (c_n + i d_n)(4/x)^{n+1/2} \qquad x > 4. \tag{61}$$

The numerical values of the coefficients a_n, b_n, c_n and d_n are given by Boersma [11], and for the reader's convenience, are reproduced here in Table 1. The maximum error for the first approximation is 1.6×10^{-9} while the maximum error for the second approximation is only 0.5×10^{-9}. Other approximations for the Fresnel integral and for integrals of the Fresnel integral, using asymptotic and/or series expansions or rational functions can be found in Abramowitz and Stegun [12].

Table 1 Numerical values of coefficients for the Fresnel integral computations (Boersma [11])

a_0 = +1.595769140	b_0 = -0.000000033	c_0 = +0.000000000	d_0 = +0.199471140
a_1 = -0.000001702	b_1 = +4.255387524	c_1 = -0.024933975	d_1 = +0.000000023
a_2 = -6.808568854	b_2 = -0.000092810	c_2 = +0.000003936	d_2 = -0.009351341
a_3 = -0.000576361	b_3 = -7.780020400	c_3 = +0.005770956	d_3 = +0.000023006
a_4 = +6.920691902	b_4 = -0.009520895	c_4 = +0.000689892	d_4 = +0.004851466
a_5 = -0.016898657	b_5 = +5.075161298	c_5 = -0.009497136	d_5 = +0.001903218
a_6 = -3.050485660	b_6 = -0.138341947	c_6 = +0.011948809	d_6 = -0.017122914
a_7 = -0.075752419	b_7 = -1.363729124	c_7 = -0.006748873	d_7 = +0.029064067
a_8 = +0.850663781	b_8 = -0.403349276	c_8 = +0.000246420	d_8 = -0.027928955
a_9 = -0.025639041	b_9 = +0.702222016	c_9 = +0.002102967	d_9 = +0.016497308
a_{10} = -0.150230960	b_{10} = -0.216195929	c_{10} = -0.001217930	d_{10} = -0.005598515
a_{11} = +0.034404779	b_{11} = +0.019547031	c_{11} = +0.000233939	d_{11} = +0.000838386

Evaluation of the $J_0(x;\mu,\rho)$ Integral

The $J_0(\tau_0,\tau;\mu,\rho)$ integral defined in (44) can also be written as

$$J_0(\tau_0,\tau;\mu,\rho) = J_0(\tau;\mu,\rho) - J_0(\tau_0;\mu,\rho) \tag{62}$$

where the general form of the $J_0(x;\mu,\rho)$ integral is

$$J_0(x;\mu,\rho) = \sqrt{\pi/|\rho|}\, s(x)\int_0^x \exp\left(i\,|\mu|u^2/2\right) \overline{E}\left(\sqrt{|\rho|/\pi}\, u\right) du. \tag{63}$$

The τ-method of Lanczos, described above, is used for the evaluation of $\overline{E}\left(\sqrt{|\rho|/\pi}\,u\right)$. Approximations for small and large arguments yield the following expressions:

$$\overline{E}\left(\sqrt{|\rho|/\pi}\,u\right) = E_2\left(|\rho|\,u^2/2\right) = \exp\left(-i|\rho|u^2/2\right)\sum_{n=0}^{11}(a_n + i\,b_n)\left(|\rho|/8\right)^{n+1/2} u^{2n+1} \qquad 0 \le \left(|\rho|\,u^2/2\right) \le 4 \tag{64}$$

$$\overline{E}\left(\sqrt{|\rho|/\pi}\,u\right) = E_2\left(|\rho|\,u^2/2\right) = (1-i)/2 + \exp\left(-i|\rho|u^2/2\right)\sum_{n=0}^{11}(c_n + i\,d_n)\left(8/|\rho|\right)^{n+1/2}(1/u)^{2n+1} \qquad \left(|\rho|\,u^2/2\right) > 4. \tag{65}$$

Substituting (64) and (65) into (63) gives us two cases for the $J_0(x;\mu,\rho)$ integral. For clarity they will be referred to as $J_0(x;\mu,\rho)_{small}$ and $J_0(x;\mu,\rho)_{large}$. The small and large (argument) cases of $J_0(x;\mu,\rho)$ are defined as

$$J_0(x;\mu,\rho) = J_0(x_s;\mu,\rho)_{small} + J_0(x_s,x;\mu,\rho)_{large} \tag{66}$$

where

$$J_0(x_s;\mu,\rho)_{small} = \int_0^{x_s}\exp\left(i\,|\mu|u^2/2\right)\overline{E}\left(\sqrt{|\rho|/\pi}\,u\right)du = \sum_{n=0}^{11}(a_n + i\,b_n)\left(|\rho|/8\right)^{n+1/2}\int_0^{x_s}\exp\left(i\,|\lambda|u^2/2\right)u^{2n+1}\,du \qquad 0 \le \left(|\rho|\,u^2/2\right) \le 4 \tag{67}$$

$$J_0(x_s,x;\mu,\rho)_{large} = \int_{x_s}^{x} \exp\left(i|\mu|u^2/2\right)\overline{E}\left(\sqrt{|\rho|/\pi}\,u\right)du$$

$$= \frac{(1-i)}{2}\int_{x_s}^{x} \exp\left(i|\mu|u^2/2\right)du + \sum_{n=0}^{11}(c_n + i d_n)\left(\frac{8}{|\rho|}\right)^{n+\frac{1}{2}}$$

$$\times \int_{x_s}^{x} \exp\left(i|\lambda|u^2/2\right)(1/u)^{2n+1}du$$

$$\left(|\rho|\,u^2/2\right) > 4 \qquad (68)$$

To evaluate (67) we are interested in integrals of the general form:

$$I_{un}(x;\lambda) = \int_0^x \exp\left(i|\lambda|u^2/2\right)u^n\,du \qquad (69)$$

$$n = 1,3,5,\ldots,23.$$

It is noted that this integral is the complex conjugate of the integral given in (12). The subscript u, in the notation $I_{un}(x;\lambda)$, is used to signify that the term u^n is 'upstairs' in the integral. Integrals of this form are evaluated using the recurrence formula

$$I_{un}(x;\lambda) = \frac{-i\,x^{n-1}}{|\lambda|}\exp\left(i|\lambda|x^2/2\right) + \frac{i\,(n-1)}{|\lambda|}I_{un-2}(x;\lambda). \qquad (70)$$

This relationship holds for n=2,3,4,... and therefore it is necessary to compute the first two terms corresponding to n=0 and n=1 by other means. These first two terms are given by

$$I_{u0}(x;\lambda) = \sqrt{\pi/|\lambda|}\;s(x)\;\overline{E}^*\left(\sqrt{|\lambda|/\pi}\,x\right)$$
$$I_{u1}(x;\lambda) = \left(-i/|\lambda|\right)\left[\exp\left(i|\lambda|x^2/2\right)-1\right]. \qquad (71)$$

Similarly, to evaluate (68) we are interested in integrals of the general form:

$$I_{dn}(x_s,x;\lambda) = I_{dn}(x_s;\lambda) - I_{dn}(x;\lambda) \qquad (72)$$

where

$$I_{dn}(x;\lambda) = \int_x^{\infty} \exp\left(i|\lambda|u^2/2\right)(1/u)^n\,du \qquad (73)$$

$$n = 1,3,5,\ldots,23$$

The subscript d, in the notation $I_{dn}(x;\lambda)$, is used to signify that the term u^n is 'downstairs' in the integral. Integrals of this form are evaluated with the recurrence formula

$$I_{dn}(x;\lambda) = \exp\left(i|\lambda|x^2/2\right)\Big/\left[(n-1)\,x^{n-1}\right] + \frac{i|\lambda|}{(n-1)}I_{dn-2}(x;\lambda). \qquad (74)$$

This relationship holds for n=2,3,4,... and therefore it is again necessary to compute the first two terms corresponding to n=0 and n=1 by other means. These first two terms are given by

$$I_{d0}(x;\lambda) = \int_x^{\infty} \exp\left(i|\lambda|u^2/2\right)du$$
$$= I_{u0}(\infty;\lambda) - I_{u0}(x;\lambda) \qquad (75)$$
$$I_{d1}(x;\lambda) = (1/2)\,E_i\left(|\lambda|x^2/2\right)$$

where $E_i(x)$ denotes the Exponential integral function defined by

$$E_i(x) = \int_x^{\infty}\left[\exp(i\,u)/u\right]du. \qquad (76)$$

Approximations and tabulated values of this function can be found in [12].

It is noted that for negative values of λ the integrals merely become the conjugate of the given integrals.

$$I_{un}^*(x;\lambda) = I_{un}(x;-\lambda), \quad I_{dn}^*(x;\lambda) = I_{dn}(x;-\lambda) \qquad (77)$$

Integration of the $J_0(x;\mu,\rho)$ Integral

To complete the solution for the transverse inertial velocity, the integration of the $J_0(x;\mu,\rho)$ integral is presented next. We will first consider the case of $J_0(x;\mu,\rho)_{small}$. It follows from equations (67) and (68) that the integral of $J_0(x;\mu,\rho)_{small}$ can be reduced to the computation of the integral of $I_{un}(x;\lambda)$. The integral of $I_{un}(x;\lambda)$ is given by the following recurrence formula:

$$\int_0^x I_{un}(u;\lambda)\,du = \frac{-i}{|\lambda|}I_{un-1}(x;\lambda) + \frac{i\,(n-1)}{|\lambda|}\int_0^x I_{un-2}(u;\lambda)\,du \qquad (78)$$

This relationship holds for n=2,3,4,... . The remaining cases, corresponding to n=0 and n=1, can be computed using

$$\int_0^x I_{u0}(u;\lambda)\,du = \sqrt{\pi/|\lambda|}\; s(x)\int_0^x E^*\left(\sqrt{|\lambda|/\pi}\,u\right) du \qquad (79)$$

$$\int_0^x I_{u1}(u;\lambda)\,du = \left(i/|\lambda|\right)\left[x - I_{u0}(x;\lambda)\right] \qquad (80)$$

where the integral of $\overline{E}^*(x)$ is given in (56).

Now consider the evaluation of $J_0(x;\mu,\rho)_{large}$. It follows from equations (68) and (73) that the integral of $J_0(x;\mu,\rho)_{large}$ can be reduced to the computation of the integral of $I_{dn}(x;\lambda)$. The integral of $I_{dn}(x;\lambda)$ is given by the following recurrence formula:

$$\int_x^\infty I_{dn}(u;\lambda)\,du = \frac{1}{(n-1)}\left[I_{dn-1}(x;\lambda) + i\,|\lambda|\int_x^\infty I_{dn-2}(u;\lambda)\,du\right] \qquad (81)$$

This relationship holds for n=2,3,4,... The case n=1 is given by the expression:

$$\int_x^\infty I_{d1}(u;\lambda)\,du = x\,I_{d1}(x;\lambda) - I_{d0}(x;\lambda) \qquad (82)$$

The last case, corresponding to n=0, requires a form of the integral of $\overline{E}^*(x)$ given in (56).

This completes the analytic solution for the transverse velocity.

Discussion of Axial Velocity Solution

The equation for the axial acceleration can be written in complex notation as

$$a_z = (i/2m)\left[f^*\varphi(\tau) - f\varphi^*(\tau)\right] + (f_z/m) \qquad (83)$$

where the corresponding axial velocity expression is represented by

$$v_z(\tau) = v_{z0} + (i/2m)\left[f^*\int_{\tau_0}^{\tau}\varphi(u)\,du - f\int_{\tau_0}^{\tau}\varphi^*(u)\,du\right] + (f_z/m)(\tau-\tau_0), \qquad v_{z0} \equiv v_z(\tau_0). \qquad (84)$$

Since the integrals in (84) are small (even though f is not) and the f_z term dominates for large spin-up durations, $v_z(\tau)$ can be approximated [5] by

$$v_z(\tau) = v_{z0} + (f_z/m)(\tau-\tau_0). \qquad (85)$$

The integration of the small terms in (84) involves integrating $\varphi(\tau)$ directly, which we recall from (27) as

$$\varphi(\tau) = \varphi_0 \exp\left(-i\,\lambda\tau^2/2\right) + \lambda \exp\left(-i\,\lambda\tau^2/2\right) I_\varphi(\tau_0,\tau;\lambda,\rho).$$

The direct integration of $\varphi(\tau)$ follows a similar development to the transverse velocity solution. The details involve the integration of $I_\varphi(\tau_0,\tau;\lambda,\rho)$, as in the transverse case, and include the addition of an exponential term. This, however, presents no further difficulty in the derivation of the solution, since the resulting expression consists of terms found in the previous formulation.

Numerical Results

Numerical examples are used to demonstrate the accuracy of the analytic solution. Tsiotras and Longuski [4] have verified the accuracy of the analytic solutions for $\varphi_x, \varphi_y, \varphi_z$ and $\omega_x, \omega_y, \omega_z$, where they used numerical values corresponding to a spin-up maneuver of the Galileo spacecraft, as an example. As in [4], we consider the following inertia parameters to be representative of the Galileo spacecraft (in all-spin mode).

$$I_x = 2985\,kg\cdot m^2, I_y = 2729\,kg\cdot m^2, I_z = 4183\,kg\cdot m^2 \qquad (86)$$

The following initial conditions are assumed

$$\omega(\tau_0) = 0,\; \varphi(\tau_0) = 0,\; \varphi_z(\tau_0) = 0$$
$$v(\tau_0) = 0,\; v_z(\tau_0) = 0 \qquad (87)$$

The torques generated about the body axes, are given by

$$M_x = -1.253\,N\cdot m, M_y = -1.494\,N\cdot m, M_z = 13.5\,N\cdot m. \qquad (88)$$

The constant body-fixed forces are given by

$$f_x = 7.660\,N,\; f_y = -6.428\,N,\; f_z = 10.0\,N. \qquad (89)$$

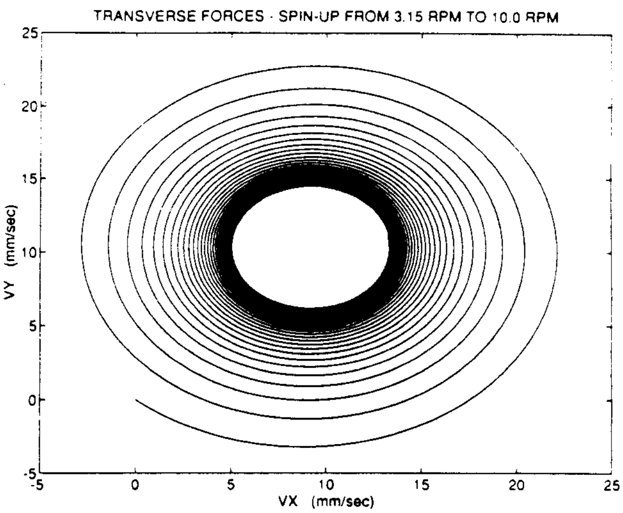

Fig. 2 Exact and analytic solutions for $v(\tau)$.

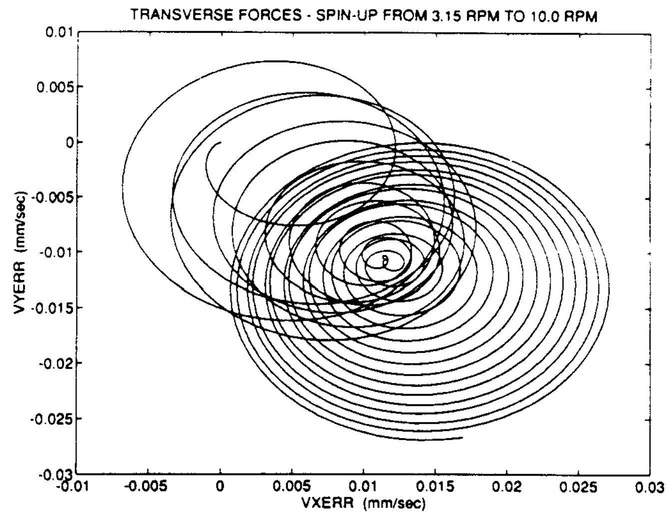

Fig. 3 Exact minus analytic solutions for $v(\tau)$.

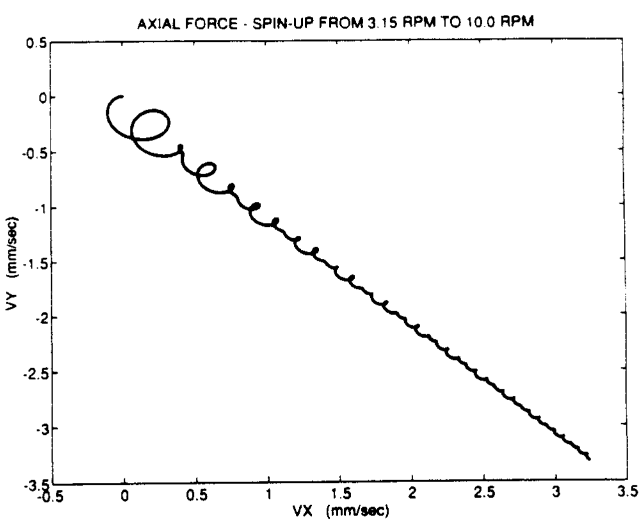

Fig. 4 Exact and analytic solutions for $v(\tau)$.

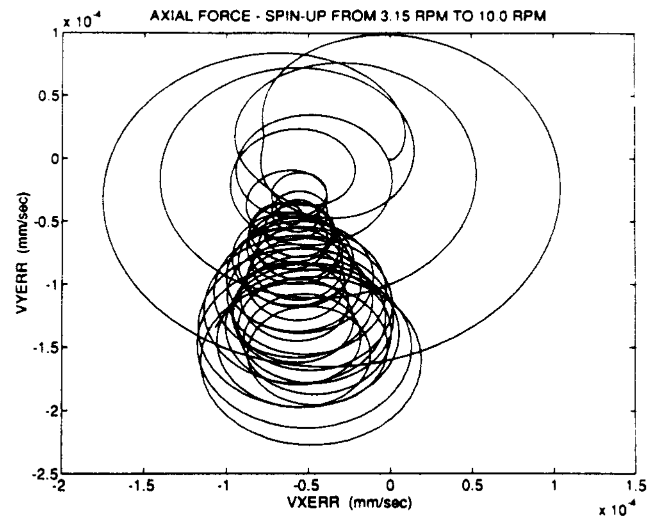

Fig. 5 Exact minus analytic solutions for $v(\tau)$.

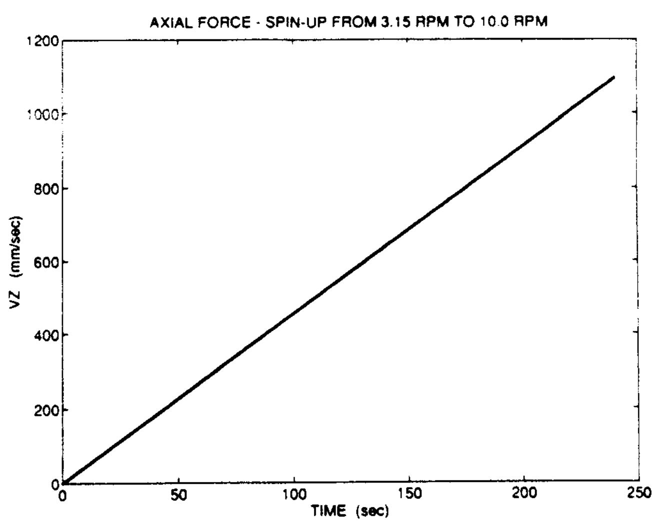

Fig. 6 Exact and analytic solutions for $v_z(\tau)$.

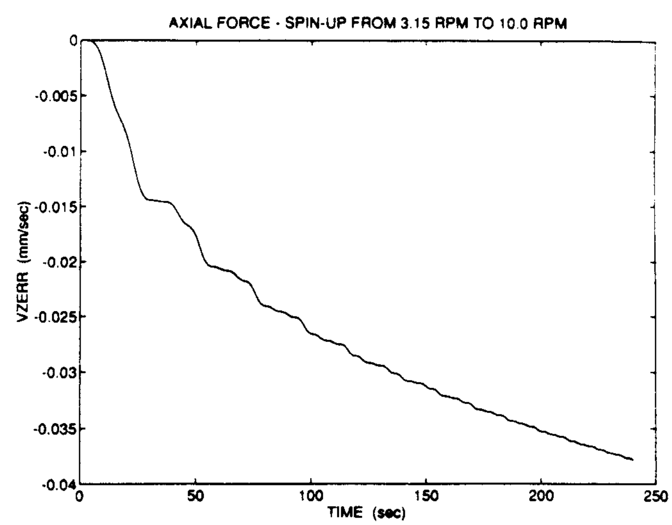

Fig. 7 Exact minus analytic solutions for $v_z(\tau)$.

For the purpose of illustration, we consider a spin-up maneuver from $\omega_z(\tau_0) = 3.15\,\text{rpm}$ to $\omega_z(\tau) = 10.0\,\text{rpm}$. Since the transverse velocity solution is a linear combination of the transverse body-fixed forces and the axial body-fixed force, numerical studies for these cases will be demonstrated independently. The analytic solution for the inertial velocity is compared to the "exact" solution which is found by numerical integration of equations (1), (21), and (46) using equation (47) for the "exact" A matrix.

Figures 2 and 3 compare the exact solution of $v(\tau)$ with the analytic solution for the case of transverse body-fixed forces. (Here we use the values of f_x and f_y given in (89), but set $f_z = 0$.) In Fig. 2 both exact and analytic solutions are represented, but they are indistinguishable from one another. Their difference is given in Fig. 3.

Figures 4 and 5 compare the exact solution of $v(\tau)$ with the analytic solution for the case of an axial body-fixed force. (Here we assume that $f_x = f_y = 0$ and $f_z = 10.0\,\text{N}$.) In Fig. 4 both exact and analytic solutions are displayed, but they too are indistinguishable. Their difference is presented in Fig. 5. For the same case, Figs. 6 and 7 compare the exact solution of $v_z(\tau)$ with the analytic solution (85). In Fig. 6 both exact and analytic solutions are presented with their difference given in Fig. 7.

Conclusions

An analytic solution has been derived for the inertial velocities of a thrusting, spinning rocket. The complex representation enables the solution to take a compact form. The solution assumes exact axial symmetry in order to write the solution for the angular velocity about the spinning axis in a linear form, but keeps the distinction of the moments of inertia in the other two equations for the angular velocities. A small angle approximation allows the Euler angles to be given as a solution of a linear, time-varying system with the expression for the angular velocities acting as the forcing function. The solution for the transverse velocity is given in terms of Fresnel integrals and other integrals that can be solved for in terms of simple recurrence formulas. Current and previous research indicates that such analytic solutions are extremely helpful in capturing the fundamental behavior of the motion, which cannot be deduced from numerical simulations.

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References

[1] Longuski, J. M., Kia, T., and Breckenridge, W. G., "Annihilation of Angular Momentum Bias During Thrusting and Spinning-Up Maneuvers," *J. Astron. Sci.*, Vol. 37, No. 4, 1989, pp. 433-450.

[2] Longuski, J. M., "Solution of Euler's Equations of Motion and Eulerian Angles for Near Symmetric Rigid Bodies Subject to Constant Moments," AIAA Paper 80-1642, Aug. 1980.

[3] Longuski, J. M., "Real Solutions for the Attitude Motion of a Self-Excited Rigid Body," *Acta Astron.*, Vol. 25, No. 3, 1991, pp. 131-140.

[4] Tsiotras, P., and Longuski, J. M., "A Complex Analytic Solution for the Attitude Motion of a Near-Symmetric Rigid Body under Body-Fixed Torques," *Celest. Mech., Vol.* 51, 1991, pp. 281-301.

[5] Hintz, G. R., and Longuski, J. M., "Error Analysis for the Delivery of a Spinning Probe to Jupiter," *J. of Guid., Cont., and Dyn.*, Vol. 8, No. 3, 1985, pp. 384-390.

[6] Klumpe, E. W., and Longuski, J. M., "Secular Solution for Delta-V During Spin Rate Change Maneuvers of Rigid Body Spacecraft," AIAA Paper 84-2011, Aug. 1984.

[7] Longuski, J. M., and Tsiotras, P., "Analytic Solutions for a Spinning Rigid Body Subject to Time-Varying Body-Fixed Torques. Part I: Constant Axial Torque," *J. of Applied Mechanics*, Vol. 60, No. 4, 1993, pp. 970-975.

[8] Tsiotras, P., and Longuski, J. M., "Analytic Solutions for a Spinning Rigid Body Subject to Time-Varying Body-Fixed Torques. Part II: Time-Varying Axial Torque," *J. of Applied Mechanics*, Vol. 60, No. 4, 1993, pp. 976-981.

[9] Tsiotras, P., "Analytic Theory and Control of the Motion of Spinning Rigid Bodies," Ph.D. Thesis, School of Aeronautics and Astronautics, Purdue University, West Lafayette, IN, 1993.

[10] Lanczos, C., *Applied Analysis*, Prentice Hall, Englewood Cliffs, NJ, 1956.

[11] Boersma, J., "Computation of Fresnel Integrals," *Math. Comp.*, Vol. 14, 1960, pp. 380.

[12] Abramowitz, M., and Stegun, I. A., *A Handbook of Mathematical Functions*, Dover Publications Inc., New York, 1972.