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Invariant Manifold Techniques for Attitude Control of Symmetric Spacecraft

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Abstract

In this paper we consider the problem of attitude stabilization of an axially symmetric spacecraft using two pairs of gas jet actuators to generate control torques about two axes spanning the two-dimensional plane orthogonal to the axis of symmetry. Under the assumption that the initial spin rate about the symmetry axis is zero, and using a new kinematic formulation, we construct an invariant manifold for the closed loop system with a specific feedback law. Using this manifold we derive a stabilizing control law which achieves arbitrary reorientation of the spacecraft.

1 Introduction

The problem of attitude stabilization of a rotating rigid body has recently been the subject of active research [1, 2, 3, 4, 5, 6, 9]. A complete mathematical description of the problem was first given by Crouch [1], where he provided necessary and sufficient conditions for the controllability of a rigid body in the cases of one, two, or three independent control torques. Recently Byrnes and Isidori [4] established that a rigid spacecraft controlled by two pairs of gas jet actuators cannot be asymptotically stabilized to an equilibrium using a continuously differentiable, i.e., smooth or $\mathcal{C}^1$, feedback control law. The attitude stabilization problem of a *symmetric* rigid spacecraft using only two control torques spanning the two-dimensional plane orthogonal to the symmetry axis was considered in [5]. The complete dynamics fail to be controllable or even accessible in this case; thus, the methodologies of [1] and [4] are not applicable. However, the spacecraft dynamics are strongly accessible and small time locally controllable in a restricted sense, namely when the spin rate remains zero; however any stabilizing control has to be *nonsmooth*. In [5] such a nonsmooth control strategy was developed, which achieves arbitrary reorientation of the spacecraft, for the restricted case of zero spin rate. This nonsmooth control law is based on previous results on the stabilization of nonholonomic mechanical systems [7].

In this paper we consider again the problem treated in [5]. Without loss of generality, we assume that the two acting control torques are along principal axes. In addition to its theoretical interest, this problem has considerable practical importance, since it corresponds to spacecraft control when one of the actuators fail.

First, we simplify the kinematics of the problem using the complex kinematic Riccati equation developed in [8] and [9]. When written in terms of the new kinematic parameter in the complex plane, the kinematic equations have a very simple and compact form. Using this formulation of the kinematics, we obtain a stabilizing control law for the restricted problem of zero spin rate, which allows arbitrary reorientation of the spacecraft. The control law thus derived is especially simple and elegant and avoids the successive switchings of [5]. Although we only consider a specific control problem of practical interest, the main purpose of the paper is more general, namely, to expose the new formulation of the kinematics and to illustrate how it can facilitate the design of stabilizing control laws for attitude control problems. Other results on spin-axis stabilization of a symmetric spacecraft subject to two control torques using this kinematic formulation are reported in [9].

2 Dynamics and Kinematics

Euler's equations of motion, for a symmetric body with no external torques about the symmetry axis, can be written as follows

$$\dot{\omega}_1 = a\omega_2\omega_3 + u_1 \tag{1a}$$
$$\dot{\omega}_2 = -a\omega_3\omega_1 + u_2 \tag{1b}$$
$$\dot{\omega}_3 = 0 \tag{1c}$$

where $a \triangleq (I_2 - I_3)/I_1$, $u_1 \triangleq M_1/I_1$ and $u_2 \triangleq M_2/I_2$. Here M_1, M_2 are the external torques due to control inputs u_1 and u_2, respectively. The positive scalars I_1, I_2, I_3 are the principal moments of inertia of the body with respect to its mass center; I_3 corresponds to the axis of symmetry and $I_1 = I_2$. The scalars $\omega_1, \omega_2, \omega_3$ denote the components of the body angular velocity vector w.r.t. the body principal axes.

It should be clear from equation (1c) that, for a symmetric body, the value of the component of the angular velocity ω_3 along the symmetry axis cannot be affected by the control. In fact, the value of ω_3 remains constant for all times. Clearly, as already mentioned, system (1) is not controllable.

Introducing the complex variables

$$\omega \triangleq \omega_1 + i\,\omega_2, \qquad u \triangleq u_1 + i\,u_2 \tag{2}$$

one can rewrite (1a)-(1b) in the compact form

$$\dot{\omega} = -ia\omega_{30}\omega + u \tag{3}$$

where $\omega_{30} \triangleq \omega_3(0)$.

Equations (1) describe the dynamics of a rotating body in space. A complete description of the attitude motion also requires a description of the kinematics. Using a 3-2-1 Eulerian angle sequence for the description of the orientation one has the associated kinematic equations [10]

$$\dot{\phi} = \omega_1 + (\omega_2 \sin\phi + \omega_3 \cos\phi)\tan\theta \tag{4a}$$
$$\dot{\theta} = \omega_2 \cos\phi - \omega_3 \sin\phi \tag{4b}$$
$$\dot{\psi} = (\omega_2 \sin\phi + \omega_3 \cos\phi)\sec\theta \tag{4c}$$

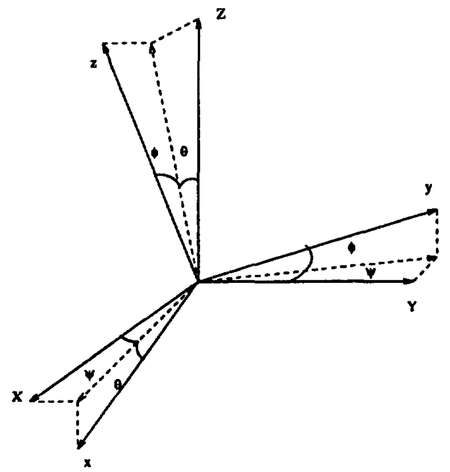

Figure 1: Eulerian angle sequence 3-2-1.

The Eulerian angles provide a local coordinate system for the rotation group $SO(3)$ which is the configuration space of the attitude motion, when $SO(3)$ is taken with its manifold structure. The equations (4) exhibit a singularity at $\theta = \pm\pi/2$. We therefore restrict the subsequent discussion to the set $\mathcal{M}$ defined by $\mathcal{M} = \{(\phi,\theta,\psi) \in \mathbb{R}^3 : \phi,\psi \in (-\pi,\pi], \theta \in (-\pi/2,\pi/2)\}$, Using this parameterization of $SO(3)$, the orientation of the local body-fixed reference frame with respect to the inertial reference frame is found by first rotating the body about its 3-axis through an angle ψ, then rotating about its 2-axis by an angle θ and finally rotating about its 1-axis by an angle ϕ; see Fig. 1.

In [8] an alternative formulation of the kinematics is presented, which simplifies equations (4) and which is used in [9] to derive asymptotically stabilizing control laws for the *reduced* system of equations (3) and (4a)-(4b). Consideration of this reduced system is possible because ψ is an ignorable variable for the system of equations (4). For this reduced system, stabilization about the origin corresponds to stabilization of the symmetry axis, with the body orientation about this axis (described by ψ) being indeterminate.

Following [8, 9], we introduce the complex kinematic variable, $w = w_1 + iw_2$, defined by

$$w \triangleq \frac{\sin\phi\cos\theta + i\sin\theta}{1+\cos\phi\cos\theta} \tag{5}$$

One can readily show that w satisfies the following complex differential equation

$$\dot{w} + i\,\omega_3 w = \frac{\omega}{2} + \frac{\bar{\omega}}{2}w^2 \tag{6}$$

where ω is defined in (2) and the bar denotes complex conjugate. This is a scalar Riccati equation with time-varying coefficients. More about transformation (5), which, incidently, is not restricted to the particular Eulerian angle set used here, can be found in [8, 9]. In fact, since equation (6) can be derived directly from Poisson's equations using the method of stereographic projection of the unit sphere $S^2 = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : x_1^2+x_2^2+x_3^2 = 1\}$ on the complex plane, it is independent of the particular parametrization of $SO(3)$. The differential equation for ψ in the (ω, w) space is given by

$$\dot{\psi} = \frac{i}{2}(\omega - \bar{\omega})\frac{(w+\bar{w})(1+|w|^2)}{(1+w^2)(1+\bar{w}^2)} \tag{7}$$

In the next section we will introduce another variable s, such that (w_1, w_2, s) are local coordinates of $SO(3)$, and are locally diffeomorphic to (ϕ,θ,ψ). In [9] asymptotically stabilizing control laws for the reduced system of equations (3) and (6) were derived, for both the cases of zero and nonzero constant spin rate ω_3. These feedback control laws correspond to asymptotic stabilization about the origin in the $(\omega_1,\omega_2,\phi,\theta)$ state space. In the extended $(\omega_1,\omega_2,\phi,\theta,\psi)$ state space this corresponds to stabilization about the one-dimensional manifold $\mathcal{N} = \{(\omega_1,\omega_2,\phi,\theta,\psi) : \omega_1=\omega_2=\phi=\theta=0\}$ rather than an isolated equilibrium. Feedback stabilization about a reduced equilibrium manifold has received attention recently, since it appears to be an important extension of stabilization about an equilibrium, yielding bounded trajectories [4]. In the next section we extend the results of [9] to include stabilization to the origin of the extended (complete) system of equations (3) and (4), for the restricted case of $\omega_3(0)=0$. In this case, as was shown in [5], the dynamics is strongly accessible and small time locally controllable at any equilibrium. Thus arbitrary reorientation of the spacecraft can be achieved if $\omega_3(0)=0$; if $\omega_3(0)\neq 0$, reorientation of the spacecraft is not possible. However, smooth stabilization about the one-dimensional manifold $\mathcal{N}$ is always possible, regardless of the value of $\omega_3(0)$ [9].

3 Feedback Control Strategy

Stabilization of the Kinematics

Assuming *a priori* that $\omega_{30} = 0$, the system to be driven to the origin takes the form

$$\dot{\omega}_1 = u_1 \tag{8a}$$
$$\dot{\omega}_2 = u_2 \tag{8b}$$

with

$$\dot{\phi} = \omega_1 + \omega_2\sin\phi\tan\theta \tag{9a}$$
$$\dot{\theta} = \omega_2\cos\phi \tag{9b}$$
$$\dot{\psi} = \omega_2\sin\phi\sec\theta \tag{9c}$$

or in the complex notation introduced previously,

$$\dot{\omega} = u \tag{10a}$$
$$\dot{w} = \frac{\omega}{2} + \frac{\bar{\omega}}{2}w^2 \tag{10b}$$
$$\dot{\psi} = \frac{i}{2}(\omega-\bar{\omega})\frac{(w+\bar{w})(1+|w|^2)}{(1+w^2)(1+\bar{w}^2)} \tag{10c}$$

These equations have the form of a cascade system so, as in [9], we concentrate first on the problem of stabilization of the subsystem (10b)-(10c), regarding ω as control input, and then we implement this control law through the integrator (10a).

In [9] we have the following result.

Lemma 3.1 *The feedback control law*

$$\omega = -\kappa w \tag{11}$$

($\kappa > 0$) globally exponentially stabilizes the subsystem (10b) with rate of decay $\kappa/2$.

With the control law (11) the closed-loop subsystem corresponding to (10b) takes the form

$$\dot{w} = -\frac{\kappa}{2}(1+|w|^2)w \tag{12}$$

and the corresponding closed-loop system in the (ϕ, θ, ψ) variables takes the form

$$\dot{\phi} = -\kappa \frac{\sin\phi}{\cos\theta(1+\cos\phi\cos\theta)} \tag{13a}$$

$$\dot{\theta} = -\kappa \frac{\sin\theta\cos\phi}{(1+\cos\phi\cos\theta)} \tag{13b}$$

$$\dot{\psi} = -\kappa \frac{\sin\phi\tan\theta}{(1+\cos\phi\cos\theta)} \tag{13c}$$

As $t \to \infty$ the Eulerian angles ϕ, θ go to zero, but ψ tends to some unspecified value. We will show next that, given any initial conditions $\phi(0), \theta(0), \psi(0)$, we can calculate this final value of ψ. In addition, by requiring the final value of ψ to be zero, we construct an invariant manifold that can be used to derive stabilizing control laws for the complete kinematics (9).

Eliminating time from (13) we obtain

$$\frac{d\psi}{d\phi} = \sin\theta, \qquad \frac{d\theta}{d\phi} = \frac{\sin\theta\cos\theta}{\tan\phi} \tag{14}$$

Integrating the last equation yields

$$\ln\frac{\sin\phi}{\sin\phi_0} = \ln\frac{\tan\theta}{\tan\theta_0} \tag{15}$$

or

$$\tan\theta = a_0 \sin\phi \tag{16}$$

where $a_0 \triangleq \tan\theta_0 / \sin\phi_0$ with $\phi_0 \triangleq \phi(0), \theta_0 \triangleq \theta(0)$. From this equation, along with the first of equations (14), we have

$$\frac{d\psi}{d\phi} = \frac{a_0 \sin\phi}{\sqrt{1 + a_0^2 \sin^2\phi}} \tag{17}$$

Letting $\psi_0 \triangleq \psi(0)$ and integrating from $(\phi_0, \theta_0, \psi_0)$ to the origin, we find that

$$-\psi_0 = \arcsin(p_0 \cos(\phi_0)) - \arcsin(p_0) \tag{18}$$

where $p_0 \triangleq a_0/\sqrt{1+a_0^2}$. For the case when $\phi_0 = 0$ (and $-\pi/2 < \theta_0 < \pi/2$) it can be easily shown that the previous equation simplifies to the statement that $\psi_0 = 0$. The choice of ϕ as the new independent variable is justified by the fact that it decreases monotonically to zero. Indeed, from (13a) and (5) one can write the differential equation for ϕ as $\dot{\phi} = -\kappa w_1/\cos^2\theta$. Now $\theta \in (-\pi/2, \pi/2)$ and if $\phi = 0$ for some time $\bar{t} < \infty$ we have $w_1(\bar{t}) = 0$. However, this is not possible because from (12) the magnitude of w_1 decreases monotonically (exponentially) to zero. The phase portrait of the system of equations (13a)-(13b) is depicted in Fig. 2.

Equation (18) yields an expression for the initial conditions from which the feedback $\omega = -\kappa w$ will drive system (13) to the origin. We now introduce the manifold

$$\mathcal{S} \triangleq \{ (\phi, \theta, \psi) \in \mathcal{M} : s(\phi, \theta, \psi) = 0 \} \tag{19}$$

where the the function $s : \mathcal{M} \to \mathbf{R}$ is defined by

$$s(\phi, \theta, \psi) \triangleq \psi + \arcsin(p\cos\phi) - \arcsin(p) \tag{20a}$$

$$p = a/\sqrt{1+a^2}\ , \quad a = \tan\theta/\sin\phi \tag{20b}$$

Clearly, $\mathcal{S}$ is an invariant manifold for system (13). Moreover, by construction, every trajectory on $\mathcal{S}$ (with $\omega = -\kappa w$) satisfies $\lim_{t\to\infty}(\phi(t), \theta(t), \psi(t)) = 0$. It is therefore advantageous to consider the utility of this manifold

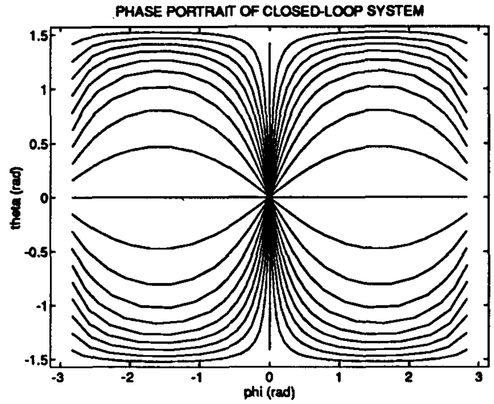

Figure 2: Phase portrait of reduced system.

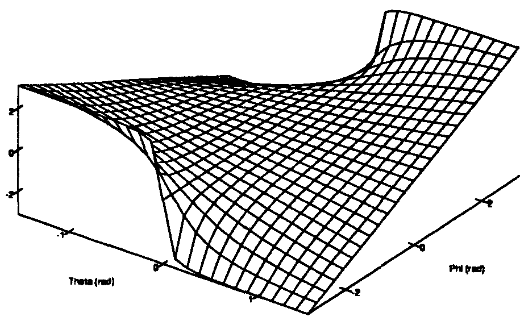

Figure 3: The two-dimensional manifold $\mathcal{S}$.

in achieving stabilization to the origin $\phi = \theta = \psi = 0$. It is interesting to note that $\mathcal{S}$ is independent of the control gain κ, therefore once on this manifold, *any* positive κ will lead to the origin. The manifold $\mathcal{S}$ is shown in Fig. 3.

The derivative of s along trajectories of (13), can be computed to be

$$\dot{s} = -w_2\omega_1 + w_1\omega_2 = Im(\omega\bar{w}) \tag{21}$$

As expected, the choice of $\omega = -\kappa w$ maintains $\dot{s} \equiv 0$ and, once on $\mathcal{S}$, the trajectories remain there.

In order to render $\mathcal{S}$ an *attracting* manifold, we restrict consideration to $w \neq 0$ and propose the following control law

$$\omega = -\kappa w - i\frac{\mu}{\bar{w}}s \tag{22}$$

with $\mu > 0$. With this control law the closed-loop system becomes

$$\dot{w} = -\frac{\kappa}{2}(1+|w|^2)w - \frac{i}{2}\mu s(\frac{1}{\bar{w}} - w) \tag{23a}$$

$$\dot{s} = -\mu s \tag{23b}$$

with $(w, s) \in (\mathbf{C}\backslash\{0\}) \times \mathbf{R}$. Moreover, $\omega + \kappa w \to 0$ as $s \to 0$. Therefore, as long as $w \neq 0$ the control law (22) drives all trajectories of subsystem (23) to the origin, for arbitrary initial conditions. Care should be taken in implementing (22) because the control law is not defined at points where $w = 0$ and $s \neq 0$. We will return to this point shortly. We have now the following result regarding system (23).

Theorem 3.1 *Consider the closed-loop system (23) with*

$$\mu > \kappa/2$$

and consider any initial condition $(w(0), s(0)) \in \mathbf{C} \times \mathbf{R}$ with $w(0) \neq 0$. Then the following hold.

(i) $w(t) \neq 0$ for all $t \geq 0$.

(ii) The trajectory $(w(\cdot), s(\cdot))$ is bounded and

$$\lim_{t\to\infty} (w(t), s(t)) = 0.$$

(iii) The control history $\omega(\cdot)$ is bounded and has bounded derivative.

Proof. We first show that if $w(0) \neq 0$ then $w(t) \neq 0$ for all $t \geq 0$.

Since

$$\frac{d}{dt}|w|^2 = 2Re(\dot{w}\bar{w})$$

one readily obtains that $|w|^2$ satisfies the differential equation

$$\frac{d}{dt}|w|^2 = -\kappa|w|^2(1 + |w|^2) \qquad (24)$$

Using the transformation $v \triangleq 1/|w|^2$ one can integrate (24) to obtain

$$|w(t)| = \left(\frac{1}{c_0 e^{\kappa t} - 1}\right)^{1/2} \qquad (25)$$

where $c_0 \triangleq (|w(0)|^2 + 1)/|w(0)|^2$. Clearly $w(t) \neq 0$ for all $t \geq 0$ and $\lim_{t\to\infty} w(t) = 0$. In fact, the magnitude of $w(t)$ is bounded between two exponential functions as follows

$$|w(0)|e^{-\frac{\kappa}{2}t} \geq |w(t)| > c_0^{-1/2}e^{-\frac{\kappa}{2}t} \quad \forall t \geq 0 \qquad (26)$$

From (23b), it should be clear that

$$s(t) = s(0)e^{-\mu t}\ ;$$

hence, $s(\cdot)$ is bounded and $\lim_{t\to\infty} s(t) = 0$.

We now show that $\omega(\cdot)$ and $\dot{\omega}(\cdot)$ are bounded. Let $\nu \triangleq s/\bar{w}$. Since $c_0 > 1$ one has that

$$\begin{aligned} |\nu(t)| &= |s(t)|/|w(t)| \leq |s(0)|e^{-\mu t}(c_0 e^{\kappa t} - 1)^{1/2} \\ &< |s(0)|c_0^{1/2}e^{-(\mu-\kappa/2)t} \qquad \forall t \geq 0 \end{aligned}$$

Since $\lambda \triangleq \mu - \kappa/2 > 0$, ν decays exponentially to zero with rate at least equal to λ. Thus $\lim_{t\to\infty} \nu(t) = 0$ and ν is bounded by $|\nu(t)| < |s(0)|c_0^{1/2} \triangleq \beta_1$ for all $t \geq 0$. Hence ω is bounded as

$$|\omega(t)| < \kappa|w(0)| + \mu|s(0)|c_0^{1/2} \triangleq \beta_2 \qquad \forall t \geq 0$$

Direct calculation shows that the derivative $\dot{\nu}$ is bounded as

$$|\dot{\nu}(t)| < (\mu + \beta_4)|s(0)|c_0^{1/2} \triangleq \beta_3 \qquad \forall t \geq 0$$

where $\beta_4 \triangleq \kappa(1 + |w(0)|^2)/2 + \mu(|s(0)| + \beta_1 c_0^{1/2})/2$ and the derivative of w is bounded as

$$|\dot{w}(t)| \leq \frac{\kappa}{2}(1 + |w(0)|^2)|w(0)| + \frac{\mu}{2}(\beta_1 + |s(0)||w(0)|) \triangleq \beta_5$$

Therefore the derivative of ω is bounded as $|\dot{\omega}(t)| < \kappa\beta_5 + \mu\beta_3$, for all $t \geq 0$. This completes the proof. ∎

Choosing therefore the gain μ in (22) such that $\mu > \kappa/2$, one has that, if $w(0) \neq 0$ then $w(t) \neq 0$ for all $t \in [0, \infty)$, and that $\lim_{t\to\infty}(w(t), s(t)) = 0$ which implies that $\lim_{t\to\infty}(\phi(t), \theta(t), \psi(t)) = 0$. If on the other hand $w(0) = 0$ (and $s(0) \neq 0$) the control law has to be slightly modified, as will be shown next.

Stabilization of the Complete System

This section contains the main results of the paper. Let $\mathcal{X}$ be the open set $\mathbf{C} \times (\mathbf{C}\backslash\{0\}) \times \mathbf{R}$. Given any compact subset $\mathcal{W} \subset \mathcal{X}$, we present a controller which generates u for the full system (10) and which has the property that for any initial condition in $\mathcal{W}$, the resulting trajectory converges asymptotically to zero.

The proposed controller is given by

$$u = u_s = -\frac{\kappa}{2}(\omega + \bar{\omega}w^2) - i\mu\, g(\omega, w, s) - \alpha(\omega + \kappa w + i\frac{\mu}{\bar{w}}s) \qquad (27)$$

where

$$g(\omega, w, s) \triangleq \frac{Im(\omega\bar{w})}{\bar{w}} - \frac{s}{2\bar{w}^2}(\bar{\omega} + \omega\bar{w}^2) \qquad (28)$$

and the scalars α, κ, μ are chosen to satisfy

$$\kappa > 0, \qquad \mu > \kappa, \qquad \alpha > (\kappa + \beta)/2, \qquad (29)$$

where β satisfies

$$|\omega + \kappa w + i\mu s/\bar{w}|(1 + |w|^2)^{1/2}/|w| \leq \beta \quad \forall\, (\omega, w, s) \in \mathcal{W} \qquad (30)$$

The main idea behind the proposed control law is to approximately implement control law (22) through the integrator (10a), by choosing the gain α "large enough". Indeed, introducing the variable

$$z \triangleq \omega + \kappa w + i\mu\frac{s}{\bar{w}} \qquad (31)$$

we can rewrite the closed-loop system in the form

$$\dot{z} = -\alpha z \qquad (32a)$$

$$\dot{w} = -\frac{\kappa}{2}(1 + |w|^2)w - \frac{i}{2}\mu s(\frac{1}{\bar{w}} - w) + \frac{z}{2} + \frac{\bar{z}}{2}w^2 \qquad (32b)$$

$$\dot{s} = -\mu s + Im(z\bar{w}) \qquad (32c)$$

Note that for large α the z-subsystem can be considered as a boundary layer system for equations (32). The outer layer, corresponding to $z = 0$, is in fact the system (23). Therefore, for large α one expects the overall system to behave like the system of Theorem 3.1. This statement is made precise in the following theorem.

Theorem 3.2 *Consider the closed-loop system (32) with α, μ and κ satisfying (29) and consider any initial condition $(\omega(0), w(0), s(0)) \in \mathcal{W}$. Then the following hold.*

(i) $w(t) \neq 0$ for all $t \geq 0$; hence the control law (27)-(30) is well-defined for all $t \geq 0$.

(ii) The trajectory $(z(\cdot), w(\cdot), s(\cdot))$ is bounded and

$$\lim_{t\to\infty} (z(t), w(t), s(t)) = 0$$

(iii) The control history $u(\cdot)$ is bounded and satisfies $\lim_{t\to\infty} u(t) = 0$.

Proof. We first show that $w(t) \neq 0$ for all $t \geq 0$. From (32b), the magnitude of w obeys the equation

$$\frac{d}{dt}|w|^2 = -(1 + |w|^2)(\kappa|w|^2 - Re(z\bar{w})) \qquad (33)$$

From definition (30) of β and condition (29) on α we have

$$|z(t)| = |z(0)|e^{-\alpha t} \leq \beta\left(\frac{|w(0)|^2}{|w(0)|^2 + 1}\right)^{1/2} e^{-\frac{\kappa+\beta}{2}t}$$

Recalling (24) and (26), the last inequality implies that $|z(t)| \leq \beta|\tilde{w}(t)|$ for all $t \geq 0$, where $\tilde{w}$ satisfies the differential equation

$$\frac{d}{dt}|\tilde{w}|^2 = -(\kappa+\beta)(1+|\tilde{w}|^2)|\tilde{w}|^2, \quad \tilde{w}(0) = w(0) \quad (34)$$

Since $|Re(z\bar{w})| \leq |z||w|$ and $|z| \leq \beta|\tilde{w}|$ it follows from (33) that

$$\frac{d}{dt}|w|^2 \geq -(1+|w|^2)(\kappa|w|^2 + \beta|w||\tilde{w}|) \quad (35)$$

From (34) and (35), $|w(t)| \leq |\tilde{w}(t)|$ implies that

$$\frac{d}{dt}|w(t)|^2 \geq \frac{d}{dt}|\tilde{w}(t)|^2$$

Since $|w(0)| = |\tilde{w}(0)|$, it follows that $|w(t)| \geq |\tilde{w}(t)|$ for all $t \geq 0$. Now, according to (34), $\tilde{w}(t) \neq 0$ for $t \geq 0$. Thus $w(t) \neq 0$ for $t \geq 0$.

Next we show that $\lim_{t\to\infty} w(t) = 0$. Since $|w(t)| \geq |\tilde{w}(t)|$ for all $t \geq 0$, where $\tilde{w}$ decays exponentially with rate $(\kappa+\beta)/2$, we have

$$\frac{|z(t)|}{|w(t)|} \leq \frac{|z(t)|}{|\tilde{w}(t)|} < \beta\, e^{-(\alpha - \frac{\kappa+\beta}{2})t} \quad \forall t \geq 0 \quad (36)$$

Since $\alpha > (\kappa+\beta)/2$ this term is exponentially decreasing. Moreover, since $Re(z\bar{w}) \leq |z||w|$, from equation (33) one obtains

$$\frac{d}{dt}|w|^2 \leq -(1+|w|^2)|w|^2(\kappa - |z|/|w|) \quad (37)$$

From (36), $\lim_{t\to\infty}(|z(t)|/|w(t)|)=0$; hence one can readily show that $\lim_{t\to\infty} w(t) = 0$. Since w is continuous, it must also be bounded.

We now show that the variable $\eta = s/|w|^2$ is bounded and asymptotically converges to zero. The evolution of η is governed by

$$\dot{\eta} = -(\mu-\kappa)\eta + \kappa|w|^2\eta - (1+|w|^2)Re(\frac{z}{w})\eta + Im(\frac{z}{w}) \quad (38)$$

Since $\lim_{t\to\infty} w(t) = 0$, $\lim_{t\to\infty} z(t)/w(t) = 0$, and $\mu - \kappa > 0$, one can now readily show that $\eta(\cdot)$ is bounded and $\lim_{t\to\infty} \eta(t) = 0$.

It now follows that $s(\cdot)$ is bounded and converges to zero asymptotically. We have therefore shown that the solutions of (32) are bounded, and that $\lim_{t\to\infty}(z(t), w(t), s(t)) = 0$.

Since $\lim_{t\to\infty} \eta(t) = 0$, one also has from (31) that $\lim_{t\to\infty} \omega(t) = 0$. It is easy to check that g in (27) is bounded and tends to zero as $t \to \infty$. Therefore, u is bounded and $\lim_{t\to\infty} u(t) = 0$, as claimed. This completes the proof. ∎

Corollary 3.1 *Under the hypotheses of Theorem 3.2 we have that* $\lim_{t\to\infty}(\omega_1(t), \omega_2(t), \phi(t), \theta(t), \psi(t)) = 0$.

So far, we have demonstrated that for initial conditions with $w(0) \neq 0$, it is possible to construct a control that will drive system (8)-(9) to the origin, with (ϕ, θ, ψ) avoiding the one-dimensional manifold

$$\mathcal{N}' \triangleq \{(\phi,\theta,\psi) : \phi = \theta = 0, \psi \neq 0\}$$

The previous methodology cannot be used if the initial condition is such that $\phi(0)=\theta(0)=0$ and $\psi(0)\neq 0$ ($w(0) = 0$ and $s(0)\neq 0$). Linearization of system (10a)-(10b) about $w = 0$ however, gives

$$\dot{\omega} = u \quad (39a)$$
$$\dot{w} = \frac{\omega}{2} \quad (39b)$$

The linearized system about $w = 0$ is completely controllable, and by choosing, for example, a constant control $u = u_c \in \mathbf{C}$, one can move away from $\mathcal{N}'$. Once away from $\mathcal{N}'$ one can use the control (27) to drive the system to the origin. We summarize the control strategy to drive to the origin $\omega_1 = \omega_2 = \phi = \theta = \psi = 0$ from arbitrary initial conditions.

$$u = \begin{cases} u_c \in \mathbf{C} & \text{if } w = 0 \text{ and } s \neq 0 \\ u_s(\omega, w, s) & \text{if otherwise} \end{cases} \quad (40)$$

Remark 1 The reader should not be misled to think that the integration of equations (13) is a coincidence attributable to the particular Eulerian angles used. In fact, the complete integration of the system is a direct consequence of the integrability of the new kinematic equation (6) for w (which is independent of the Eulerian angle set used) under the feedback control $\omega = -\kappa w$. To make this point clear, in the following equation we give, without proof, an expression for the manifold $\mathcal{S}$ as a function of (w, ψ) instead of (ϕ, θ, ψ).

$$\begin{aligned} \mathcal{S} &= \{ (w,\psi) \in \mathbf{C} \times (-\pi, \pi] : \tilde{s}(w,\psi) = 0 \} \\ \tilde{s}(w,\psi) &= \psi + \arctan(Re(w^2)/Im(w^2)) \\ &- \arctan((|w|^4 + Re(w^2))/Im(w^2)) \end{aligned}$$

This can be derived directly from equations (12) and (7), without the need to resort to equations (13).

Remark 2 Notice that equation (5) establishes a smooth change of coordinates (i.e., a diffeomorphism) between (w_1, w_2) and (ϕ, θ), for all (ϕ, θ) which do not correspond to an "upside-down" configuration of the rigid body $(\theta = 0, \phi = \pi)$; in this case $w = \infty$. This permits the use of equation (6) instead of equations (4a)-(4b) in stabilization problems, since stability of w implies in particular that $w(t) < \infty$ for all $t > 0$. Of course, one has to take into consideration the case when *initially* the rigid body has this singular configuration; however, we can always avoid this problem by simply turning the thrusters on to move away from this initial orientation.

As a final remark we note that the difference between the current control strategy and the control strategy of [5], where the same problem is investigated, is that in [5] the authors intend to render the one-dimensional manifold $\mathcal{N}'$ attractive, whereas in the current work we intend to avoid $\mathcal{N}'$ and render attractive the invariant manifold $\mathcal{S}$. We have demonstrated a control strategy that for initial conditions $(\phi_0, \theta_0, \psi_0) \notin \mathcal{N}'$ drives the system to the origin with the proper choice of feedback gains. If the initial conditions are on $\mathcal{N}'$ then one must first move *away* from $\mathcal{N}'$ in order to go to the origin, i.e., the control strategy is nonlocal in nature. This can be achieved either by the methodology described earlier, or by the techniques of [5].

4 Numerical Example

We illustrate the previous ideas with a numerical example. Consider a large angle maneuver of a symmetric spacecraft that is initially at rest ($\omega_1(0) = \omega_2(0) = 0$) and with initial orientation given by $\phi(0) = \pi, \theta(0) = 0.25\pi, \psi = -0.5\pi$. This initial data is taken from [5] for comparison. The controller is given by equation (27). The feedback gains are chosen as $\kappa = 0.5, \mu = 1.25$ and $\alpha = 2$. The results are shown in Figs. 4 and 5. This problem corresponds to an initial condition not on the manifold $\mathcal{S}$ (see Fig. 3). In the first five seconds the configuration is driven to $\mathcal{S}$ and once on $\mathcal{S}$ the system exhibits exponential decay to the origin as predicted by Lemma 3.1.

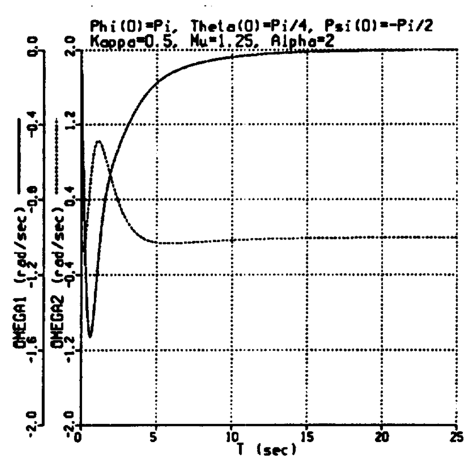

Figure 4: Angular velocities ω_1 and ω_2.

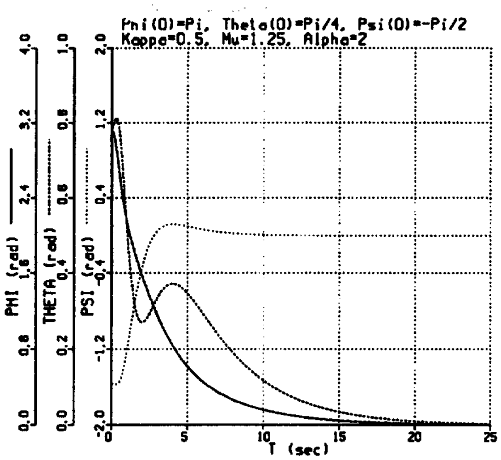

Figure 5: Eulerian angles ϕ, θ and ψ.

5 Conclusions

We have demonstrated a control strategy that achieves arbitrary reorientation of a rotating spacecraft, when the two available control torques span the two dimensional plane perpendicular to the axis of symmetry, under the assumption that the initial spin-rate is zero. Integrating the complete closed-loop system of the kinematic equations we have constructed a two-dimensional manifold, which is used to derive a feedback control that drives the complete system to the origin from arbitrary initial conditions. The control design methodology is based on a novel formulation of the attitude kinematics, which promises to be extremely useful for control and/or stabilization purposes.

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