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# SECULAR SOLUTION FOR DELTA-V DURING SPIN RATE CHANGE MANEUVERS OF RIGID BODY SPACECRAFT

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## Abstract

Analytic expressions have been found for Euler's Equations of Motion and for the Eulerian Angles for both symmetric and near symmetric rigid bodies under the influence of arbitrary constant body-fixed torques. These solutions have been used to solve for the secular terms in the translational ΔV equations in inertial space. This secular ΔV solution is of interest in application to spinning spacecraft in that it describes the average direction of the ΔV of the spacecraft during a spin-up maneuver. Numerical integration of the governing differential equations has verified that the secular ΔV solution is valid for large time and is accurate in many physical situations including spin-up maneuvers of the Galileo spacecraft.

## I. Introduction

The Galileo spacecraft is a dual-spin spacecraft designed to explore Jupiter and its moons. It is scheduled to be launched from the space shuttle in May of 1986 to arrive at Jupiter in August of 1988. The spacecraft's prime mission is to send a probe into the Jovian atmosphere. After the probe has been released the spacecraft will perform the orbit insertion maneuver, sending it into orbit around Jupiter. It will spend the next twenty months collecting scientific data on Jupiter and its moons and transmitting it to Earth.

The spacecraft is usually in dual-spin mode, but during ΔV maneuvers the mode is changed to single-spin by locking the rotor and stator together. It is then spun up to 1.05 rad/s, from a nominal of 0.306 rad/s, prior to an axial ΔV burn. This increases the stability margin and accuracy of the maneuver. After the burn, the spacecraft is spun down to 0.306 rad/s. The thruster configuration is illustrated in Fig. 1 where S2A and -S1A are the primary spinup and spindown thrusters, respectively, and S2B and -S1B are backups. Because of certain design considerations, (such as plume impingement problems, redundancy and expense of thrusters), the Galileo spacecraft does not have coupled thrusters. Thus, during a spinup or spindown manuever, there will be not only a constant body-fixed torque about the spin axis, but also small undesired constant torques about the transverse axes. The undesired torques cause the orientation of the angular momentum vector to change in inertial space.

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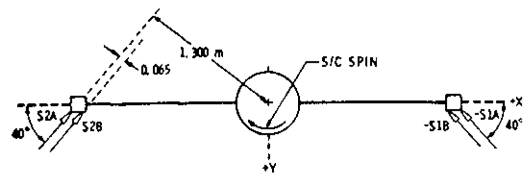


Fig. 1 Spin thruster configuration.

Analytic expressions are found for Euler's Equations of Motion and for the Eulerian Angles for both symmetric and near symmetric rigid bodies under the influence of arbitrary constant body-fixed torques.[1] These solutions provide the body-fixed angular velocities and the attitude of the body, respectively, as functions of time. They are of special interest in applications to spinning spacecraft because they include the effect of time varying spin rate. Thus, they have been applied to spin-up and spin-down maneuvers as well as to error analysis for thruster misalignments. In order to complete the analysis of these maneuvers, one final set of differential equations must be integrated: those corresponding to the ΔV's imparted to the spacecraft during the spin-up or spin-down maneuver.

The main purpose in solving these equations is found in their applications to satellites and deep space probes. Even though numerical solutions are easily found by computer simulations analytic solutions can provide deeper insight and understanding and can be used in obtaining quick solutions, error analyses, and compact algorithms for on-board computations. In space applications certain simplifying, yet realistic, assumptions can be made so that valuable approximate analytic solutions can be found.

The following sections include a brief description of the previous work, the analytic development of the secular solution for the ΔV equations, and simulation results which validate the accuracy of the solution.

## II. Background and Summary of Previous Work

### Solution of Euler's Equations of Motion

Euler's equations of motion of a rigid body are:

$$M_x = I_x\dot{\omega}_x + (I_z-I_y)\omega_y\omega_z$$
$$M_y = I_y\dot{\omega}_y + (I_x-I_z)\omega_z\omega_x$$
$$M_z = I_z\dot{\omega}_z + (I_y-I_x)\omega_x\omega_y \quad (1)$$

An accurate approximate analytic solution to Eqs. (1) is obtained for near-symmetric rigid bodies subject to arbitrary constant moments by assuming:

$$\omega_z \approx M_z\, t\, / I_z + \omega_{z_0}. \tag{2}$$

When $I_x = I_y$ then Eq. (2) is exact but when $I_x \approx I_y$ the approximation provides very useful accurate solutions, particularly when $\omega_x$ and $\omega_y$ *are small*, which is usually the case for spin-stabilized spacecraft. The solutions for $\omega_x$ and $\omega_y$ are given in Appendix 1.

## Approximate Solution for the Eulerian Angles

The kinematic equations of motion for a type 1:3-1-2 Euler angle rotation are[2]:

$$\dot{\phi}_x = \omega_x \cos\phi_y + \omega_z \sin\phi_y$$

$$\dot{\phi}_y = \omega_y - (\omega_z \cos\phi_y - \omega_x \sin\phi_y)\tan\phi_x$$

$$\dot{\phi}_z = (\omega_z \cos\phi_y - \omega_x \sin\phi_y)\sec\phi_x. \tag{3}$$

A highly accurate approximate analytic solution for the *Eulerian Angles for a near*-symmetric rigid body has been found and the main restrictions on the solution are that two of the Eulerian angles ($\phi_x$, $\phi_y$) and the parameter, $\dot{\omega}_z/\omega_z^2$, must remain small. This solution is given in Appendix 2.

## Solution for the Angular Momentum Vector

With the analytic results for the angular velocities, $\omega_x$, $\omega_y$, and $\omega_z$, and type 1:3-1-2 Euler angles, $\phi_x$, $\phi_y$, and $\phi_z$, the approximate analytic solution for the angular momentum vector in inertial space can be constructed easily:

$$\begin{Bmatrix} H_x \\ H_y \\ H_z \end{Bmatrix} = A \begin{Bmatrix} I_x\omega_x \\ I_y\omega_y \\ I_z\omega_z \end{Bmatrix} \tag{4}$$

where A is the transformation matrix based upon the Eulerian angles. The nominal pointing history of the angular momentum vector, $\vec{H}$, during the spinup maneuver is shown in Fig. 2, which was generated from Eq. (4).[3] Since the angles involved are very small, the quantities $H_x/H_z$ and $H_y/H_z$ are used to describe the orientation of the angular momentum vector in inertial space.

It is interesting to note that the radial distance of the spiral path exhibited in Fig. 2 from its center can be approximated accurately by the heuristic relation:

$$\rho(t) = (M_x^2 + M_y^2)^{1/2} / I_z \omega_z^2(t) \tag{5}$$

The center point of the spiral is also approximated by the simple heuristic relations:

$$H_x/H_z = -M_y/I_z\omega_{z_0}^2$$

$$H_y/H_z = M_x/I_z\omega_{z_0}^2. \tag{6}$$

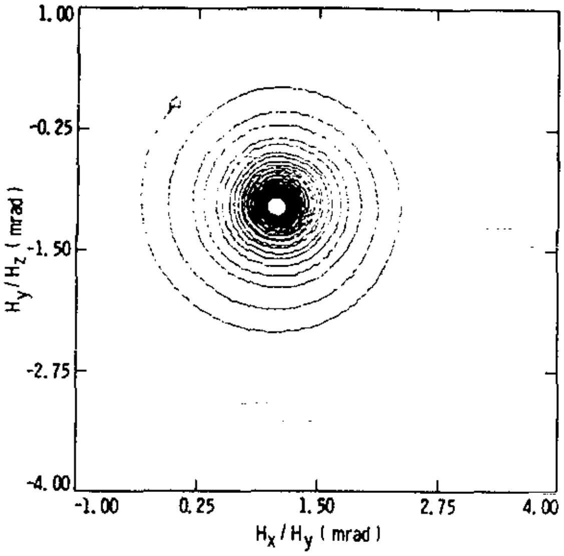


Fig. 2 Nominal orientation of the angular momentum vector in inertial space during spin-up maneuver.

These solutions were inspired by the constant spin rate case, $M_z = 0$, in which they can be easily derived for small angles $\phi_x$ and $\phi_y$. In this situation the angular momentum vector precesses about the direction given in Eq. (6) in a closed curve.

It was further reasoned that for very large durations of the spinup maneuver, as time approaches infinity, the direction of $\Delta\vec{V}$ should approach the average direction of $\vec{H}$, or the center point of the spiral in Fig. 2. Since the angles involved in this case are also very small, the quantities $\Delta V_x/\Delta V_z$ and $\Delta V_y/\Delta V_z$ will be used to describe the orientation of the $\Delta V$ vector in inertial space. It is the intent of this paper to show that by ignoring periodic terms and concentrating on the secular terms, the approximate solutions for the $\Delta V$ equations for very large times will approach a limiting direction comparable to Eq. (6) or in other words:

$$\Delta V_x/\Delta V_z \approx -M_y/I_z\omega_{z_0}^2$$

$$\Delta V_y/\Delta V_z \approx M_x/I_z\omega_{z_0}^2. \tag{7}$$

# III. Secular Solutions for the ΔV Equations

## Acceleration Equations

Let $F_x$, $F_y$, $F_z$ be the body fixed forces due to he spin thruster and m be the mass of the spacecraft. Then for Type 1:3-1-2 Eulerian angles, $\phi_x$, $\phi_y$, $\phi_z$, the acceleration components in inertial space are:

$$\begin{Bmatrix} a_x \\ a_y \\ a_z \end{Bmatrix} = A \begin{Bmatrix} f_x/m \\ f_y/m \\ f_z/m \end{Bmatrix} \tag{8}$$

where A is the transformation matrix based upon the Eulerian angles. During the spinup maneuver the terms $f_x$, $f_y$, and $\phi_z$ can be large while $f_z$, $\phi_x$ and $\phi_y$ are typically small. In the case of Galileo $f_z$ is a small term encompassing thruster misalignment, plume impingement, and wobble. By allowing $\phi_x$ and $\phi_y$ to be small Eqs. (8) reduce to:

$$a_x = \frac{f_x}{m}\cos\phi_z - \frac{f_y}{m}\sin\phi_z - \frac{f_y}{m}\left(\phi_y\cos\phi_z + \phi_x\sin\phi_z\right)$$

$$a_y = \frac{f_x}{m}\sin\phi_z + \frac{f_y}{m}\cos\phi_z + \frac{f_z}{m}\left(\phi_y\sin\phi_z - \phi_x\cos\phi_z\right)$$

$$a_z = \frac{-f_x}{m}\phi_y + \frac{f_y}{m}\phi_x + \frac{f_z}{m}, \tag{9}$$

where, according to Appendix 2,

$$\phi_x(t) = \phi_{xo}\cos[\alpha(\tfrac{1}{2}\bar{a}t^2 + \bar{b}t)] + \phi_{yo}\sin[\alpha(\tfrac{1}{2}\bar{a}t^2 + \bar{b}t)] + U\,\mathrm{sgn}\bar{a}\,[W_{ys}(\tfrac{1}{2}\bar{a}t^2 + \bar{b}t) + W_{xc}(\tfrac{1}{2}\bar{a}t^2 + \bar{b}t)]$$

$$\phi_y(t) = -\phi_{xo}\sin[\alpha(\tfrac{1}{2}\bar{a}t^2 + \bar{b}t)] + \phi_{yo}\cos[\alpha(\tfrac{1}{2}\bar{a}t^2 + \bar{b}t)] + U\,\mathrm{sgn}\bar{a}\,[W_{yc}(\tfrac{1}{2}\bar{a}t^2 + \bar{b}t) - W_{xs}(\tfrac{1}{2}\bar{a}t^2 + \bar{b}t)]$$

$$\phi_z(t) = \frac{M_z}{2I_z}t^2 + \omega_{zo}t + \phi_{zo}. \tag{10}$$

It is reasonable to assume that the large time solution would be independent of the initial conditions therefore Eqs. (10) can be reduced to:

$$\phi_x(t) = U\,\mathrm{sgn}\bar{a}\,[W_{ys}(\tfrac{1}{2}\bar{a}t^2 + \bar{b}t) + W_{xc}(\tfrac{1}{2}\bar{a}t + \bar{b}t)]$$

$$\phi_y(t) = U\,\mathrm{sgn}\bar{a}\,[W_{yc}(\tfrac{1}{2}\bar{a}t^2 + \bar{b}t) - W_{xs}(\tfrac{1}{2}\bar{a}t^2 + \bar{b}t)]$$

$$\phi_z(t) = \frac{M_z}{2I_z}t^2 + \omega_{zo}t, \tag{11}$$

where u, $\bar{a}$, and $\bar{b}$ are as defined in Appendix 1. The W functions can be expressed as follows:

$$W_{ys}(\bar{\tau}) = k_{y1}J_{cs}(\bar{\tau}, 1, 0, -\alpha, \alpha\bar{\tau}) + k_{y2}J_{ss}(\bar{\tau}, 1, 0, -\alpha, \alpha\bar{\tau}) + k_{y3}J_{cs}\left(\bar{\tau}, \mathrm{sgn}\bar{a}, \frac{\bar{b}^2}{2\bar{a}}\mathrm{sgn}\bar{a}, -\alpha, \alpha\bar{\tau}\right) + k_{y4}J_{ss}\left(\bar{\tau}, \mathrm{sgn}\bar{a}, \frac{\bar{b}^2}{2\bar{a}}\mathrm{sgn}\bar{a}, -\alpha, \alpha\bar{\tau}\right) + k_{y5}F_s(\bar{\tau}) + k_{y6}G_s(\bar{\tau})$$

$$W_{xc}(\bar{\tau}) = k_{x1}J_{cc}(\bar{\tau}, 1, 0, -\alpha, \alpha\bar{\tau}) + k_{x2}J_{cs}(\bar{\tau}, -\alpha, \alpha\bar{\tau}, 1, 0) + k_{x3}J_{cc}\left(\bar{\tau}, \mathrm{sgn}\bar{a}, \frac{\bar{b}^2}{2\bar{a}}\mathrm{sgn}\bar{a}, -\alpha, \alpha\bar{\tau}\right) + k_{x4}J_{cs}\left(\bar{\tau}, -\alpha, \alpha\bar{\tau}, \mathrm{sgn}\bar{a}, \frac{\bar{b}^2}{2\bar{a}}\mathrm{sgn}\bar{a}\right) + k_{x5}F_c(\bar{\tau}) + k_{x6}G_c(\bar{\tau})$$

$$W_{yc}(\bar{\tau}) = k_{y1}J_{cc}(\bar{\tau}, 1, 0, -\alpha, \alpha\bar{\tau}) + k_{y2}J_{cs}(\bar{\tau}, -\alpha, \alpha\bar{\tau}, 1, 0) + k_{y3}J_{cc}\left(\bar{\tau}, \mathrm{sgn}\bar{a}, \frac{\bar{b}^2}{2\bar{a}}\mathrm{sgn}\bar{a}, -\alpha, \alpha\bar{\tau}\right) + k_{y4}J_{cs}\left(\bar{\tau}, -\alpha, \alpha\bar{\tau}, \mathrm{sgn}\bar{a}, \frac{\bar{b}^2}{2\bar{a}}\mathrm{sgn}\bar{a}\right) + k_{y5}F_c(\bar{\tau}) + k_{y6}G_c(\bar{\tau})$$

$$W_{xs}(\bar{\tau}) = k_{x1}J_{cs}(\bar{\tau}, 1, 0, -\alpha, \alpha\bar{\tau}) + k_{x2}J_{ss}(\bar{\tau}, 1, 0, -\alpha, \alpha\bar{\tau}) + k_{x3}J_{cs}\left(\bar{\tau}, \mathrm{sgn}\bar{a}, \frac{\bar{b}^2}{2\bar{a}}\mathrm{sgn}\bar{a}, -\alpha, \alpha\bar{\tau}\right) + k_{x4}J_{ss}\left(\bar{\tau}, \mathrm{sgn}\bar{a}, \frac{\bar{b}^2}{2\bar{a}}\mathrm{sgn}\bar{a}, -\alpha, \alpha\bar{\tau}\right) + k_{x5}F_s(\bar{\tau}) + k_{x6}G_s(\bar{\tau}) \tag{12}$$

and the J functions can in turn be defined as:

$$J_{cc}(\bar{\tau}, K_1, K_2, K_3, K_4) = L_c(\bar{\tau}, K_1 - K_3, K_2 - K_4) + L_c(\bar{\tau}, K_1 + K_3, K_2 + K_4)$$

$$J_{cs}(\bar{\tau}, K_1, K_2, K_3, K_4) = L_s(\bar{\tau}, K_1+K_3, K_2+K_4) - L_s(\bar{\tau}, K_1-K_3, K_2-K_4) \tag{13}$$

$$J_{ss}(\bar{\tau}, K_1, K_2, K_3, K_4) = L_c(\bar{\tau}, K_1-K_3, K_2-K_4) - L_c(\bar{\tau}, K_1+K_3, K_2+K_4)$$

where for large time:

$$L_c(\bar{\tau}, h_1, h_2) = \frac{-\sin h_2}{2h_1 |\bar{b}|}$$

$$L_s(\bar{\tau}, h_1, h_2) = \frac{\cos h_2}{2h_1 |\bar{b}|} \tag{14}$$

From Ref. (1) it can be seen tha $G_s$ and $G_c$ are negligible in comparison to $F_s$ and $F_c$ and by considering only the highest order terms can be written as:

$$F_s(\bar{\tau}) = \frac{\left(\frac{1}{\pi}\right)\left(\frac{\pi}{2}\right)^{1/2}}{\sqrt{2|\bar{a}|}}\left[\sin(t_1)\mathrm{Cos}(t_0, t_1) - \cos(t_1)\mathrm{Sin}(t_0, t_1)\right]$$

$$F_c(\bar{\tau}) = \frac{\left(\frac{1}{\pi}\right)\left(\frac{\pi}{2}\right)^{1/2}}{\sqrt{2|\bar{a}|}}\left[\cos(t_1)\mathrm{Cos}(t_0, t_1) + \sin(t_1)\mathrm{Sin}(t_0, t_1)\right] \tag{15}$$

where

$$t_1 = \alpha\bar{\tau} + \frac{a\bar{b}^2}{2\bar{a}},$$

$$t_0 + \frac{\alpha\bar{b}^2}{2\bar{a}},$$

$$\alpha = \frac{1}{\sqrt{\lambda_1 \lambda_2}},$$

$$\mathrm{Cos}(t_0, t_1) = \int_{t_0}^{t_1} \frac{\cos(s)\,ds}{s},$$

$$\mathrm{Sin}(t_0, t_1) = \int_{t_0}^{t_1} \frac{\sin(s)\,ds}{s}.$$

Asymptotic expansions can be written for Eq. (15) and by allowing time to become very large and by keeping higher order terms:

$$F_s(\bar{\tau}) = \frac{-\cos(\alpha\bar{\tau})}{2\sqrt{\pi|\bar{a}|t_0}},$$

$$F_c(\bar{\tau}) = \frac{\sin(\alpha\bar{\tau})}{2\sqrt{\pi|\bar{a}|}\; t_0}. \tag{16}$$

Equation (12) can now be re-written as:

$$W_{y_5}(\bar{\tau}) = \frac{K_{y_1}\,\alpha\cos(\alpha\bar{\tau})}{|\bar{b}|(1-\alpha^2)} + \frac{K_{y_2}\sin(\alpha\bar{\tau})}{|\bar{b}|(1-\alpha^2)} + \frac{K_{y_3}}{2|\bar{b}|(1-\alpha^2)}\left[(\mathrm{sgn}\bar{a} + \alpha)\cos\left(\frac{\bar{b}^2}{2\bar{a}}\mathrm{sgn}\bar{a} + \alpha\bar{\tau}\right) - (\mathrm{sgn}\bar{a} - \alpha)\cos\left(\frac{\bar{b}^2}{2\bar{a}}\mathrm{sgn}\bar{a} - \alpha\bar{\tau}\right)\right]$$
$$- \frac{K_{y_4}}{2|\bar{b}|(1-\alpha^2)}\left[(\mathrm{sgn}\bar{a} - \alpha)\sin\left(\frac{\bar{b}^2}{2\bar{a}}\mathrm{sgn}\bar{a} - \alpha\bar{\tau}\right) - (\mathrm{sgn}\bar{a} + \alpha)\sin\left(\frac{\bar{b}^2}{2\bar{a}}\mathrm{sgn}\bar{a} + \alpha\bar{\tau}\right)\right] - \frac{K_{y_5}\cos(\alpha\bar{\tau})}{2\sqrt{\pi|\bar{a}|}\; t_0}$$

$$W_{xc}(\bar{\tau}) = \frac{-K_{x_1}\,\alpha\sin(\alpha\bar{\tau})}{|\bar{b}|(1-\alpha^2)} + \frac{K_{x_2}\cos(\alpha\bar{\tau})}{|\bar{b}|(1-\alpha^2)} - \frac{K_{x_3}}{2|\bar{b}|(1-\alpha^2)}\left[(\mathrm{sgn}\bar{a}-\alpha)\sin\left(\frac{\bar{b}^2}{2\bar{a}}\mathrm{sgn}\bar{a} - \alpha\bar{\tau}\right) + (\mathrm{sgn}\bar{a} + \alpha)\sin\left(\frac{\bar{b}^2}{2\bar{a}}\mathrm{sgn}\bar{a} + \alpha\bar{\tau}\right)\right]$$
$$+ \frac{K_{x_4}}{2|\bar{b}|(1-\alpha^2)}\left[(\mathrm{sgn}\bar{a} + \alpha)\cos\left(\frac{\bar{b}^2}{2\bar{a}}\mathrm{sgn}\bar{a} + \alpha\bar{\tau}\right) + (\mathrm{sgn}\bar{a} - \alpha)\cos\left(\frac{\bar{b}^2}{2\bar{a}}\mathrm{sgn}\bar{a} - \alpha\tau\right)\right] + \frac{K_{x_5}\sin(\bar{\alpha}\tau)}{2\sqrt{\pi|\bar{a}|}\; t_0}$$

$$W_{yc}(\bar{\tau}) - \frac{K_{y_1}\,\alpha\sin(\alpha\bar{\tau})}{|\bar{b}|(1-\alpha^2)} + \frac{K_{y_2}\cos(\alpha\bar{\tau})}{|\bar{b}|(1-\alpha^2)} - \frac{K_{y_3}}{2|\bar{b}|(1-\alpha^2)}\left[(\mathrm{sgn}\bar{a} - \alpha)\sin\left(\frac{\bar{b}^2}{2\bar{a}}\mathrm{sgn}\bar{a} - \alpha\bar{\tau}\right) + (\mathrm{sgn}\bar{a} + \alpha)\sin\left(\frac{\bar{b}^2}{2\bar{a}}\mathrm{sgn}\bar{a} + \alpha\bar{\tau}\right)\right] \tag{17}$$

$$+ \frac{K_{y_4}}{2|\bar{b}|\ (1-\alpha^2)} \left[ (\mathrm{sgn}\bar{a} + \alpha)\ \cos\left(\frac{\bar{b}^2}{2\bar{a}}\mathrm{sgn}\bar{a} + \alpha\bar{\tau}\right) \right.$$

$$\left. + (\mathrm{sgn}\bar{a} - \alpha)\ \cos\left(\frac{\bar{b}^2}{2\bar{a}}\ \mathrm{sgn}\bar{a} - \alpha\bar{\tau}\right)\right] + \frac{K_{y_5} \sin(\alpha\bar{\tau})}{2\sqrt{\pi|\bar{a}|}\ t_0}$$

$$W_{xs}(\bar{\tau}) = \frac{K_{x_1}\alpha\ \cos(\alpha\bar{\tau})}{|\bar{b}|\ (1-\alpha^2)} + \frac{K_{x_2}\ \sin(\alpha\bar{\tau})}{|\bar{b}|\ (1-\alpha^2)}$$

$$+ \frac{K_{x_3}}{2|\bar{b}|\ (1-\alpha^2)} \left[ (\mathrm{sgn}\bar{a} + \alpha)\ \cos\left(\frac{\bar{b}^2}{2\bar{a}}\ \mathrm{sgn}\bar{a} + \alpha\bar{\tau}\right) \right.$$

$$\left. - (\mathrm{sgn}\bar{a} - \alpha)\ \cos\left(\frac{\bar{b}^2}{2\bar{a}}\ \mathrm{sgn}\bar{a} - \alpha\bar{\tau}\right)\right]$$

$$- \frac{K_{x_4}}{2|\bar{b}|\ (1-\alpha^2)} \left[ (\mathrm{sgn}\bar{a} - \alpha)\ \sin\left(\frac{\bar{b}^2}{2\bar{a}}\ \mathrm{sgn}\bar{a} - \alpha\bar{\tau}\right) \right.$$

$$\left. - (\mathrm{sgn}\bar{a} + \alpha)\ \sin\left(\frac{\bar{b}^2}{2\bar{a}}\ \mathrm{sgn}\bar{a} + \alpha\bar{\tau}\right)\right] - \frac{K_{x_5}\ \cos(\alpha\bar{\tau})}{2\sqrt{\pi|\bar{a}|}\ t_0}$$

Taking the expressions for the constant coefficients $K_{x1}$ through $K_{x5}$ and $K_{y1}$ through $K_{y5}$, found in Appendix 1, substituting them into the above equation, and then combining the $W$ functions in accordance with Eqs. (11) yields:

$$W_{ys}(\bar{\tau}) + W_{xc}(\bar{\tau}) = \frac{(\mathrm{sgn}\bar{a} + \alpha)\ \cos\left(\frac{\bar{b}^2}{2\bar{a}}\ \mathrm{sgn}\bar{a} + \alpha\bar{\tau}\right)}{2|\bar{b}|\ (1-\alpha^2)} \cdot$$

$$u\mathrm{sgn}\bar{a}\left[\frac{c}{|\bar{b}|}\left(\mathrm{sgn}\bar{a} - \sqrt{\frac{\lambda_2}{\lambda_1}}\right)\cos\left(\frac{\bar{b}^2}{2\bar{a}}\right)\right.$$

$$\left. - \frac{d\mathrm{sgn}\bar{a}}{|\bar{b}|}\left(\mathrm{sgn}\bar{a} - \sqrt{\frac{\lambda_1}{\lambda_2}}\right)\sin\left(\frac{\bar{b}^2}{2\bar{a}}\right)\right]$$

$$- \frac{(\mathrm{sgn}\bar{a} - \alpha)\ \cos\left(\frac{\bar{b}^2}{2\bar{a}}\ \mathrm{sgn}\bar{a} - \alpha\bar{\tau}\right)}{2|\bar{b}|\ (1-\alpha^2)} \cdot$$

$$u\mathrm{sgn}\bar{a}\left[\frac{-c}{|\bar{b}|}\left(\mathrm{sgn}\bar{a} + \sqrt{\frac{\lambda_2}{\lambda_1}}\right)\cos\left(\frac{\bar{b}^2}{2a}\right)\right.$$

$$\left. - \frac{d\mathrm{sgn}\bar{a}}{|\bar{b}|}\left(\mathrm{sgn}\bar{a} + \sqrt{\frac{\lambda_1}{\lambda_2}}\right)\sin\left(\frac{\bar{b}^2}{2a}\right)\right]$$

$$- \frac{(\mathrm{sgn}\bar{a} - \alpha)\ \sin\left(\frac{\bar{b}^2}{2\bar{a}}\ \mathrm{sgn}\bar{a} - \alpha\bar{\tau}\right)}{2|\bar{b}|\ (1-\alpha^2)} \cdot$$

$$u\mathrm{sgn}\bar{a}\left[\frac{d}{|\bar{b}|}\left(\mathrm{sgn}\bar{a} + \sqrt{\frac{\lambda_1}{\lambda_2}}\right)\cos\left(\frac{\bar{b}^2}{2a}\right)\right.$$

$$\left. - \frac{c\mathrm{sgn}\bar{a}}{|\bar{b}|}\left(\mathrm{sgn}\bar{a} + \sqrt{\frac{\lambda_2}{\lambda_1}}\right)\sin\left(\frac{\bar{b}^2}{2a}\right)\right]$$

$$+ \frac{(\mathrm{sgn}\bar{a} + \alpha)\ \sin\left(\frac{\bar{b}^2}{2\bar{a}}\ \mathrm{sgn}\bar{a} + \alpha\bar{\tau}\right)}{2|\bar{b}|\ (1-\alpha^2)} \cdot$$

$$u\mathrm{sgn}\bar{a}\left[\frac{d}{|\bar{b}|}\left(\mathrm{sgn}\bar{a} - \sqrt{\frac{\lambda_1}{\lambda_2}}\right)\cos\left(\frac{\bar{b}^2}{2a}\right)\right.$$

$$\left. + \frac{c\mathrm{sgn}\bar{a}}{|\bar{b}|}\left(\mathrm{sgn}\bar{a} - \sqrt{\frac{\lambda_2}{\lambda_1}}\right)\sin\left(\frac{\bar{b}^2}{2a}\right)\right]$$

$$- \frac{\mathrm{sgn}b}{2|\bar{a}|} \cdot \left[\frac{\lambda_2 C}{\sqrt{\lambda_1\lambda_2}}\right] \frac{\cos\ (\alpha\bar{\tau})}{t_0}$$

$$- \frac{\mathrm{sgn}b}{2|\bar{a}|} \left[\frac{\lambda_1\ d}{\sqrt{\lambda_1\lambda_2}}\right] \frac{\sin\ (\alpha\bar{\tau})}{t_0}, \qquad (18)a$$

$$W_{yc}(\bar{\tau}) - W_{xs}(\bar{\tau}) = -\frac{(\mathrm{sgn}\bar{a}-\alpha)\ \sin\ \left(\frac{\bar{b}^2}{2\bar{a}}\ \mathrm{sgn}\bar{a} - \alpha\bar{\tau}\right)}{2|\bar{b}|\ (1-\alpha^2)} \cdot$$

$$u\mathrm{sgn}\bar{a}\left[\frac{-c}{|\bar{b}|}\left(\mathrm{sgn}\bar{a} - \sqrt{\frac{\lambda_2}{\lambda_1}}\right)\cos\left(\frac{\bar{b}^2}{2a}\right)\right.$$

$$\left. - \frac{d\mathrm{sgn}\bar{a}}{|\bar{b}|}\left(\mathrm{sgn}\bar{a} + \sqrt{\frac{\lambda_1}{\lambda_2}}\right)\sin\left(\frac{\bar{b}^2}{2a}\right)\right]$$

$$- \frac{(\mathrm{sgn}\bar{a} + \alpha)\ \sin\left(\frac{\bar{b}^2}{2\bar{a}}\ \mathrm{sgn}\bar{a} + \alpha\bar{\tau}\right)}{2|\bar{b}|\ (1-\alpha^2)} \cdot$$

$$u\mathrm{sgn}\bar{a}\left[\frac{c}{|\bar{b}|}\left(\mathrm{sgn}\bar{a} - \sqrt{\frac{\lambda_2}{\lambda_1}}\right)\cos\left(\frac{\bar{b}^2}{2a}\right)\right.$$

$$\left. - \frac{d\mathrm{sgn}\bar{a}}{|\bar{b}|}\left(\mathrm{sgn}\bar{a} - \sqrt{\frac{\lambda_1}{\lambda_2}}\right)\sin\left(\frac{\bar{b}^2}{2a}\right)\right]$$

$$+ \frac{(\mathrm{sgn}\bar{a} + \alpha)\ \cos\left(\frac{\bar{b}^2}{2\bar{a}}\ \mathrm{sgn}\bar{a} + \alpha\bar{\tau}\right)}{2|\bar{b}|\ (1-\alpha^2)} \cdot$$

$$u\mathrm{sgn}\bar{a}\left[\frac{d}{|\bar{b}|}\left(\mathrm{sgn}\bar{a} - \sqrt{\frac{\lambda_1}{\lambda_2}}\right)\cos\left(\frac{\bar{b}^2}{2a}\right)\right.$$

$$\left. + \frac{c\mathrm{sgn}\bar{a}}{|\bar{b}|}\left(\mathrm{sgn}\bar{a} - \sqrt{\frac{\lambda_2}{\lambda_1}}\right)\sin\left(\frac{\bar{b}^2}{2a}\right)\right]$$

$$+ \frac{(\operatorname{sgn}\bar{a} - \alpha)\cos\left(\frac{\bar{b}^2}{2\bar{a}}\operatorname{sgn}\bar{a} - \alpha\bar{\tau}\right)}{2|\bar{b}|\,(1-\alpha^2)} \cdot$$

$$u\operatorname{sgn}\bar{a}\left[\frac{d}{|\bar{b}|}\left(\operatorname{sgn}\bar{a} + \sqrt{\frac{\lambda_1}{\lambda_2}}\right)\cos\left(\frac{\bar{b}^2}{2\bar{a}}\right)\right.$$

$$\left. - \frac{c\operatorname{sgn}\bar{a}}{|\bar{b}|}\left(\operatorname{sgn}\bar{a} + \sqrt{\frac{\lambda_2}{\lambda_1}}\right)\sin\left(\frac{\bar{b}^2}{2\bar{a}}\right)\right]$$

$$+ \frac{\operatorname{sgn} b}{2|\bar{a}|} \cdot \left[\frac{\lambda_2 C}{\sqrt{\lambda_1 \lambda_2}}\right] \frac{\sin(\alpha\bar{\tau})}{t_0}$$

$$- \frac{\operatorname{sgn} b}{2|\bar{a}|}\left[\frac{\lambda_1 d}{\sqrt{\lambda_1\lambda_2}}\right]\frac{\cos(\alpha\bar{\tau})}{t_0} \quad . \qquad (18)b$$

By expanding Eqs. (18) completely and then reducing them by eliminating the terms that cancel the following equations result:

$$W_{ys}(\bar{\tau}) - W_{xc}(\bar{\tau}) = \frac{u\operatorname{sgn}\bar{a}}{\bar{b}^2}\left[M_x \frac{(I_z - I_x)}{I_z I_x}\cos(\alpha\bar{\tau}) + M_y \frac{(I_z - I_y)}{I_z I_y}\sin(\alpha\bar{\tau})\right] - \frac{u\operatorname{sgn}\bar{a}}{\omega_{zo}^2}\left[\frac{c}{\lambda_1}\cos(\alpha\bar{\tau}) + \frac{d}{\lambda_2}\sin(\alpha\bar{\tau})\right] \qquad (19)$$

$$W_{yc}(\bar{\tau}) - W_{xs}(\bar{\tau}) = \frac{u\operatorname{sgn}\bar{a}}{I_z\omega_{zo}^2}\left[\frac{M_y\cos(\alpha\bar{\tau})}{(I_z - I_x)} I_x - \frac{M_x\sin(\alpha\bar{\tau})}{(I_z - I_y)} I_y\right] + \frac{u\operatorname{sgn}\bar{a}}{\omega_{zo}^2}\left[\frac{c}{\lambda_1}\sin(\alpha\bar{\tau}) - \frac{d}{\lambda_2}\cos(\alpha\bar{\tau})\right].$$

Eqs. (19) can be substituted into Eqs. (11) and reduced further to obtain:

$$\phi_x(\bar{\tau}) = \frac{-M_x\cos(\alpha\bar{\tau}) - M_y\sin(\alpha\bar{\tau})}{I_z\,\omega_{zo}^2},$$

$$\phi_y(\bar{\tau}) = \frac{M_x\sin(\alpha\bar{\tau}) - M_y\cos(\alpha\bar{\tau})}{I_z\,\omega_{zo}^2}, \qquad (20)$$

$$\phi_z(\bar{\tau}) = \frac{M_z}{2I_z}t^2 + \omega_{zo}t = \alpha\bar{\tau}$$

Eqs. (20) can be substituted into the acceleration equation, Eqs. (9), and by retaining only the secular terms yield:

$$a_x = \frac{-f_z}{m}\frac{M_y}{I_z\omega_{zo}^2},$$

$$a_y = \frac{f_z}{m}\frac{M_x}{I_z\omega_{zo}^2}, \qquad (21)$$

$$a_z = \frac{f_z}{m},$$

each of which is a constant. The $\Delta V$ equations can be obtained by integrating the above equations with respect to time and by taking the ratios of $\Delta V_x/\Delta V_z$ and $\Delta V_y/\Delta V_z$ the dependence upon time drops out to result in:

$$\frac{\Delta V_x}{\Delta V_z} = \frac{-M_y}{I_z\,\omega_{zo}^2},$$

$$\frac{\Delta V_y}{\Delta V_z} = \frac{M_x}{I_z\omega_{zo}^2}, \qquad (22)$$

which are the approximate solutions for the average direction of $\Delta\vec{V}$ as applying constant moments to a near symmetric rigid body. Eqs. (22) can also be interpreted as the limiting direction of $\Delta\vec{V}$ as a result of doing a spin-up maneuver for a large time.

## IV. Numerical Simulations and Results

A computer simulation was set up using the Advanced Continuous Simulation Language, (ACSL), where Eqs. (1), (3), and (8) were integrated numerically with respect to time. The following values for the moments, inertia properties and initial conditions were input into the simulation and correspond to using the S2A spin thruster to spin up Galileo just prior to the Jupiter Orbit Insertion, (JOI), maneuver:

$M_x$ = −0.486 Nm
$M_y$ = −0.579 Nm
$M_z$ = 12.22 Nm

$I_x$ = 3640 kg $m^2$
$I_y$ = 3256 kg $m^2$
$I_z$ = 5371 kg $m^2$

$\omega_{z_0}$ = 0.306 $s^{-1}$

These values, when substituted into Eqs. (22), yield:

$\Delta V_x/\Delta V_z$ = 1.15 mrad

$\Delta V_y/\Delta V_z$ = −0.966 mrad

The following plots, in addition to Fig. (2), are the results of that simulation.

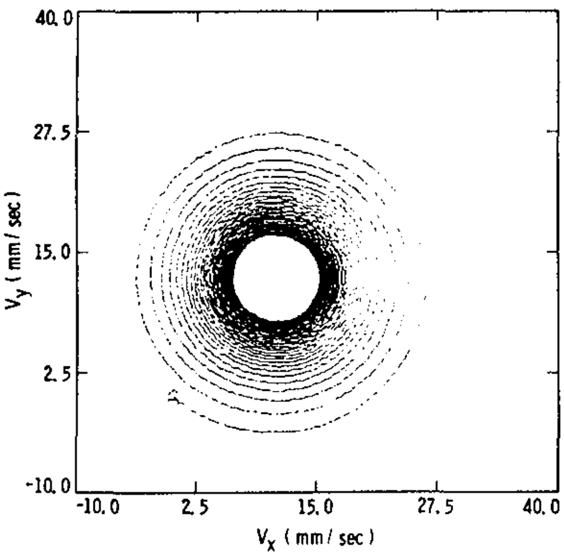


Fig. 3 Transverse delta-V's in inertial space during spin-up maneuver.

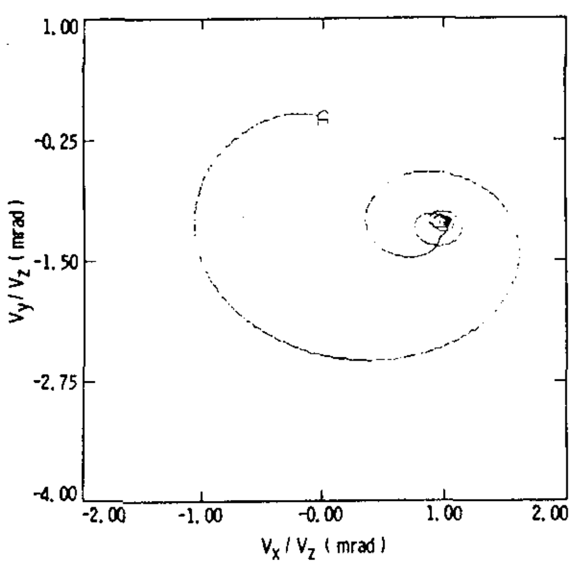


Fig. 4 Nominal orientation of the delta-V vector in inertial space during spin-up maneuver.

As was mentioned previously, Fig. (2) illustrates the pointing history of the H vector of Galileo during the pre-JOI spin-up maneuver. Fig. (3) illustrates that during this spin-up maneuver the lateral velocity of the spacecraft is bounded and Fig. (4) shows the pointing history of the ΔV vector and that it is converging upon the point mentioned above. It is interesting to note that the center of the center spiral in Fig. (2) is along the same direction as the spiral in Fig. (2) is along the same direction as the spiral center in Fig. (4). This confirms the previous assumption that the limiting direction of the ΔV vector as a result of doing a spin-up maneuver for large time coincides with the average H vector direction.

## V. Conclusions

Analytic expressions have been found for Euler's Equations of Motion and for the Eulerian Angles for both symmetric and near symmetric rigid bodies under the influence of arbitrary constant body-fixed torques. These solutions were then used to develop a secular solution for the change in velocity imparted to the rigid body as a result of these body-fixed torques and subsequently this secular solution was confirmed by a numerical time simulation.

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## Appendix 1

Solution for $\omega_x$ and $\omega_y$

$$\omega_x(t) = \omega_{xo}\cos\left(\frac{1}{2}\bar{a}t^2 + \bar{b}t\right) - \sqrt{\frac{\lambda_1}{\lambda_2}}\,\omega_{yo}\sin\left(\frac{1}{2}\bar{a}t^2 + \bar{b}t\right)$$

$$+ u\,\mathrm{sgn}\bar{a}\left[\frac{\tilde{S}}{\sqrt{|2\bar{a}|}}\left\{\frac{\lambda_1 d}{\sqrt{\lambda_1\lambda_2}}\cos\left[\left(\frac{1}{2}\bar{a}t^2 + \bar{b}t + \frac{\bar{b}^2}{2\bar{a}}\right)\mathrm{sgn}\bar{a}\right] + \frac{\bar{a}c}{|\bar{a}|}\sin\left[\left(\frac{1}{2}\bar{a}t^2 + \bar{b}t + \frac{\bar{b}^2}{2\bar{a}}\right)\mathrm{sgn}\bar{a}\right]\right\}\right.$$

$$\left. + \frac{\tilde{C}}{\sqrt{|2\bar{a}|}}\left\{\frac{\bar{a}c}{|\bar{a}|}\cos\left[\left(\frac{1}{2}\bar{a}t^2 + \bar{b}t + \frac{\bar{b}^2}{2\bar{a}}\right)\mathrm{sgn}\bar{a}\right] - \frac{\lambda_1 d}{\sqrt{\lambda_1\lambda_2}}\sin\left[\left(\frac{1}{2}\bar{a}t^2 + \bar{b}t + \frac{\bar{b}^2}{2\bar{a}}\right)\mathrm{sgn}\bar{a}\right]\right\}\right]$$

where

$$\tilde{C} = \int_{\frac{\bar{b}^2}{2\bar{a}}\mathrm{sgn}\bar{a}}^{\left(\frac{1}{2}\bar{a}t^2 + \bar{b}t + \frac{\bar{b}^2}{2\bar{a}}\right)\mathrm{sgn}\bar{a}} \frac{\cos s}{\sqrt{s}}\,ds,$$

$$\tilde{S} = \int_{\frac{\bar{b}^2}{2\bar{a}}\mathrm{sgn}\bar{a}}^{\left(\frac{1}{2}\bar{a}t^2 + \bar{b}t + \frac{\bar{b}^2}{2\bar{a}}\right)\mathrm{sgn}\bar{a}} \frac{\sin s}{\sqrt{s}}\,ds$$

$$\lambda_1 = \frac{I_z - I_y}{I_x}, \quad \lambda_2 = \frac{I_z - I_x}{I_y}$$

$$\bar{a} = \sqrt{\lambda_1\lambda_2}\,\frac{M_z}{I_z}, \quad \bar{b} = \sqrt{\lambda_1\lambda_2}\,\omega_{z0}$$

$$c = \frac{M_x}{I_x}, \quad d = \frac{M_y}{I_y}, \quad x = \omega_x, \quad y = \omega_y$$

and

$u = 1$ for pure spin up ($\bar{a}$ and $\bar{b}$ same sign)

$= -1$ for pure spin down ($\bar{a}$ and $\bar{b}$ opposite sign and $0 \le t \le -b/a$)

$\tilde{C}$ and $\tilde{S}$ can be written in terms of Fresnel integrals

$$\tilde{C} = \sqrt{2\pi}\left[C\left(\sqrt{\frac{2}{\pi}\left(\frac{\bar{a}t^2}{2} + \bar{b}t + \frac{\bar{b}^2}{2\bar{a}}\right)}\right) - C\left(\sqrt{\frac{\bar{b}^2}{\pi\bar{a}}}\right)\right]$$

$$\tilde{S} = \sqrt{2\pi}\left[S\left(\sqrt{\frac{2}{\pi}\left(\frac{\bar{a}t^2}{2} + \bar{b}t + \frac{\bar{b}^2}{2\bar{a}}\right)}\right) - S\left(\sqrt{\frac{\bar{b}^2}{\pi\bar{a}}}\right)\right]$$

where C and S are the Fresnel integrals

$$C(z) = \int_0^z \cos\left(\frac{\pi}{2}t^2\right)dt$$

$$S(z) = \int_0^z \sin\left(\frac{\pi}{2}t^2\right)dt$$

By writing the Fresnel integrals in terms of the f and g functions

$$C(x) = \frac{1}{2} + f(x)\sin\left(\frac{\pi}{2}x^2\right) - g(x)\cos\left(\frac{\pi}{2}x^2\right)$$

$$S(x) = \frac{1}{2} - f(x)\cos\left(\frac{\pi}{2}x^2\right) - g(x)\sin\left(\frac{\pi}{2}x^2\right)$$

the solution for $\omega_x$ can be simply written as

$$\omega_x(\xi) = k_{x1}c_1 + k_{x2}s_1 + k_{x3}c_2 + k_{x4}s_2 + k_{x5}f + k_{x6}g$$

where

$$k_{x1} = \omega_{x0}, \quad k_{x2} = -\sqrt{\frac{\lambda_1}{\lambda_2}}\,\omega_{y0}$$

$$k_{x3} = u\,\mathrm{sgn}\bar{a}\sqrt{\frac{\pi}{|\bar{a}|}}\cdot\left[\frac{\lambda_1 d}{\sqrt{\lambda_1\lambda_2}}\left(\frac{1}{2} - s_0\right) + \frac{\bar{a}c}{|\bar{a}|}\left(\frac{1}{2} - c_0\right)\right]$$

$$k_{x4} = u\ \mathrm{sgn}\bar{a}\ \sqrt{\frac{\pi}{|\bar{a}|}}$$

$$\left[\frac{\bar{a}c}{|\bar{a}|}\left(\frac{1}{2} - S_0\right) - \frac{\lambda_1 d}{\sqrt{\lambda_1\lambda_2}}\left(\frac{1}{2} - C_0\right)\right]$$

$$k_{x5} = u\ \mathrm{sgn}\bar{a}\ \sqrt{\frac{\pi}{|\bar{a}|}}\ \left[-\frac{\lambda_1 d}{\sqrt{\lambda_1\lambda_2}}\right],$$

$$k_{x6} = u\ \mathrm{sgn}\bar{a}\ \sqrt{\frac{\pi}{|\bar{a}|}}\ \left[-\frac{\bar{a}c}{|\bar{a}|}\right]$$

$$c_1 = \cos\xi, \quad s_1 = \sin\xi$$

$$c_2 = \cos\left[\left(\xi + \frac{\bar{b}^2}{2\bar{a}}\right)\mathrm{sgn}\bar{a}\right],$$

$$s_2 = \sin\left[\left(\xi + \frac{\bar{b}^2}{2\bar{a}}\right)\mathrm{sgn}\bar{a}\right]$$

$$C_0 = C\left(\sqrt{\frac{\bar{b}^2}{\pi\bar{a}}\ \mathrm{sgn}\bar{a}}\right)$$

$$S_0 = S\left(\sqrt{\frac{\bar{b}^2}{\pi\bar{a}}\ \mathrm{sgn}\bar{a}}\right)$$

$$g = g\left(\sqrt{\frac{2}{\pi}\left[\xi + \frac{\bar{b}^2}{2\bar{a}}\right]\mathrm{sgn}\bar{a}}\right)$$

$$f = f\left(\sqrt{\frac{2}{\pi}\left[\xi + \frac{\bar{b}^2}{2\bar{a}}\right]\mathrm{sgn}\bar{a}}\right)$$

The solution for $\omega_y$ is analogous to the solution for $\omega_x$ and can be found from the $\omega_x$ solution by replacing

$(\lambda_1/\lambda_2)^{1/2}$ with $(\lambda_2/\lambda_1)^{1/2}$,

$c$ with $d$,

$\omega_{xo}$ with $\omega_{yo}$,

$\omega_{yo}$ with $-\omega_{xo}$,

and $d$ with $-c$.

## Appendix 2

### Solution for $\phi_x$ and $\phi_y$

$$\phi_x(t) = \phi_{x0}\cos\left[\alpha\left(\frac{1}{2}\bar{a}t^2 + \bar{b}t\right)\right]$$

$$+ \phi_{y0}\sin\left[\alpha\left(\frac{1}{2}\bar{a}t^2 + \bar{b}t\right)\right]$$

$$+ u\ \mathrm{sgn}\bar{a}\left[W_{ys}\left(\frac{1}{2}\bar{a}t^2 + \bar{b}t\right) + W_{xc}\left(\frac{1}{2}\bar{a}t^2 + \bar{b}t\right)\right]$$

$$\phi_y(t) = \phi_{y0}\cos\left[\alpha\left(\frac{1}{2}\bar{a}t^2 + \bar{b}t\right)\right]$$

$$- \phi_{x0}\sin\left[\alpha\left(\frac{1}{2}\bar{a}t^2 + \bar{b}t\right)\right]$$

$$+ u\ \mathrm{sgn}\bar{a}\left[W_{yc}\left(\frac{1}{2}\bar{a}t^2 + \bar{b}t\right) - W_{xs}\left(\frac{1}{2}\bar{a}t^2 + \bar{b}t\right)\right]$$

where

$$\alpha = \frac{1}{\sqrt{\lambda_1\lambda_2}}$$

$$W_{ys}(\bar{\tau}) = \int_0^{\bar{\tau}} \frac{\omega_y(\xi)\sin(\alpha\bar{\tau} - \alpha\xi)}{\sqrt{\bar{b}^2 + 2\bar{a}\xi}}\,d\xi$$

$$W_{xc}(\bar{\tau}) = \int_0^{\bar{\tau}} \frac{\omega_x(\xi)\cos(\alpha\bar{\tau} - \alpha\xi)}{\sqrt{\bar{b}^2 + 2\bar{a}\xi}}\,d\xi$$

$$W_{yc}(\bar{\tau}) = \int_0^{\bar{\tau}} \frac{\omega_y(\xi)\cos(\alpha\bar{\tau} - \alpha\xi)}{\sqrt{\bar{b}^2 + 2\bar{a}\xi}}\,d\xi$$

$$W_{xs}(\bar{\tau}) = \int_0^{\bar{\tau}} \frac{\omega_x(\xi)\sin(\alpha\bar{\tau} - \alpha\xi)}{\sqrt{\bar{b}^2 + 2\bar{a}\xi}}\,d\xi$$

$$\bar{\tau} = (\lambda_1\lambda_2)^{1/2}\left[(at^2/2) + bt\right]$$

where

$$W_{ys}(\bar{\tau}) = k_{y1}J_{cs}(\bar{\tau}, 1, 0, -\alpha, \alpha\bar{\tau}) + k_{y2}J_{ss}(\bar{\tau}, 1, 0, -\alpha, \alpha\bar{\tau})$$

$$+ k_{y3}J_{cs}\left(\bar{\tau},\ \mathrm{sgn}\bar{a},\ \frac{\bar{b}^2}{2\bar{a}}\mathrm{sgn}\bar{a},\ -\alpha,\ \alpha\bar{\tau}\right)$$

$$+ k_{y4}J_{ss}\left(\bar{\tau},\ \mathrm{sgn}\bar{a},\ \frac{\bar{b}^2}{2\bar{a}}\mathrm{sgn}\bar{a},\ -\alpha,\ \alpha\bar{\tau}\right)$$

$$+ k_{y5}F_s(\bar{\tau}) + k_{y6}G_s(\bar{\tau})$$

$$W_{xc}(\bar{\tau}) = k_{x1}J_{cc}(\bar{\tau}, 1, 0, -\alpha, \alpha\bar{\tau}) + k_{x2}J_{cs}(\bar{\tau}, -\alpha, \alpha\bar{\tau}, 1, 0)$$

$$+ k_{x3}J_{cc}\left(\bar{\tau},\ \mathrm{sgn}\bar{a},\ \frac{\bar{b}^2}{2\bar{a}}\mathrm{sgn}\bar{a},\ -\alpha,\ \alpha\bar{\tau}\right)$$

$$+ k_{x4}J_{cs}\left(\bar{\tau},\ -\alpha,\ \alpha\bar{\tau},\ \mathrm{sgn}\bar{a},\ \frac{\bar{b}^2}{2\bar{a}}\mathrm{sgn}\bar{a}\right)$$

$$+ k_{x5}F_s(\bar{\tau}) + k_{x6}G_s(\bar{\tau})$$

where

$$J_{cc}(\bar{\tau}, k_1, k_2, k_3, k_4) = \int_0^{\bar{\tau}} \frac{\cos(k_1\xi + k_2)\cos(k_3\xi + k_4)}{\sqrt{\bar{b}^2 + 2\bar{a}\,\xi}}\, d\xi$$

$$J_{cs}(\bar{\tau}, k_1, k_2, k_3, k_4) = \int_0^{\bar{\tau}} \frac{\cos(k_1\xi + k_2)\sin(k_3\xi + k_4)}{\sqrt{\bar{b}^2 + 2\bar{a}\,\xi}}\, d\xi$$

$$J_{ss}(\bar{\tau}, k_1, k_2, k_3, k_4) = \int_0^{\tau} \frac{\sin(k_1\xi + k_2)\sin(k_3\xi + k_4)}{\sqrt{\bar{b}^2 + 2\bar{a}\,\xi}}\, d\xi$$

$$F_c(\bar{\tau}) = \int_0^{\bar{\tau}} \frac{f\left(\sqrt{\frac{2}{\pi}\left[\xi + \frac{\bar{b}^2}{2\bar{a}}\right]\mathrm{sgn}\bar{a}}\right)\cos(\alpha\bar{\tau} - \alpha\xi)}{\sqrt{\bar{b}^2 + 2\bar{a}\,\xi}}\, d\xi$$

$$F_s(\bar{\tau}) = \int_0^{\bar{\tau}} \frac{f\left(\sqrt{\frac{2}{\pi}\left[\xi + \frac{\bar{b}^2}{2\bar{a}}\right]\mathrm{sgn}\bar{a}}\right)\sin(\alpha\bar{\tau} - \alpha\xi)}{\sqrt{\bar{b}^2 + 2\bar{a}\,\xi}}\, d\xi$$

$$G_c(\bar{\tau}) = \int_0^{\bar{\tau}} \frac{g\left(\sqrt{\frac{2}{\pi}\left[\xi + \frac{\bar{b}^2}{2\bar{a}}\right]\mathrm{sgn}\bar{a}}\right)\cos(\alpha\bar{\tau} - \alpha\xi)}{\sqrt{\bar{b}^2 + 2\bar{a}\,\xi}}\, d\xi$$

$$G_s(\bar{\tau}) = \int_0^{\bar{\tau}} \frac{g\left(\sqrt{\frac{2}{\pi}\left[\xi + \frac{\bar{b}^2}{2\bar{a}}\right]\mathrm{sgn}\bar{a}}\right)\sin(\alpha\bar{\tau} - \alpha\xi)}{\sqrt{\bar{b}^2 + 2\bar{a}\,\xi}}\, d\xi$$

Define the integrals $L_c$ and $L_s$:

$$L_c(\bar{\tau}, h_1, h_2) = \frac{1}{2}\int_0^{\bar{\tau}} \frac{\cos(h_1\xi + h_2)}{\sqrt{\bar{b}^2 + 2\bar{a}\,\xi}}\, d\xi$$

$$L_s(\bar{\tau}, h_1, h_2) = \frac{1}{2}\int_0^{\bar{\tau}} \frac{\sin(h_1\xi + h_2)}{\sqrt{\bar{b}^2 + 2\bar{a}\,\xi}}\, d\xi$$

Then, by well-known trigonometric identities:

$$J_{cc}(\bar{\tau}, k_1, k_2, k_3, k_4) = L_c(\bar{\tau}, k_1 - k_3, k_2 - k_4) + L_c(\bar{\tau}, k_1 + k_3, k_2 + k_4)$$

$$J_{cs}(\bar{\tau}, k_1, k_2, k_3, k_4) = L_s(\bar{\tau}, k_1 + k_3, k_2 + k_4) - L_s(\bar{\tau}, k_1 - k_3, k_2 - k_4)$$

$$J_{ss}(\bar{\tau}, k_1, k_2, k_3, k_4) = L_c(\bar{\tau}, k_1 - k_3, k_2 - k_4) - L_c(\bar{\tau}, k_1 + k_3, k_2 + k_4)$$

The integrals $L_c$ and $L_s$ can be expressed solely in terms of Fresnel Integrals $C_2$ and $S_2$

$$L_c(\bar{\tau}, h_1, h_2) = \frac{1}{2h_1} \sqrt{\pi \left| \frac{h_1}{\bar{a}} \right|}$$

$$\cdot \ \mathrm{sgn} \frac{\bar{a}}{h_1} \left\{ \cos t_2 \left[ C_2(t_1) - C_2(t_0) \right] \right.$$

$$\left. - \sin t_2 \left[ S_2(t_1) - S_2(t_0) \right] \right\}$$

$$L_s(\bar{\tau}, h_1, h_2) = \frac{\sqrt{2\pi}}{2h_1} \sqrt{\left| \frac{h_1}{2\bar{a}} \right|}$$

$$\cdot \left\{ \cos t_2 \left[ S_2(t_1) - S_2(t_0) \right] \right.$$

$$\left. + \sin t_2 \left[ C_2(t) - C_2(t_0) \right] \right\} \tag{40}$$

where

$$t_0 = \frac{h_1 \bar{b}^2}{2\bar{a}} \ \mathrm{sgn} \frac{\bar{a}}{h_1}, \quad t_1 = h_1 \left( \bar{\tau} + \frac{\bar{b}^2}{2\bar{a}} \right) \mathrm{sgn} \frac{\bar{a}}{h_1},$$

$$t_2 = \left( h_2 - \frac{h_1 \bar{b}^2}{2\bar{a}} \right) \mathrm{sgn} \frac{\bar{a}}{h_1}$$

$$C_2(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \frac{\cos t}{\sqrt{t}} \, dt$$

$$S_2(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \frac{\sin t}{\sqrt{t}} \, dt$$

The $C_2$ and $S_2$ Fresnel integrals are related to C and S by a change of argument

$$C(x) = C_2\left(\frac{\pi}{2} x^2\right)$$

$$S(x) = S_2\left(\frac{\pi}{2} x^2\right)$$

Next, the unknown integrals $F_c$, $F_s$, $G_c$ and $G_s$ must be evaluated by asymptotic expansion since they cannot be expressed explicitly in terms of known functions. The asymptotic expansions of the f and g functions are

$$\pi z f(z) \sim 1 + \sum_{m=1}^{\infty} (-1)^m \frac{1 \cdot 3 \cdot \ldots (4m-1)}{(\pi z^2)^{2m}}$$

$$\pi z g(z) \sim \sum_{m=0}^{\infty} (-1)^m \frac{1 \cdot 3 \cdot \ldots (4m+1)}{(\pi z^2)^{2m+1}}$$

**Only two terms of the expansions** will be used

$$f(z) \approx \frac{1}{\pi z} - \frac{3}{\pi^3 z^5}$$

$$g(z) \approx \frac{1}{\pi^2 z^3} - \frac{15}{\pi^4 z^7}$$

to obtain

$$F_c(\bar{\tau}) \approx \frac{1}{2\sqrt{\pi |\bar{a}|}} \left\{ \left[ \cos u_1 \cos_1(u_0, u_1) \right. \right.$$

$$\left. + \sin u_1 \sin_1(u_0, u_1) \right] - 3 \left(\frac{\alpha}{2}\right)^2$$

$$\left. \cdot \left[ \cos u_1 \cos_3(u_0, u_1) = \sin u_1 \sin_3(u_0, u_1) \right] \right\}$$

$$F_s(\bar{\tau}) \approx \frac{1}{2\sqrt{\pi |\bar{a}|}} \left\{ \left[ \sin u_1 \cos_1(u_0, u_1) \right. \right.$$

$$\left. - \cos u_1 \sin_1(u_0, u_1) \right] - 3 \left(\frac{\alpha}{2}\right)^2$$

$$\left. \cdot \left[ \sin u_1 \cos_3(u_0, u_1) - \cos u_1 \sin_3(u_0, u_1) \right] \right\}$$

$$G_c(\bar{\tau}) \approx \frac{\alpha}{4\sqrt{\pi |\bar{a}|}} \left\{ \left[ \cos u_1 \cos_2(u_0, u_1) \right. \right.$$

$$\left. + \sin u_1 \sin_2(u_0, u_1) \right] - 15 \left(\frac{\alpha}{2}\right)^2$$

$$\left. \cdot \left[ \cos u_1 \cos_4(u_0, u_1) + \sin u_1 \sin_4(u_0, u_1) \right] \right\}$$

$$G_s(\bar{\tau}) \approx \frac{\alpha}{4\sqrt{\pi|\bar{a}|}} \Bigg\{ \Big[ \sin u_1 \cos_2(u_0, u_1) - \cos u_1 \sin_2(u_0, u_1) \Big] - 15 \left(\frac{\alpha}{2}\right)^2 \cdot \Big[ \sin u_1 \cos_4(u_0, u_1) - \cos u_1 \sin_4(u_0, u_1) \Big] \Bigg\}$$

where

$$u_1 = \alpha\bar{\tau} + \frac{\alpha\bar{b}^2}{2\bar{a}} , \quad u_0 = \frac{\alpha\bar{b}^2}{2\bar{a}} , \quad \alpha = \frac{1}{\sqrt{\lambda_1\lambda_2}}$$

and the definitions have been used

$$\sin_m(t_0, t_1) \equiv \int_{t_0}^{t_1} \frac{\sin s}{s^m} ds$$

$$\cos_m(t_0, t_1) \equiv \int_{t_0}^{t_1} \frac{\cos s}{s^m} ds$$

$\sin_m(t_0, t_1)$ and $\cos_m(t_0, t_1)$ can be reduced to $\sin_1(t_0, t_1)$ and $\cos_1(t_0, t_1)$ by repeated application of the formulas:

$$\cos_m(u_0, u_1) = \frac{1}{1-m}\left[u_1^{1-m} \cos u_1 - u_0^{1-m} \cos u_0\right] + \frac{1}{1-m} \sin_{m-1}(u_0, u_1)$$

$$\sin_m(u_0, u_1) = \frac{1}{1-m}\left[u_1^{1-m} \sin u_1 - u_0^{1-m} \sin u_0\right] - \frac{1}{1-m} \cos_{m-1}(u_0, u_1)$$

$\cos_1(u_0, u_1)$ and $\sin_1(u_0, u_1)$ are expressible in terms of the sine and cosine integrals

$$\cos_1(u_0, u_1) = C_i(u_1) - C_i(u_0)$$

$$\sin_1(u_0, u_1) = S_i(u_1) - S_i(u_0)$$

where

$$C_i(z) \equiv \gamma + \ln z + \int_0^z \frac{\cos t - 1}{t} dt$$

$$S_i(z) \equiv \int_0^z \frac{\sin t}{t} dt$$

The solution for $\phi_y$ is analogous to the solution for $\phi_x$ and can be found from the $\phi_x$ solution by replacing

$\phi_{xo}$ with $\phi_{yo}$

$\phi_{xo}'$ with $-\phi_{xo}$

$\omega_y$ with $-\omega_x$

$\omega_x$ with $\omega_y$

$W_{ys}$ with $-W_{xs}$

$W_{xc}$ with $W_{yc}$