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ERROR ANALYSIS OF ANALYTIC SOLUTIONS FOR SELF-EXCITED NEAR-SYMMETRIC RIGID BODIES: A NUMERICAL STUDY

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Abstract

To perform a complete and exhaustive error analysis by computer simulation is tedious, time consuming, and expensive, especially when several parameters must be perturbed (M_x, M_y, M_z, I_x, I_y, I_z). Providing heuristic and/or analytic formulas for the errors is far easier, more succinct, and much more elegant.

In this paper, we provide both an analytic error analysis of the analytic solutions for self-excited rigid bodies, and supporting simulation tests that help validate and calibrate the final error equations. These error equations give general guidelines for the accuracy and applicability of the analytic solutions for whatever parameters the user might encounter.

Several computer plots are provided to show the accuracy of a nominal case (using Galileo spacecraft parameters) and to provide a few cases of extreme deviation, demonstrating the use of the error equations.

In development of the error equations, simplified analytic solutions were derived for Eulerian rates, angles, and angular momentum vectors directly from the original, highly complicated analytic solutions. The compact result for the angular momentum vector is identical to the heuristic equations deduced from computer simulations published in a previous paper.

I. Introduction

In a series of articles since 1980, the authors have given approximate analytic and heuristic solutions for self-excited near symmetric rigid bodies. The analytic solution was first reported in 1980[1], which prompted a second paper that demonstrated the incorrectness of the general Bodewadt solution for the Eulerian angles.[2] Further study of the analytic solutions revealed a highly simplified and useful heuristic solution[3], particularly in the annihilation of the angular momentum vector drift.[4] Although particular cases have been studied, until now the accuracy of the solutions had not been tested.

In this paper we present the analytic error bounds for the analytic solutions and their applications. Furthermore, a simplified analytic solution for the Eulerian rates and angles and the simulation results are presented.

II. Euler's Equations

Euler's equations of motion of a rigid body with principal axes at the center of mass are

$$\left.\begin{aligned} M_x &= I_x\dot{\omega}_x + (I_z - I_y)\,\omega_y\omega_z \\ M_y &= I_y\dot{\omega}_y + (I_x - I_z)\,\omega_x\omega_z \\ M_z &= I_z\dot{\omega}_z + (I_y - I_x)\,\omega_x\omega_y \end{aligned}\right\} \quad (1)$$

Bodewadt obtained an exact analytic solution of these equations for a symmetric rigid body subject to constant arbitrary torques in 1952.[5,6] The solution allows for variable spin rate and is given in terms of the Fresnel integrals. Unfortunately, for near symmetric rigid bodies such as the Galileo spacecraft, this solution does not provide enough accuracy for performance analysis. This is especially true when the solution is integrated again in the kinematic equations to find the Eulerian angles. This problem can be circumvented by the following assumption for near symmetric bodies[1]

$$\omega_z \approx (M_z/I_z)\,t + \omega_{z0} \quad (2)$$

As shown in Appendix 1, the Eulerian rates may be expressed as follows:

$$\omega_x(\xi) = k_1c_1 + k_2s_2 + k_3c_2 + k_4s_2 + k_5f + k_6g \quad (3)$$

A simple solution for the Eulerian rates may be obtained by assuming asymptotic series expansions for the Fresnel integrals embedded in coefficients k_3 and k_4, and by approximating functions f and g by the first term of their respective asymptotic expansions[1]

$$f(z) = 1/\pi z \quad (4)$$

$$g(z) = 1/\pi^2 z^3 \quad (5)$$

$$S_0 = 1/2 - \cos(\pi z^2/2)/\pi z - \sin(\pi z^2/2)/\pi^2 z^3 \quad (6)$$

$$C_0 = 1/2 + \sin(\pi z^2/2)/\pi z - \cos(\pi z^2/2)/\pi^2 z^3 \quad (7)$$

where

$$z = \pi M_z/\left[\sqrt{\lambda_1\lambda_2}\;(I_z\omega_{z0}^2)\right] \ll 1$$

Substituting Equations (4) through (7) into Equation (3), and assuming zero initial rates results in the following simplified equation for the Eulerian rate:

$$\omega_x(t) = \left\{\sqrt{\lambda_1/\lambda_2}\; d\left[\cos(\sqrt{\lambda_1\lambda_2}\;\phi_z) - 1\right] + c\sin(\sqrt{\lambda_1\lambda_2}\;\phi_z)\right\}/\sqrt{\lambda_1\lambda_2}\,\omega_{z0} \quad (8)$$

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Similarly,

$$\omega_y(t) = \left\{-\sqrt{\lambda_2/\lambda_1}\ c\left[\cos\left(\sqrt{\lambda_1\lambda_2}\ \phi_z\right) - 1\right] + d\sin\left(\sqrt{\lambda_1\lambda_2}\ \phi_z\right)\right\}/\sqrt{\lambda_1\lambda_2}\omega_{z0} \quad (9)$$

When $I_x = I_y$, as in the Bodewadt solution, these solutions are exact for any spin change commencing from any initial spin rate, but when $I_x \approx I_y$, the approximation provides very useful, accurate solutions, particularly when ω_x and ω_y are small, which is usually the case for spin-stabilized spacecraft. For near symmetric cases, the neglected term $(I_y - I_x)\omega_x\omega_y$ introduces error into the ω_z solution. The error contribution may be found by integrating the term over the spin change maneuver period

$$\varepsilon_{\omega z} = \int\left[(I_x - I_y)\ \omega_x\omega_y/I_z\right]dt \quad (10)$$

An upper bound for the error is derived using the simplified solutions for ω_x and ω_y, given above. These expressions are substituted into Equation (10) resulting in

$$\varepsilon_{\omega z} = \int\left[(I_x - I_y)/I_z\lambda_1\lambda_2\omega_{z0}^2\right]\left\{-cd\cos\left(2\sqrt{\lambda_1\lambda_2}\ \phi_z\right) + 2cd\cos\left(\sqrt{\lambda_1\lambda_2}\ \phi_z\right) + \mathcal{E}\sin\left(2\sqrt{\lambda_1\lambda_2}\ \phi_z\right) - 2\mathcal{E}\sin\left(\sqrt{\lambda_1\lambda_2}\ \phi_z\right) - dc\right\}dt \quad (11)$$

where

$$\mathcal{E} = \left(\sqrt{\lambda_1/\lambda_2}\ d^2 - \sqrt{\lambda_2/\lambda_1}\ c^2\right)/2$$

Since all but one of the terms under the integral are sinusoidal, it is clear that for large values of t, product dc becomes the dominant factor. Hence, the model error for the Eulerian angular velocity is bounded by the following expression:

$$|\varepsilon_{\omega z}| \le dcT\left[(I_x - I_y)/I_z\lambda_1\lambda_2\omega_{z0}^2\right] \quad (12)$$

Substituting for the spin change period T in Equation (12) results in the desired error equation

$$|\varepsilon_{\omega z}| \le dc\ (\omega_{zf} - \omega_{z0})\ (I_x - I_y)/\lambda_1\lambda_2\omega_{z0}^2 M_z \quad (13)$$

III. Euler's Angles

The kinematic equations of motion, for a type 1:3-1-2 Euler angle rotation[7] are

$$\left.\begin{aligned}\dot\phi_x &= \omega_x\cos\phi_y + \omega_z\sin\phi_y\\ \dot\phi_y &= \omega_y - (\omega_z\cos\phi_y - \omega_x\sin\phi_y)\tan\phi_x\\ \dot\phi_z &= (\omega_z\cos\phi_y - \omega_x\sin\phi_y)\sec\phi_x\end{aligned}\right\} \quad (14)$$

An approximate solution for the Eulerian angles is presented in Appendix 2. The main restrictions in the solutions are that two of the Eulerian angles, ω_x and ω_y, must remain small so that the approximations

$$\sin\phi \approx 0, \quad \cos\phi \approx 1$$

hold, and that

$$M_z\ /\ (I_z\omega_{z0}^2) \ll 1 \quad (15)$$

This last requirement is not essential since it results from choosing the asymptotic series expansion. A series expansion in terms of ω_{z0} may be developed to circumvent this limitation. In that case, only the small angle restriction applies.

Following the same procedure and reasoning used for the simplified Eulerian rates, the simplified solutions for the Eulerian angles are obtained.

$$\phi_x = -\left[M_y\sin\phi_z + M_x(\cos\phi_z - 1)\right]/I_z\omega_{z0}^2 \quad (16)$$

$$\phi_y = -\left[M_x\sin\phi_z - M_y(\cos\phi_z - 1)\right]/I_z\omega_{z0}^2 \quad (17)$$

The following procedure is used to derive an upper bound on the magnitude of the above angles

$$|\phi_x^2 + \phi_y^2| = \left|(1/I_z\omega_{z0}^2)^2\left[2(M_x^2 + M_y^2) - 2\cos\phi_z\ (M_x^2 + M_y^2)\right]\right| \le 4(M_x^2 + M_y^2)/(I_z\omega_{z0}^2)^2 \quad (18)$$

Hence

$$|\phi|_{max} \le 2\sqrt{(M_x^2 + M_y^2)}/\ (I_z\omega_{z0}^2) \quad (19)$$

There are two sources of error for the symmetric rigid bodies. One source is the small angle approximation, which may be checked by limiting the size of the two Eulerian angles to a reasonable limit. An error which may be larger is caused by omitting the $\phi_y\omega_x$ from the kinematic equation describing ϕ_z angle

$$\varepsilon_{\phi zs} = \int\varepsilon_{\phi z}\ dt\ dt = \int(-\phi_y\omega_x)\ dt \quad (20)$$

An upper bound for the angular rates may be derived from Euler's equations of motions

$$\omega_x \le M_y/(I_x - I_z)\omega_z$$

$$\omega_y \le M_x/(I_z - I_y)\omega_z \quad (21)$$

Similarly, from Euler's kinematic equations

$$\phi_y \le M_y/I_z\omega_{z0}^2$$

Substitution of Equations (21) into Equation (20) results in the desired error term

$$\varepsilon_{\phi zs} \leq 2\left[M_y(\omega_{zf} - \omega_{z0}) \sqrt{(M_x^2 + M_y^2)}\right] / \left[M_z(I_z - I_x)\omega_{z0}^3\right] \tag{22}$$

In addition to the above error, the nonsymmetric model has an error contribution caused by the inaccuracy in modeling the Eulerian rates discussed

$$\varepsilon_{\phi zn} = \iint \varepsilon_{\omega z} \, dt \, dt = \iint \left[(I_y - I_x)\omega_x\omega_y / I_z\right] dt \, dt \tag{23}$$

Substituting Equations (21) into Equation (23) and integrating ω_z^{-2} over time results in the following error equation

$$\varepsilon_{\phi zn} \leq \left[(I_y - I_x) / (I_x - I_z)(I_z - I_y)\right] M_x M_y \left[(\omega_{zf} - \omega_{z0}) I_z / M_z^2 \omega_{z0}\right] \tag{24}$$

Total angular error may be defined as the root-sum square of the above two terms

$$\varepsilon_{\phi z}^2 \leq \varepsilon_{\phi zn}^2 + \varepsilon_{\phi zs}^2 \tag{25}$$

IV. Angular Momentum Vector

With the analytic results for the angular velocities, ω_x, ω_y, and ω_z and the Type 1:3-1-2 Euler angles, ϕ_x, ϕ_y and ϕ_z, the appropriate analytic solution for the angular momentum vector in the inertial space can be easily constructed

$$\overline{H} = A \, \overline{I\omega} \tag{26}$$

where

$$\overline{H} = \begin{bmatrix} H_x & H_y & H_z \end{bmatrix}^T$$

$$\overline{I\omega} = \begin{bmatrix} I_x\omega_x & I_y\omega_y & I_z\omega_z \end{bmatrix}^T$$

where A is the transformation matrix based on the Eulerian angles, and superscript T denotes the transpose of the matrix.

Using the small angle approximation, the A matrix may be defined as

$$A \approx \begin{bmatrix} \cos\phi_z & -\sin\phi_z & (\phi_y\cos\phi_z + \phi_x\sin\phi_z) \\ \sin\phi_z & \cos\phi_z & (\phi_y\sin\phi_z - \phi_x\cos\phi_z) \\ -\phi_y & \phi_x & 1 \end{bmatrix} \tag{27}$$

and the angular momentum vector as

$$\left.\begin{aligned} H_x &\approx I_x\omega_x\cos\phi_z - I_y\omega_y\sin\phi_z + I_z\omega_z(\phi_y\cos\phi_z + \phi_x\sin\phi_z) \\ H_y &\approx I_x\omega_x\sin\phi_z + I_y\omega_y\cos\phi_z + I_z\omega_z(\phi_y\sin\phi_z - \phi_x\cos\phi_z) \\ H_z &\approx I_z\omega_z \end{aligned}\right\} \tag{28}$$

Substituting for ϕ_x and ϕ_y from Equations (16) and (17) reveals the secular terms in the angular momentum vector

$$H_x \approx \left[I_x\omega_x + M_y\omega_z/\omega_{z0}^2\right]\cos\phi_z - M_y\omega_z/\omega_{z0}^2 - \left[I_y\omega_y + M_x\omega_z/\omega_{z0}^2\right]\sin\phi_z \tag{29}$$

$$H_y \approx \left[I_x\omega_x + M_y\omega_z/\omega_{z0}^2\right]\sin\phi_z + M_x\omega_z/\omega_{z0}^2 - \left[I_y\omega_y + M_x\omega_z/\omega_{z0}^2\right]\cos\phi_z \tag{30}$$

that result in the following angular momentum pointing

$$H_x/H_z = -M_y/I_z\omega_{z0}^2 \tag{31}$$

$$H_y/H_z = M_x/I_z\omega_{z0}^2 \tag{32}$$

and

$$\angle H = \sqrt{(H_x/H_z)^2 + (H_y/H_z)^2} = \sqrt{(M_x^2 + M_y^2)} / I_z\omega_{z0}^2 = (\phi_{max})/2 \tag{33}$$

In analyzing the error, it should be noted that since the transformation is exact, it does not contribute to the total pointing error, but inherent error embedded in the calculated values of the Eulerian rates and angles may cause unacceptably large errors in the calculated values of the angular momentum pointing. An error model may be derived by perturbing the Eulerian rates and angles

$$\hat{\omega} = \overline{\omega} + \overline{\varepsilon}_\omega$$

and

$$\hat{\phi} = \overline{\phi} + \overline{\varepsilon}_\phi \tag{34}$$

The perturbed values can then be substituted in Equation (22) to obtain the perturbed momentum vector

$$\hat{H} = (A + \varepsilon A)(\overline{I\omega} + \overline{I\varepsilon}_\omega) \approx \overline{H} + A\,\overline{I\varepsilon}_\omega + \varepsilon A\,\overline{I\omega} \tag{35}$$

The difference between the perturbed and the nominal H-vector is defined as the error vector

$$\overline{\epsilon H} = \hat{H} - \overline{H} = A\overline{I\epsilon_{\omega}} + \epsilon A\,\overline{I\omega} \tag{36}$$

The orientation of the angular momentum vector may be defined as follows:

$$\begin{aligned} |\epsilon(H_x/H_z)|^2 \le\ & \epsilon_{\phi x}^2 + \epsilon_{\phi y}^2 \\ & + \left[(I_x\epsilon_{\omega x})^2 + (I_y\epsilon_{\omega y})^2\right]/(I_z\omega_z)^2 \\ & + (\phi_{xmax}^2 + \phi_{xmax}^2)\ (\epsilon_{\omega z}/\omega_z)^2 \\ & + \Big\{(I_x\omega_x)^2 + (I_y\omega_y)^2\ /(I_z\omega_z)^2 \\ & + (\phi_{xmax}^2 + \phi_{xmax}^2)\Big\}\ |\epsilon_{\phi z}|^2 \end{aligned} \tag{37}$$

Similar results were found for the y component of the momentum vector.

The angular momentum vector error equation may be defined as the root-sum square of the pointing errors on x and y components

$$\begin{aligned} |\epsilon\angle H|^2 \le \Big\{ & 4\epsilon_{\phi xmax}^2 + 4\phi_{xmax}^2\ (\epsilon_{\omega z}/\omega_z)^2 \\ & + \left[(I_x\epsilon_{\omega x})^2 + (I_y\epsilon_{\omega y})^2\right]/(I_z\omega_z)^2 \\ & + \Big\{\left[(I_x\omega_x)^2 + (I_y\omega_y)^2\right]/(I_z\omega_z)^2 \\ & + 4\ \phi_{xmax}^2\Big\}\left|\epsilon_{\phi z}\right|^2\Big\} \end{aligned} \tag{38}$$

V. Results

The main restriction in deriving the analytic solutions was that the size of two Eulerian angles ϕ_x and ϕ_y should remain small. This implies that in order to obtain good analytical results, in addition to Equation 15, the following inequality should hold

$$\sqrt{(M_x^2 + M_y^2)}\ \ /\ \ (I_z\ \omega_{z0}^2) \le \delta \tag{39}$$

where δ is a small positive value. Equations (13), (25), and (38) define the error bounds for the simulation.

To test the results, it was decided to limit Equations (15) and (39) to 0.1. Furthermore, it was decided that the predicted angular momentum pointing error should not exceed the predicted pointing defined by Equation (33), that is

$$\frac{\epsilon\ \angle H}{\angle H} \le 1$$

As a nominal case, the Galileo spacecraft's physical parameters (see Table 1) were used in a computer simulation. The results shown in Figs. 1 through 5 show that for the nominal case, the analytical solutions are very close approximations of the exact numerical simulations. Following the guidelines set by this paper, similar simulations were performed for extreme transverse axis torques and extreme asymmetrical orientations on the spacecraft. These results are shown in Figs 6 through 11.

Simulation results in each case were compared to the error expressions and the result was tabulated (see Table 2). In general, the model is very accurate in predicting the behavior of the near-symmetric rigid bodies with transverse torques less than half the spin-axis torque. Because the error equations are extremely conservative as the body becomes more asymmetric or as the transverse torques approach the spin axis torque, the error model may not predict the good cases.

Table 1 Galileo spacecraft parameters and initial conditions

Parameters	Nominal values
I_x, Kg-m	2985
I_y, Kg-m	2729
I_z, Kg-m	4183
M_x, Kg-m	-0.4757
M_y, Kg-m	-0.5669
M_z, Kg-m	13.5
ω_{x0}, rad/s	0.0
ω_{y0}, rad/s	0.0
ω_{z0}, rad/s	0.306
ω_{zf}, rad/s	1.047
ϕ_{x0}, rad	0.0
ϕ_{y0}, rad	0.0
ϕ_{z0}, rad	0.0

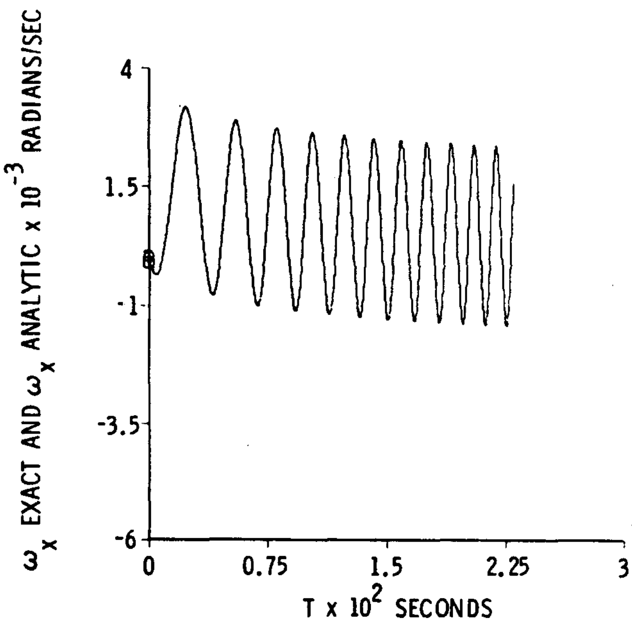

Fig. 1. Analytic and exact solution of Eulerian rate $\omega_x(t)$ for the nominal case

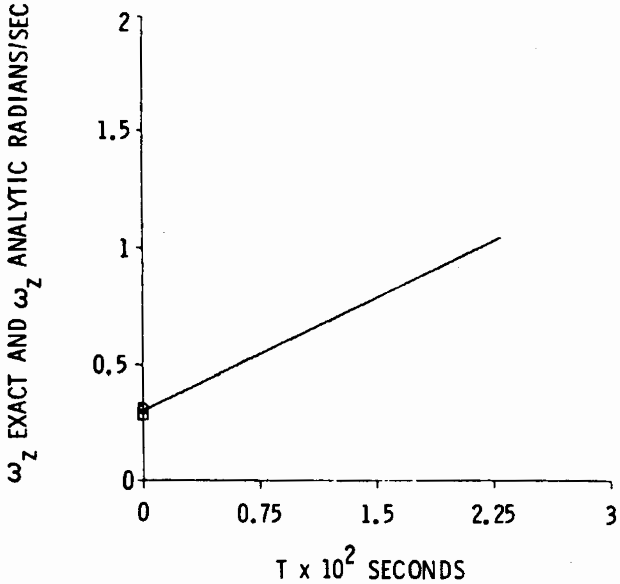

Fig. 2. Analytic and exact solution of Eulerian rate $\omega_z(t)$ for the nominal case

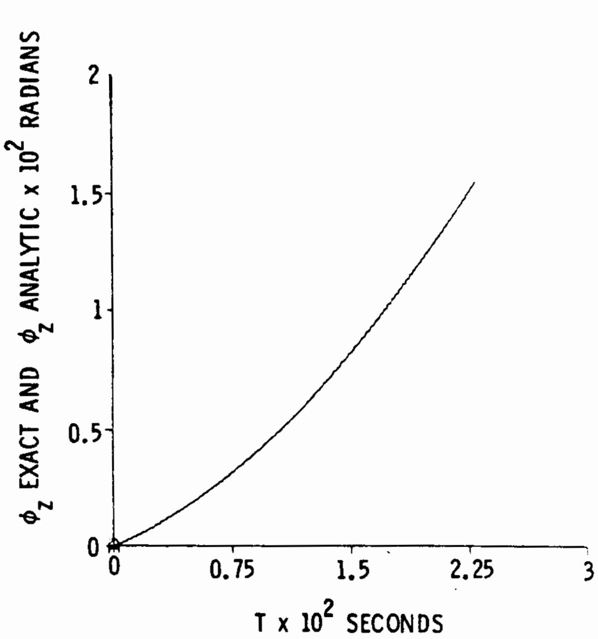

Fig. 4. Analytic and exact solution of Eulerian angle $\phi_z(t)$ for the nominal case

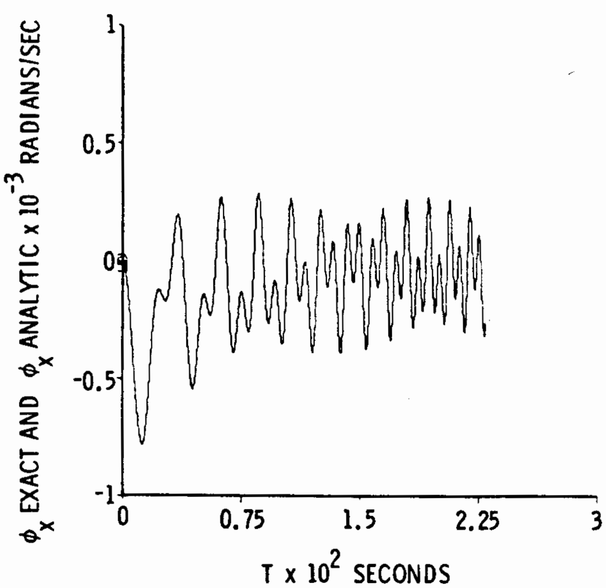

Fig. 3. Analytic and exact solution of Eulerian angle $\phi_x(t)$ for the nominal case

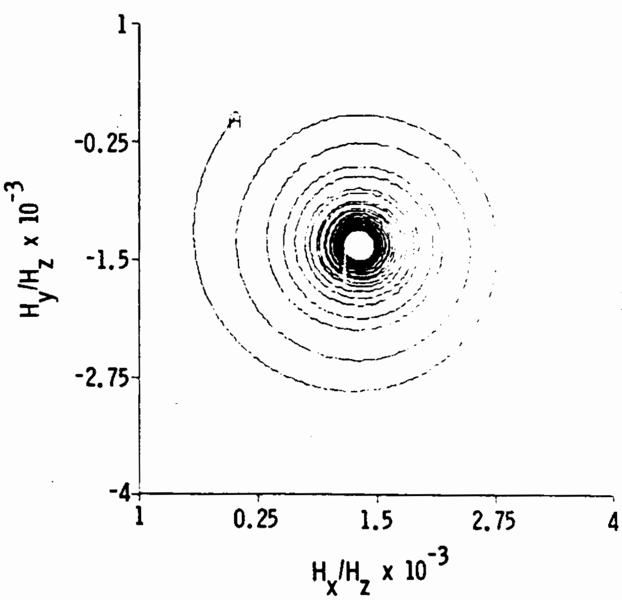

Fig. 5. Nominal orientation of the angular momentum vector in inertial space for the nominal case

VI. Conclusion

Simplified analytic expressions for the solution, the range of application and the error bounds for the analytic solutions of Euler's equations of motion and the Eulerian angles were found. These expressions were tested against the simulation results. The simplified solutions were very accurate for the near symmetric body.

The range and error bound expression may be used as fast and relatively accurate guidelines for rigid body spin change maneuvers analysis.

VII. Acknowledgments

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VIII. References

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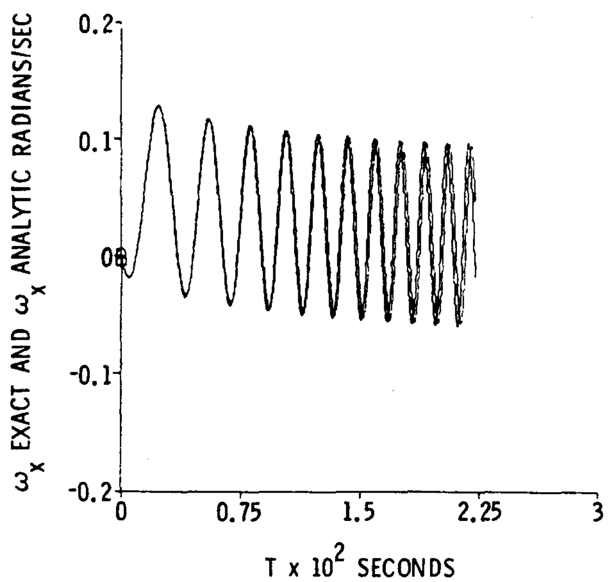

Fig. 6. Analytic and exact solution of Eulerian rate $\omega_x(t)$ for the near symmetric rigid body with large transverse torque, $M_x = M_y = -22$

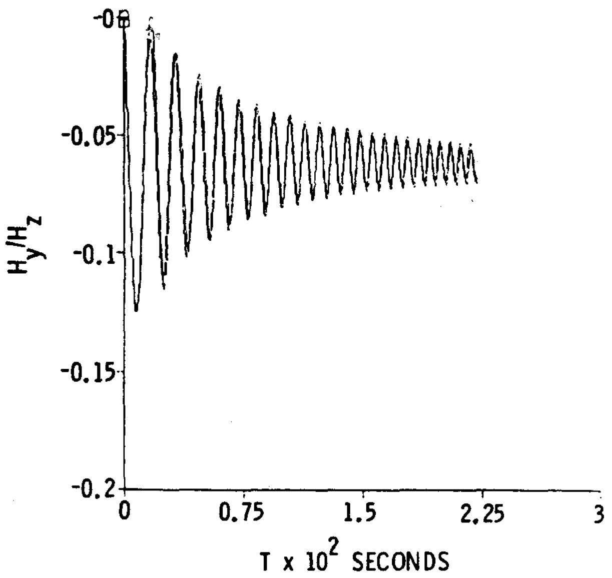

Fig. 8. Angular momentum vector orientation H_y/H_z for the near symmetric rigid body with large transverse torque, $M_x = M_y = -22$

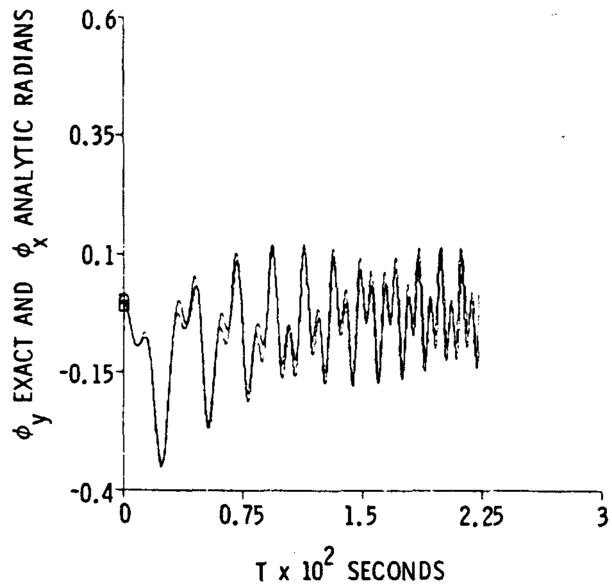

Fig. 7. Analytic and exact solution of Eulerian angle $\phi_y(t)$ for the near symmetric rigid body with large transverse torque, $M_x = M_y = -22$

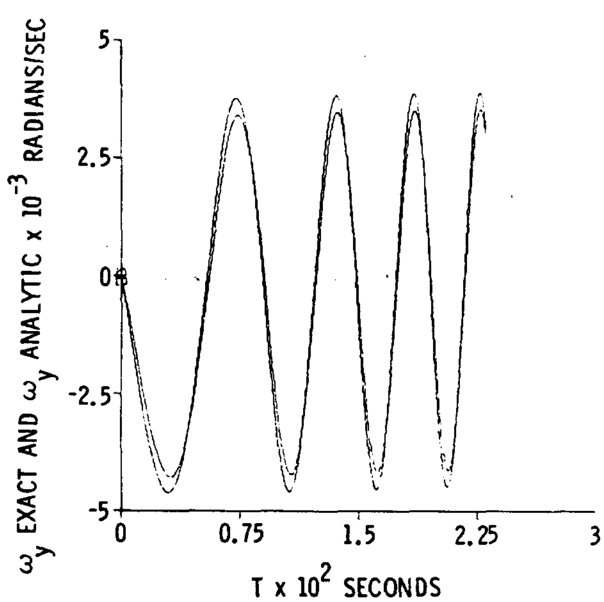

Fig. 9. Analytic and exact solution of Eulerian rate $\omega_y(t)$ for an asymmetric rigid body $I_x = 4000$

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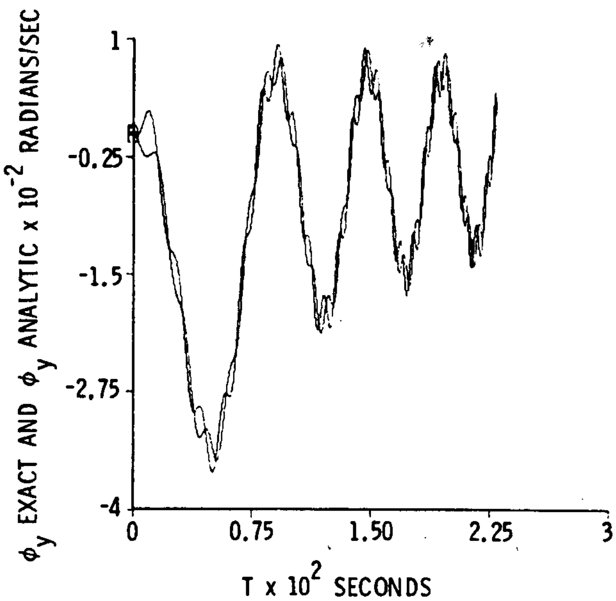

Fig. 10. Analytic and exact solution of Eulerian angle $\phi_y(t)$ for an asymmetric rigid body I_x = 4000

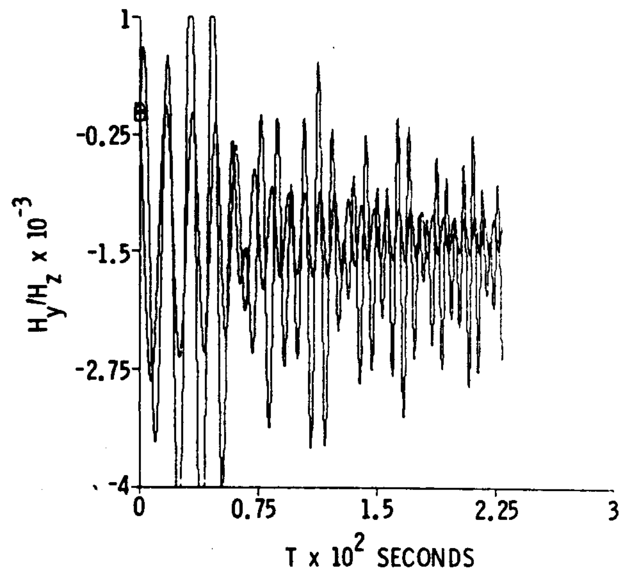

Fig. 11. Angular momentum orientation H_y/H_z for an asymmetric rigid body I_x = 4000

Table 2 Simulation results

Test case	Rate error, rad/s p/c[a]	Angle error, rad/s p/c[a]	Normalized pointing error, p/c[a]	Approximation success flag	Comments
Nominal	2.3E-5/1.5E-5	2.6E-3/4E-3	8E-3/1E-2	Succeeds	Very accurate results
$M_x = M_y$ = -5	2E-3/6.4E-4	0.23/0.034	0.695/0.012	Succeeds	Very accurate results
$M_x = M_t$ = -10	9E-3/2E-3	1.0/0.15	2.8/0.083	May fail	Borderline
$M_x = M_y$ = -20	0.034/0.012	3.7/0.74	11.08/0.28	Fails	Fails
Asymmetric I_x = 4000	7E-4/3E-4	0.07/0.03	1.22/1.85	May fail	Borderline

[a]p is the predicted, and c is the calculated value.

Appendix 1

Solution for ω_x and ω_y

$$\omega_x(t) = \omega_{xo} \cos\left(\frac{1}{2}\bar{a}t^2 + \bar{b}t\right)$$

$$-\sqrt{\frac{\lambda_1}{\lambda_2}}\,\omega_{yo} \sin\left(\frac{1}{2}\bar{a}t^2 + \bar{b}t\right)$$

$$+ u\,\mathrm{sgn}\bar{a}\left[\frac{\tilde{S}}{\sqrt{|2\bar{a}|}}\left\{\frac{\lambda_1 d}{\sqrt{\lambda_1\lambda_2}}\cos\left[\left(\frac{1}{2}\bar{a}t^2 + \bar{b}t + \frac{\bar{b}^2}{2\bar{a}}\right)\mathrm{sgn}\bar{a}\right]\right.\right.$$

$$\left.+ \frac{\bar{a}c}{|\bar{a}|}\sin\left[\left(\frac{1}{2}\bar{a}t^2 + \bar{b}t + \frac{\bar{b}^2}{2\bar{a}}\right)\mathrm{sgn}\bar{a}\right]\right\}$$

$$+ \frac{\tilde{C}}{\sqrt{|2\bar{a}|}}\left\{\frac{\bar{a}c}{|\bar{a}|}\cos\left[\left(\frac{1}{2}\bar{a}t^2 + \bar{b}t + \frac{\bar{b}^2}{2\bar{a}}\right)\mathrm{sgn}\bar{a}\right]\right.$$

$$\left.\left. - \frac{\lambda_1 d}{\sqrt{\lambda_1\lambda_2}}\sin\left[\left(\frac{1}{2}\bar{a}t^2 + \bar{b}t + \frac{\bar{b}^2}{2\bar{a}}\right)\mathrm{sgn}\bar{a}\right]\right\}\right]$$

where

$$\tilde{C} = \int_{\frac{\bar{b}^2}{2\bar{a}}\mathrm{sgn}\bar{a}}^{\left(\frac{1}{2}\bar{a}t^2 + \bar{b}t + \frac{\bar{b}^2}{2\bar{a}}\right)\mathrm{sgn}\bar{a}} \frac{\cos s}{\sqrt{s}}\,ds\,,$$

$$\tilde{S} = \int_{\frac{\bar{b}^2}{2\bar{a}}\mathrm{sgn}\bar{a}}^{\left(\frac{1}{2}\bar{a}t^2 + \bar{b}t + \frac{\bar{b}^2}{2\bar{a}}\right)\mathrm{sgn}\bar{a}} \frac{\sin s}{\sqrt{s}}\,ds$$

$$\lambda_1 = \frac{I_z - I_y}{I_x}\,, \quad \lambda_2 = \frac{I_z - I_x}{I_y}$$

$$\bar{a} = \sqrt{\lambda_1\lambda_2}\,\frac{M_z}{I_z}\,, \quad \bar{b} = \sqrt{\lambda_1\lambda_2}\,\omega_{z0}$$

$$c = \frac{M_x}{I_x}\,, \quad d = \frac{M_y}{I_y}\,, \quad x = \omega_x\,, \quad y = \omega_y$$

and

$u = 1$ for pure spin up ($\bar{a}$ and $\bar{b}$ same sign)

$= -1$ for pure spin down ($\bar{a}$ and $\bar{b}$ opposite sign and $0 \le t \le -b/\bar{a}$)

$\tilde{C}$ and $\tilde{S}$ can be written in terms of Fresnel integrals

$$\tilde{C} = \sqrt{2\pi}\left[C\left(\sqrt{\frac{2}{\pi}\left(\frac{\bar{a}t^2}{2} + \bar{b}t + \frac{\bar{b}^2}{2\bar{a}}\right)}\right) - C\left(\sqrt{\frac{\bar{b}^2}{\pi\bar{a}}}\right)\right]$$

$$\tilde{S} = \sqrt{2\pi}\left[S\left(\sqrt{\frac{2}{\pi}\left(\frac{\bar{a}t^2}{2} + \bar{b}t + \frac{\bar{b}^2}{2\bar{a}}\right)}\right) - S\left(\sqrt{\frac{\bar{b}^2}{\pi\bar{a}}}\right)\right]$$

where C and S are the Fresnel integrals

$$C(z) = \int_0^z \cos\left(\frac{\pi}{2}t^2\right)dt$$

$$S(z) = \int_0^z \sin\left(\frac{\pi}{2}t^2\right)dt$$

By writing the Fresnel integrals in terms of the f and g functions

$$C(x) = \frac{1}{2} + f(x)\sin\left(\frac{\pi}{2}x^2\right) - g(x)\cos\left(\frac{\pi}{2}x^2\right)$$

$$S(x) = \frac{1}{2} - f(x)\cos\left(\frac{\pi}{2}x^2\right) - g(x)\sin\left(\frac{\pi}{2}x^2\right)$$

the solution for ω_x can be simply written as

$$\omega_x(\xi) = k_{x1}c_1 + k_{x2}s_1 + k_{x3}c_2 + k_{x4}s_2 + k_{x5}f + k_{x6}g$$

where

$$k_{x1} = \omega_{x0}\,, \quad k_{x2} = -\sqrt{\frac{\lambda_1}{\lambda_2}}\,\omega_{y0}$$

$$k_{x3} = u\,\mathrm{sgn}\bar{a}\sqrt{\frac{\pi}{|\bar{a}|}}$$

$$\cdot\left[\frac{\lambda_1 d}{\sqrt{\lambda_1\lambda_2}}\left(\frac{1}{2} - s_0\right) + \frac{\bar{a}c}{|\bar{a}|}\left(\frac{1}{2} - c_0\right)\right]$$

$$k_{x4} = u\ \mathrm{sgn}\bar{a} \sqrt{\frac{\pi}{|\bar{a}|}}$$

$$\left[\frac{\bar{a}c}{|\bar{a}|}\left(\frac{1}{2} - S_0\right) - \frac{\lambda_1 d}{\sqrt{\lambda_1\lambda_2}}\left(\frac{1}{2} - C_0\right)\right]$$

$$k_{x5} = u\ \mathrm{sgn}\bar{a} \sqrt{\frac{\pi}{|\bar{a}|}} \left[-\frac{\lambda_1 d}{\sqrt{\lambda_1\lambda_2}}\right],$$

$$k_{x6} = u\ \mathrm{sgn}\bar{a} \sqrt{\frac{\pi}{|\bar{a}|}} \left[-\frac{\bar{a}c}{|\bar{a}|}\right]$$

$$c_1 = \cos\xi,\quad s_1 = \sin\xi$$

$$c_2 = \cos\left[\left(\xi + \frac{\bar{b}^2}{2\bar{a}}\right)\mathrm{sgn}\bar{a}\right],$$

$$s_2 = \sin\left[\left(\xi + \frac{\bar{b}^2}{2\bar{a}}\right)\mathrm{sgn}\bar{a}\right]$$

$$C_0 = C\left(\sqrt{\frac{\bar{b}^2}{\pi\bar{a}}\ \mathrm{sgn}\bar{a}}\right)$$

$$S_0 = S\left(\sqrt{\frac{\bar{b}^2}{\pi\bar{a}}\ \mathrm{sgn}\bar{a}}\right)$$

$$g = g\left(\sqrt{\frac{2}{\pi}\left[\xi + \frac{\bar{b}^2}{2\bar{a}}\right]\mathrm{sgn}\bar{a}}\right)$$

$$f = f\left(\sqrt{\frac{2}{\pi}\left[\xi + \frac{\bar{b}^2}{2\bar{a}}\right]\mathrm{sgn}\bar{a}}\right)$$

The solution for ω_y is analogous to the solution for ω_x and can be found from the ω_x solution by replacing

$(\lambda_1/\lambda_2)^{1/2}$ with $(\lambda_2/\lambda_1)^{1/2}$,

c with d,

ω_{xo} with ω_{yo},

ω_{yo} with $-\omega_{xo}$,

and d with $-c$.

Appendix 2

Solution for ϕ_x and ϕ_y

$$\phi_x(t) = \phi_{x0}\cos\left[\alpha\left(\frac{1}{2}\bar{a}t^2 + \bar{b}t\right)\right]$$

$$+ \phi_{y0}\sin\left[\alpha\left(\frac{1}{2}\bar{a}t^2 + \bar{b}t\right)\right]$$

$$+ u\ \mathrm{sgn}\bar{a}\left[W_{ys}\left(\frac{1}{2}\bar{a}t^2 + \bar{b}t\right) + W_{xc}\left(\frac{1}{2}\bar{a}t^2 + \bar{b}t\right)\right]$$

$$\phi_y(t) = \phi_{y0}\cos\left[\alpha\left(\frac{1}{2}\bar{a}t^2 + \bar{b}t\right)\right]$$

$$- \phi_{x0}\sin\left[\alpha\left(\frac{1}{2}\bar{a}t^2 + \bar{b}t\right)\right]$$

$$+ u\ \mathrm{sgn}\bar{a}\left[W_{yc}\left(\frac{1}{2}\bar{a}t^2 + \bar{b}t\right) - W_{xs}\left(\frac{1}{2}\bar{a}t^2 + \bar{b}t\right)\right]$$

where

$$\alpha = \frac{1}{\sqrt{\lambda_1\lambda_2}}$$

$$W_{ys}(\bar{\tau}) = \int_o^{\bar{\tau}} \frac{\omega_y(\xi)\sin(\alpha\bar{\tau} - \alpha\xi)}{\sqrt{\bar{b}^2 + 2\bar{a}\xi}}\,d\xi$$

$$W_{xc}(\bar{\tau}) = \int_o^{\bar{\tau}} \frac{\omega_x(\xi)\cos(\alpha\bar{\tau} - \alpha\xi)}{\sqrt{\bar{b}^2 + 2\bar{a}\xi}}\,d\xi$$

$$W_{yc}(\bar{\tau}) = \int_o^{\bar{\tau}} \frac{\omega_y(\xi)\cos(\alpha\bar{\tau} - \alpha\xi)}{\sqrt{\bar{b}^2 + 2\bar{a}\xi}}\,d\xi$$

$$W_{xs}(\bar{\tau}) = \int_o^{\bar{\tau}} \frac{\omega_x(\xi)\sin(\alpha\bar{\tau} - \alpha\xi)}{\sqrt{\bar{b}^2 + 2\bar{a}\xi}}\,d\xi$$

$$\bar{\tau} = (\lambda_1\lambda_2)^{1/2}[(at^2/2)+bt]$$

where

$$W_{ys}(\bar{\tau}) = k_{y1}J_{cs}(\bar{\tau}, 1, 0, -\alpha, \alpha\bar{\tau}) + k_{y2}J_{ss}(\bar{\tau}, 1, 0, -\alpha, \alpha\bar{\tau})$$

$$+ k_{y3}J_{cs}\left(\bar{\tau}, \text{sgn}\bar{a}, \frac{\bar{b}^2}{2\bar{a}}\text{sgn}\bar{a}, -\alpha, \alpha\bar{\tau}\right)$$

$$+ k_{y4}J_{ss}\left(\bar{\tau}, \text{sgn}\bar{a}, \frac{\bar{b}^2}{2\bar{a}}\text{sgn}\bar{a}, -\alpha, \alpha\bar{\tau}\right)$$

$$+ k_{y5}F_s(\bar{\tau}) + k_{y6}G_s(\bar{\tau})$$

$$W_{xc}(\bar{\tau}) = k_{x1}J_{cc}(\bar{\tau}, 1, 0, -\alpha, \alpha\bar{\tau}) + k_{x2}J_{cs}(\bar{\tau}, -\alpha, \alpha\bar{\tau}, 1, 0)$$

$$+ k_{x3}J_{cc}\left(\bar{\tau}, \text{sgn}\bar{a}, \frac{\bar{b}^2}{2\bar{a}}\text{sgn}\bar{a}, -\alpha, \alpha\bar{\tau}\right)$$

$$+ k_{x4}J_{cs}\left(\bar{\tau}, -\alpha, \alpha\bar{\tau}, \text{sgn}\bar{a}, \frac{\bar{b}^2}{2\bar{a}}\text{sgn}\bar{a}\right)$$

$$+ k_{x5}F_s(\bar{\tau}) + k_{x6}G_s(\bar{\tau})$$

where

$$J_{cc}(\bar{\tau}, k_1, k_2, k_3, k_4) = \int_0^{\bar{\tau}} \frac{\cos(k_1\xi + k_2)\cos(k_3\xi + k_4)}{\sqrt{\bar{b}^2 + 2\bar{a}\,\xi}}\, d\xi$$

$$J_{cs}(\bar{\tau}, k_1, k_2, k_3, k_4) = \int_0^{\bar{\tau}} \frac{\cos(k_1\xi + k_2)\sin(k_3\xi + k_4)}{\sqrt{\bar{b}^2 + 2\bar{a}\,\xi}}\, d\xi$$

$$J_{ss}(\bar{\tau}, k_1, k_2, k_3, k_4) = \int_0^{\tau} \frac{\sin(k_1\xi + k_2)\sin(k_3\xi + k_4)}{\sqrt{\bar{b}^2 + 2\bar{a}\,\xi}}\, d\xi$$

$$F_c(\bar{\tau}) = \int_0^{\bar{\tau}} \frac{f\left(\sqrt{\frac{2}{\pi}\left[\xi + \frac{\bar{b}^2}{2\bar{a}}\right]\text{sgn}\bar{a}}\right)\cos(\alpha\bar{\tau} - \alpha\xi)}{\sqrt{\bar{b}^2 + 2\bar{a}\,\xi}}\, d\xi$$

$$F_s(\bar{\tau}) = \int_0^{\bar{\tau}} \frac{f\left(\sqrt{\frac{2}{\pi}\left[\xi + \frac{\bar{b}^2}{2\bar{a}}\right]\text{sgn}\bar{a}}\right)\sin(\alpha\bar{\tau} - \alpha\xi)}{\sqrt{\bar{b}^2 + 2\bar{a}\,\xi}}\, d\xi$$

$$G_c(\bar{\tau}) = \int_0^{\bar{\tau}} \frac{g\left(\sqrt{\frac{2}{\pi}\left[\xi + \frac{\bar{b}^2}{2\bar{a}}\right]\text{sgn}\bar{a}}\right)\cos(\alpha\bar{\tau} - \alpha\xi)}{\sqrt{\bar{b}^2 + 2\bar{a}\,\xi}}\, d\xi$$

$$G_s(\bar{\tau}) = \int_0^{\bar{\tau}} \frac{g\left(\sqrt{\frac{2}{\pi}\left[\xi + \frac{\bar{b}^2}{2\bar{a}}\right]\text{sgn}\bar{a}}\right)\sin(\alpha\bar{\tau} - \alpha\xi)}{\sqrt{\bar{b}^2 + 2\bar{a}\,\xi}}\, d\xi$$

Define the integrals L_c and L_s:

$$L_c(\bar{\tau}, h_1, h_2) = \frac{1}{2}\int_0^{\bar{\tau}} \frac{\cos(h_1\xi + h_2)}{\sqrt{\bar{b}^2 + 2\bar{a}\,\xi}}\, d\xi$$

$$L_s(\bar{\tau}, h_1, h_2) = \frac{1}{2}\int_0^{\bar{\tau}} \frac{\sin(h_1\xi + h_2)}{\sqrt{\bar{b}^2 + 2\bar{a}\,\xi}}\, d\xi$$

Then, by well-known trigonometric identities:

$$J_{cc}(\bar{\tau}, k_1, k_2, k_3, k_4) = L_c(\bar{\tau}, k_1 - k_3, k_2 - k_4) + L_c(\bar{\tau}, k_1 + k_3, k_2 + k_4)$$

$$J_{cs}(\bar{\tau}, k_1, k_2, k_3, k_4) = L_s(\bar{\tau}, k_1 + k_3, k_2 + k_4) - L_s(\bar{\tau}, k_1 - k_3, k_2 - k_4)$$

$$J_{ss}(\bar{\tau}, k_1, k_2, k_3, k_4) = L_c(\bar{\tau}, k_1 - k_3, k_2 - k_4) - L_c(\bar{\tau}, k_1 + k_3, k_2 + k_4)$$

The integrals L_c and L_s can be expressed solely in terms of Fresnel Integrals C_2 and S_2

$$L_c(\bar{\tau}, h_1, h_2) = \frac{1}{2h_1} \sqrt{\pi \left| \frac{h_1}{\bar{a}} \right|} \cdot \operatorname{sgn} \frac{\bar{a}}{h_1} \left\{ \cos t_2 \left[C_2(t_1) - C_2(t_0) \right] - \sin t_2 \left[S_2(t_1) - S_2(t_0) \right] \right\} ,$$

$$L_s(\bar{\tau}, h_1, h_2) = \frac{\sqrt{2\pi}}{2h_1} \sqrt{\left| \frac{h_1}{2\bar{a}} \right|} \cdot \left\{ \cos t_2 \left[S_2(t_1) - S_2(t_0) \right] + \sin t_2 \left[C_2(t) - C_2(t_0) \right] \right\} \tag{40}$$

where

$$t_0 = \frac{h_1 \bar{b}^2}{2\bar{a}} \operatorname{sgn} \frac{\bar{a}}{h_1} , \quad t_1 = h_1 \left(\bar{\tau} + \frac{\bar{b}^2}{2\bar{a}} \right) \operatorname{sgn} \frac{\bar{a}}{h_1} ,$$

$$t_2 = \left(h_2 - \frac{h_1 \bar{b}^2}{2\bar{a}} \right) \operatorname{sgn} \frac{\bar{a}}{h_1}$$

$$C_2(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \frac{\cos t}{\sqrt{t}} \, dt$$

$$S_2(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \frac{\sin t}{\sqrt{t}} \, dt$$

The C_2 and S_2 Fresnel integrals are related to C and S by a change of argument

$$C(x) = C_2\left(\frac{\pi}{2} x^2\right)$$

$$S(x) = S_2\left(\frac{\pi}{2} x^2\right)$$

Next, the unknown integrals F_c, F_s, G_c and G_s must be evaluated by asymptotic expansion since they cannot be expressed explicitly in terms of known functions. The asymptotic expansions of the f and g functions are

$$\pi z f(z) \sim 1 + \sum_{m=1}^{\infty} (-1)^m \frac{1 \cdot 3 \cdot \ldots (4m-1)}{(\pi z^2)^{2m}}$$

$$\pi z g(z) \sim \sum_{m=0}^{\infty} (-1)^m \frac{1 \cdot 3 \cdot \ldots (4m+1)}{(\pi z^2)^{2m+1}}$$

Only two terms of the **expansions** will be used

$$f(z) \approx \frac{1}{\pi z} - \frac{3}{\pi^3 z^5}$$

$$g(z) \approx \frac{1}{\pi^2 z^3} - \frac{15}{\pi^4 z^7}$$

to obtain

$$F_c(\bar{\tau}) \approx \frac{1}{2\sqrt{\pi |\bar{a}|}} \left\{ \left[\cos u_1 \cos_1(u_0, u_1) + \sin u_1 \sin_1(u_0, u_1) \right] - 3 \left(\frac{\alpha}{2}\right)^2 \cdot \left[\cos u_1 \cos_3(u_0, u_1) + \sin u_1 \sin_3(u_0, u_1) \right] \right\}$$

$$F_s(\bar{\tau}) \approx \frac{1}{2\sqrt{\pi |\bar{a}|}} \left\{ \left[\sin u_1 \cos_1(u_0, u_1) - \cos u_1 \sin_1(u_0, u_1) \right] - 3 \left(\frac{\alpha}{2}\right)^2 \cdot \left[\sin u_1 \cos_3(u_0, u_1) - \cos u_1 \sin_3(u_0, u_1) \right] \right\}$$

$$G_c(\bar{\tau}) \approx \frac{\alpha}{4\sqrt{\pi |\bar{a}|}} \left\{ \left[\cos u_1 \cos_2(u_0, u_1) + \sin u_1 \sin_2(u_0, u_1) \right] - 15 \left(\frac{\alpha}{2}\right)^2 \cdot \left[\cos u_1 \cos_4(u_0, u_1) + \sin u_1 \sin_4(u_0, u_1) \right] \right\}$$

$$G_s(\bar{\tau}) \approx \frac{\alpha}{4\sqrt{\pi|\bar{a}|}} \left\{ \left[\sin u_1 \cos_2(u_0, u_1) - \cos u_1 \sin_2(u_0, u_1) \right] - 15 \left(\frac{\alpha}{2}\right)^2 \cdot \left[\sin u_1 \cos_4(u_0, u_1) - \cos u_1 \sin_4(u_0, u_1) \right] \right\}$$

where

$$u_1 = \alpha\bar{\tau} + \frac{\alpha\bar{b}^2}{2\bar{a}}, \quad u_0 = \frac{\alpha\bar{b}^2}{2\bar{a}}, \quad \alpha = \frac{1}{\sqrt{\lambda_1\lambda_2}}$$

and the definitions have been used

$$\sin_m(t_0, t_1) \equiv \int_{t_0}^{t_1} \frac{\sin s}{s^m}\, ds$$

$$\cos_m(t_0, t_1) \equiv \int_{t_0}^{t_1} \frac{\cos s}{s^m}\, ds$$

$\sin_m(t_0, t_1)$ and $\cos_m(t_0, t_1)$ can be reduced to $\sin_1(t_0, t_1)$ and $\cos_1(t_0, t_1)$ by repeated application of the formulas:

$$\cos_m(u_0, u_1) = \frac{1}{1-m}\left[u_1^{1-m}\cos u_1 - u_0^{1-m}\cos u_0\right] + \frac{1}{1-m}\sin_{m-1}(u_0, u_1)$$

$$\sin_m(u_0, u_1) = \frac{1}{1-m}\left[u_1^{1-m}\sin u_1 - u_0^{1-m}\sin u_0\right] - \frac{1}{1-m}\cos_{m-1}(u_0, u_1)$$

$\cos_1(u_0, u_1)$ and $\sin_1(u_0, u_1)$ are expressible in terms of the sine and cosine integrals

$$\cos_1(u_0, u_1) = C_i(u_1) - C_i(u_0)$$

$$\sin_1(u_0, u_1) = S_i(u_1) - S_i(u_0)$$

where

$$C_i(z) \equiv \gamma + \ln z + \int_0^z \frac{\cos t - 1}{t}\, dt$$

$$S_i(z) \equiv \int_0^z \frac{\sin t}{t}\, dt$$

The solution for ϕ_y is analogous to the solution for ϕ_x and can be found from the ϕ_x solution by replacing

ϕ_{xo} with ϕ_{yo}

ϕ_{xo} with $-\phi_{xo}$

ω_y with $-\omega_x$

ω_x with ω_y

W_{ys} with $-W_{xs}$

W_{xc} with W_{yc}