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SOLUTION OF EULER'S EQUATIONS OF MOTION AND EULERIAN ANGLES FOR NEAR SYMMETRIC RIGID BODIES SUBJECT TO CONSTANT MOMENTS*

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Abstract

Analytic expressions are found for Euler's Equations of Motion and for the Eulerian Angles for both symmetric and near symmetric rigid bodies under the influence of arbitrary constant body-fixed torques. These solutions provide the body-fixed angular velocities and the attitude of the body, respectively, as functions of time. They are of special interest in applications to spinning spacecraft (such as the Galileo Spacecraft to be launched in 1984) because they include the effect of time-varying spin rate. Thus they can be applied to spin-up and spin-down maneuvers as well as to error analysis for thruster misalignments. The solutions are given for arbitrary initial conditions in terms of Fresnel, Sine and Cosine Integrals. Numerical integration of the governing differential equations has verified that the approximate analytic solutions are very accurate in many physical situations of interest.

I. Introduction

The rotational motion of a rigid body is governed by Euler's equations of motion which are, in general, nonlinear. When a solution of these equations can be found it provides the body fixed angular velocities $\omega_x(t)$, $\omega_y(t)$ and $\omega_z(t)$. In order to determine the attitude of the body as a function of time, a second set of nonlinear differential equations involving the particular Eulerian angles chosen and $\omega_x(t)$, $\omega_y(t)$ and $\omega_z(t)$ must also be solved.

The main purpose in solving these equations is found in their applications to satellites and deep space probes. Even though numerical solutions are easily found by computer simulations, analytic solutions can provide deeper insight and understanding, and can be used in obtaining quick solutions, error analyses and compact algorithms for onboard computations. In space applications, certain simplifying, yet realistic assumptions can be made so that valuable approximate analytic solutions can be found.

In the first step, Euler's equations of motion are solved for near symmetric rigid bodies subject to constant moments. The assumption of near symmetry allows the heuristic assumption that $\omega_z(t)$ varies linearly with time to be used to convert the nonlinear differential equations into linear differential equations with time varying coefficients. By a proper change of the independent variable the equations are transformed into linear differential equations with constant coefficients, but with time varying forcing functions. From the impulse response, a particular solution can be generated in terms of Fresnel Integrals. The particular solution plus the easily obtained homogeneous solution provide the total heuristic solution. When the body is symmetric this becomes an exact analytic solution of $\omega_x(t)$, $\omega_y(t)$ and $\omega_z(t)$.

In the second step the Eulerian Angles are solved to determine the attitude of the body. Although a particular set of Eulerian Angles are defined in the text (the Type 1: 3-1-2 Eulerian Angles), the method can be applied to others. At this point, the most limiting assumption must be made: the two angles defining the direction of the spin axis must be small. Thus, the body must be initially spinning about the z axis or the torque about the z axis must be large relative to the other two torques. This assumption applies to many actual spacecraft and allows the nonlinear differential equations to be reduced to linear differential equations with time varying coefficients. Changing the independent variable by the same transformation used in the Euler equations of motion results in a set of linear differential equations with constant coefficients and time varying forcing functions. By expressing the Fresnel Integrals, which appear in the forcing functions, in terms of auxiliary functions, the impulse response can be obtained by series and asymptotic expansions. Thus, an approximate analytic solution is found for the Eulerian Angles.

In order to test the accuracy of these approximate analytic solutions, the ACSL (Advanced Continuous Simulation Language) was used to numerically integrate the governing differential equations and compare with the results given by the analytic solutions. Specific parameters were taken from the Galileo spacecraft, which is a dual spinner scheduled to be launched for an orbital mission of Jupiter in 1984. A number of important maneuvers occur when the stator and rotor are locked together (when the Galileo acts as a single spinner) so that the approximate analytic solution can be extremely useful if it is accurate. The computer simulations reveal that the heuristic solutions are very accurate in describing the rotational motion of the spacecraft. Hence, these solutions which have not been previously published, will find important applications not only in the maneuver analysis of the Galileo spacecraft, but also in the analysis of many other satellites and deep space probes of the future.

II. Euler's Equations of Motion

The rotational equations of motion of a rigid body with principal axes at the center of mass are

$$M_x = I_x \dot{\omega}_x + (I_z - I_y)\, \omega_y\, \omega_z$$

$$M_y = I_y \dot{\omega}_y + (I_x - I_z)\, \omega_z\, \omega_x \qquad (1)$$

$$M_z = I_z \dot{\omega}_z + (I_y - I_x)\, \omega_x\, \omega_y$$

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These are known as Euler's equations of motion. They are nonlinear in general and considered to be intractable. In the cases where a solution exists it should be noted that the time integrals of ω_x, ω_y, and ω_z do not provide any physical angles which would specify the orientation of the body. To obtain the orientation the differential equations specifying the Eulerian Angle rates in terms of ω_x, ω_y and ω_z are integrated. In this paper, Eqs. (1) are integrated for a special case corresponding to the spin up and spin down maneuvers of the Galileo Spacecraft. The solution of the Eulerian Angles is discussed in Section V.

III. Integration of Euler's Equations of Motion for Near Symmetric Body

A heuristic solution of Eqs. (1) is available by assuming

$$\omega_z = \frac{M_z}{I_z} t + \omega_{zo} \tag{2}$$

Computer simulations have indicated that Eq. (2) provides a very accurate approximate solution when

$$I_x \approx I_y \tag{3}$$

This is especially true for the spin up maneuver of the Galileo Spacecraft since ω_x and ω_y are both small (of the order 10^{-2}) and oscillate about a mean near zero. In this case the discrepancy between the approximation for ω_z and the exact solution is in the fourth or fifth digit.

Substituting Eq. (2) in Eqs. (1) gives

$$\begin{aligned} \dot{\omega}_x &= \frac{M_x}{I_x} - \frac{(I_z - I_y)}{I_x} \omega_y \left(\frac{M_z}{I_z} t + \omega_{zo}\right) \\ \dot{\omega}_y &= \frac{M_y}{I_y} - \frac{(I_x - I_z)}{I_y} \omega_x \left(\frac{M_z}{I_z} t + \omega_{zo}\right) \end{aligned} \tag{4}$$

Let

$$\lambda_1 = \frac{I_z - I_y}{I_x}$$

$$\lambda_2 = \frac{I_z - I_x}{I_y}$$

$$a = \frac{M_z}{I_z}$$

$$b = \omega_{zo}$$

$$c = \frac{M_x}{I_x}$$

$$d = \frac{M_y}{I_y}$$

$$\begin{aligned} x &= \omega_x \\ y &= \omega_y \end{aligned} \tag{5}$$

Then Eqs. (4) are written as

$$\begin{aligned} \dot{x} + \lambda_1 (at + b) y &= c \\ \dot{y} - \lambda_2 (at + b) x &= d \end{aligned} \tag{6}$$

Note that Eqs. (6) are coupled linear differential equations with time varying coefficients and constant forcing functions. The time varying coefficients can be removed by an appropriate change of the independent variable.

$$\tau = \frac{1}{2} at^2 + bt \tag{7}$$

Eq. (7) is used to put Eqs. (6) in the following form

$$\begin{aligned} \frac{dx}{d\tau} + \lambda_1 y &= \frac{c}{at + b} \\ \frac{dy}{d\tau} - \lambda_2 x &= \frac{c}{at + b} \end{aligned} \tag{8}$$

Now the original independent variable t in Eqs. (8) must be written in terms of the new variable τ. Solution of the quadratic equation from Eq. (7) gives

$$t = \frac{-b \pm \sqrt{b^2 + 2a\tau}}{a} \tag{9}$$

Great care must be exercised in choosing the correct root when the signs of a and b are arbitrary. In the case of a and b both positive, the positive sign on the square root term always applies. This represents the pure spin up maneuver in which the original spin is positive and the applied torque is positive. Similarly, when both a and b are negative, the negative sign on the square root term is used. When a is positive and b is negative, the sign of the square root term is at first negative but switches to positive for $t \geq -b/a$. This represents the spin down case until $t = -b/a$ at which point $\omega_z = 0$. Beyond this point, any additional torque serves to spin up the spacecraft; hence, the sign becomes positive. In the same manner, when a is negative and b is positive, the sign of the square root is positive for $0 \leq t \leq -b/a$ and negative for $t \geq -b/a$. With these considerations in mind, Eqs. (9) becomes

$$t = \frac{-b + u\,\mathrm{sgn}(a) \sqrt{b^2 + 2a\tau}}{a} \tag{9a}$$

and Eqs. (8) can be rewritten as

$$\begin{aligned} \frac{dx}{d\tau} + \lambda_1 y &= \frac{u\,\mathrm{sgn}(a)\, c}{\sqrt{b^2 + 2a\tau}} \\ \frac{dy}{d\tau} - \lambda_2 x &= \frac{u\,\mathrm{sgn}(a)\, d}{\sqrt{b^2 + 2a\tau}} \end{aligned} \tag{10}$$

where

$u = 1$ for spin up (a and b same sign)

$u = -1$ for spin down (a and b opposite sign) and only for $0 \leq t \leq -b/a$.

Thus, the solution of Eqs. (10) will be given for a constant value of u which corresponds to a pure spin up or a pure spin down maneuver. If the spin down maneuver solution is desired for $t > -b/a$ then the final conditions $\omega_x(-b/a)$, $\omega_y(-b/a)$ and $\omega_z(-b/a) = 0$ should be used as initial conditions with the spin up solution (u=1).

Differentiation of Eqs. (10) provides the decoupled equations

$$\frac{d^2x}{d\tau^2} + \lambda_1\lambda_2 x - u\ \mathrm{sgn}(a)\left[-\lambda_1 d(b^2 + 2a\tau)^{-1/2} - ac(b^2 + 2a\tau)^{-3/2}\right]$$

$$\frac{d^2y}{d\tau^2} + \lambda_1\lambda_2 y = u\ \mathrm{sgn}(a)\left[\lambda_2 c(b^2 + 2a\tau)^{-1/2} - ad(b^2 + 2a\tau)^{-3/2}\right] \quad (11)$$

The solution of Eqs. (11) will be obtained for all cases in which

$$\lambda_1\lambda_2 > 0 \quad (12)$$

so that the equations are stable. This implies that I_z is the largest or the smallest of the principal moments of inertia. For the Galileo Spacecraft, I_z is the largest of the principal moments of inertia. When I_z is the intermediate moment of inertia, then $\lambda_1\lambda_2 < 0$ and the Eqs. (11) are unstable. The solution of Eqs. (11) for the unstable case will not be investigated in this work.

At this point it is convenient to make a second change of variables. Let

$$\bar{\tau} = \sqrt{\lambda_1\lambda_2}\,\tau$$
$$\bar{a} = \sqrt{\lambda_1\lambda_2}\,a \quad (13)$$
$$\bar{b} = \sqrt{\lambda_1\lambda_2}\,b$$

Then Eqs. (11) become

$$\frac{d^2x}{d\bar{\tau}^2} + x = u\ \mathrm{sgn}\bar{a}\left[-\frac{\lambda_1}{\sqrt{\lambda_1\lambda_2}}d(\bar{b}^2 + 2\bar{a}\bar{\tau})^{-1/2} - \bar{a}c(\bar{b}^2 + 2\bar{a}\bar{\tau})^{-3/2}\right]$$

$$\frac{d^2y}{d\bar{\tau}^2} + y = u\ \mathrm{sgn}\bar{a}\left[\frac{\lambda_2}{\sqrt{\lambda_1\lambda_2}}c(\bar{b}^2 + 2\bar{a}\bar{\tau})^{-1/2} - \bar{a}d(\bar{b}^2 + 2\bar{a}\bar{\tau})^{-3/2}\right] \quad (14)$$

The solutions of Eqs. (14) take the form

$$x(\bar{\tau}) = A_x\cos\bar{\tau} + B_x\sin\bar{\tau} + \int_0^{\bar{\tau}} f_1(\phi)\sin(\bar{\tau} - \phi)\,d\phi \quad (15)$$

$$y(\bar{\tau}) = A_y\cos\bar{\tau} + B_y\sin\bar{\tau} + \int_0^{\bar{\tau}} f_2(\phi)\sin(\bar{\tau} - \phi)\,d\phi$$

where $f_1(\bar{\tau})$ and $f_2(\bar{\tau})$ are forcing functions of Eqs. (14)

The following integrals have been obtained for arbitrary signs on $\bar{a}$ and $\bar{b}$.

$$\int_0^{\bar{\tau}} \frac{\sin(\bar{\tau} - \phi)}{\sqrt{\bar{b}^2 + 2\bar{a}\phi}}\,d\phi = \frac{1}{\sqrt{|2\bar{a}|}}\left[\bar{C}\sin\left[\mathrm{sgn}\bar{a}\left(\bar{\tau} + \frac{\bar{b}^2}{2\bar{a}}\right)\right] - \bar{S}\cos\left[\mathrm{sgn}\bar{a}\left(\bar{\tau} + \frac{\bar{b}^2}{2\bar{a}}\right)\right]\right]$$

$$\int_0^{\bar{\tau}} \frac{\cos(\bar{\tau} - \phi)}{\bar{b}^2 + 2\bar{a}\phi}\,d\phi = \frac{\mathrm{sgn}\bar{a}}{\sqrt{|2\bar{a}|}}\left[\bar{C}\cos\left[\mathrm{sgn}\bar{a}\left(\bar{\tau} + \frac{\bar{b}^2}{2\bar{a}}\right)\right] + \bar{S}\sin\left[\mathrm{sgn}\bar{a}\left(\bar{\tau} + \frac{\bar{b}^2}{2\bar{a}}\right)\right]\right]$$

$$\int_0^{\bar{\tau}} (\bar{b}^2 + 2\bar{a}\phi)^{-3/2}\sin(\bar{\tau} - \phi)\,d\phi =$$

$$-\sqrt{\frac{1}{|2\bar{a}^3|}}\left[\bar{C}\cos\left[\mathrm{sng}\bar{a}\left(\bar{\tau} + \frac{\bar{b}^2}{2\bar{a}}\right)\right] + \bar{S}\sin\left[\mathrm{sgn}\bar{a}\left(\bar{\tau} + \frac{\bar{b}^2}{2\bar{a}}\right)\right]\right]$$

$$+ \frac{1}{|\bar{a}|\ |\bar{b}|}\sin\left[\mathrm{sgn}\bar{a}(\bar{\tau})\right]$$

where

$$\bar{C} \equiv \int_{\frac{\bar{b}^2}{2\bar{a}} \operatorname{sgn}\bar{a}}^{\left[\bar{\tau} + \frac{\bar{b}^2}{2\bar{a}}\right] \operatorname{sgn}\bar{a}} \frac{\cos s}{\sqrt{s}}\, ds \;,$$

$$\bar{S} \equiv \int_{\frac{\bar{b}^2}{2\bar{a}} \operatorname{sgn}\bar{a}}^{\left[\bar{\tau} + \frac{\bar{b}^2}{2\bar{a}}\right] \operatorname{sgn}\bar{a}} \frac{\sin s}{\sqrt{s}}\, ds \qquad (16)$$

Using Eqs. (15) and (16), the solutions of Eqs. (14) are

$$\begin{aligned} x(\bar{\tau}) = {} & A_x \cos\bar{\tau} + B_x \sin\bar{\tau} \\ & + u \operatorname{sgn}\bar{a} \left[\frac{\bar{S}}{\sqrt{|2\bar{a}|}} \left\{ \frac{\lambda_1 d}{\sqrt{\lambda_1\lambda_2}} \cos\left[\left(\bar{\tau} + \frac{\bar{b}^2}{2\bar{a}}\right) \operatorname{sgn}\bar{a}\right] \right.\right. \\ & \left. + \frac{\overline{ac}}{|\bar{a}|} \sin\left[\left(\bar{\tau} + \frac{\bar{b}^2}{2\bar{a}}\right) \operatorname{sgn}\bar{a}\right] \right\} \\ & + \frac{\bar{C}}{\sqrt{|2\bar{a}|}} \left\{ \frac{\overline{ac}}{|\bar{a}|} \cos\left[\left(\bar{\tau} + \frac{\bar{b}^2}{2\bar{a}}\right) \operatorname{sgn}\bar{a}\right] \right. \\ & \left. - \frac{\lambda_1 d}{\sqrt{\lambda_1\lambda_2}} \sin\left[\left(\bar{\tau} + \frac{\bar{b}^2}{2\bar{a}}\right) \operatorname{sgn}\bar{a}\right] \right\} \\ & \left. - \frac{\overline{ac}}{|\bar{a}|\,|\bar{b}|} \sin(\bar{\tau} \operatorname{sgn}\bar{a}) \right] \end{aligned}$$

$$\begin{aligned} y(\bar{\tau}) = {} & A_y \cos\bar{\tau} + B_y \sin\bar{\tau} \\ & + u \operatorname{sgn}\bar{a} \left[\frac{\bar{S}}{\sqrt{|2\bar{a}|}} \left\{ \frac{\overline{ad}}{|\bar{a}|} \sin\left[\left(\bar{\tau} + \frac{\bar{b}^2}{2\bar{a}}\right) \operatorname{sgn}\bar{a}\right] \right.\right. \\ & \left. - \frac{\lambda_2 c}{\sqrt{\lambda_1\lambda_2}} \cos\left[\left(\bar{\tau} + \frac{\bar{b}^2}{2\bar{a}}\right) \operatorname{sgn}\bar{a}\right] \right\} \\ & + \frac{\bar{C}}{\sqrt{|2\bar{a}|}} \left\{ \frac{\lambda_2 c}{\sqrt{\lambda_1\lambda_2}} \sin\left[\left(\bar{\tau} + \frac{\bar{b}^2}{2\bar{a}}\right) \operatorname{sgn}\bar{a}\right] \right. \\ & \left. + \frac{\overline{ad}}{|\bar{a}|} \cos\left[\left(\bar{\tau} + \frac{\bar{b}^2}{2\bar{a}}\right) \operatorname{sgn}\bar{a}\right] \right\} \\ & \left. - \frac{\overline{ad}}{|\bar{a}|\,|\bar{b}|} \sin(\bar{\tau} \operatorname{sgn}\bar{a}) \right] \qquad (17) \end{aligned}$$

Solving Eqs. (17) in terms of the original independent variable, the time t, for the initial conditions gives the final result:

$$\begin{aligned} x(t) = {} & x(0) \cos\left(\frac{1}{2}\bar{a}t^2 + \bar{b}t\right) \\ & - \sqrt{\frac{\lambda_1}{\lambda_2}}\, y(0) \sin\left(\frac{1}{2}\bar{a}t^2 + \bar{b}t\right) \\ & + u \operatorname{sgn}\bar{a} \left[\frac{\tilde{S}}{\sqrt{|2\bar{a}|}} \left\{ \frac{\lambda_1 d}{\sqrt{\lambda_1\lambda_2}} \cos\left[\left(\frac{1}{2}\bar{a}t^2 + \bar{b}t + \frac{\bar{b}^2}{2\bar{a}}\right) \operatorname{sgn}\bar{a}\right] \right.\right. \\ & \left. + \frac{\overline{ac}}{|\bar{a}|} \sin\left[\left(\frac{1}{2}\bar{a}t^2 + \bar{b}t + \frac{\bar{b}^2}{2\bar{a}}\right) \operatorname{sgn}\bar{a}\right] \right\} \\ & + \frac{\tilde{C}}{\sqrt{|2\bar{a}|}} \left\{ \frac{\overline{ac}}{|\bar{a}|} \cos\left[\left(\frac{1}{2}\bar{a}t^2 + \bar{b}t + \frac{\bar{b}^2}{2\bar{a}}\right) \operatorname{sgn}\bar{a}\right] \right. \\ & \left.\left. - \frac{\lambda_1 d}{\sqrt{\lambda_1\lambda_2}} \sin\left[\left(\frac{1}{2}\bar{a}t^2 + \bar{b}t + \frac{\bar{b}^2}{2\bar{a}}\right) \operatorname{sgn}\bar{a}\right] \right\} \right] \end{aligned}$$

$$\begin{aligned} y(t) = {} & y(0) \cos\left(\frac{1}{2}\bar{a}t^2 + \bar{b}t\right) \\ & + \sqrt{\frac{\lambda_2}{\lambda_1}}\, x(0) \sin\left(\frac{1}{2}\bar{a}t^2 + \bar{b}t\right) \\ & + u \operatorname{sgn}\bar{a} \left[\frac{\tilde{S}}{\sqrt{|2\bar{a}|}} \left\{ \frac{\overline{ad}}{|\bar{a}|} \sin\left[\left(\frac{1}{2}\bar{a}t^2 + \bar{b}t + \frac{\bar{b}^2}{2\bar{a}}\right) \operatorname{sgn}\bar{a}\right] \right.\right. \\ & \left. - \frac{\lambda_2 c}{\sqrt{\lambda_1\lambda_2}} \cos\left[\left(\frac{1}{2}\bar{a}t^2 + \bar{b}t + \frac{\bar{b}^2}{2\bar{a}}\right) \operatorname{sgn}\bar{a}\right] \right\} \\ & + \frac{\tilde{C}}{\sqrt{|2\bar{a}|}} \left\{ \frac{\lambda_2 c}{\sqrt{\lambda_1\lambda_2}} \sin\left[\left(\frac{1}{2}\bar{a}t^2 + \bar{b}t + \frac{\bar{b}^2}{2\bar{a}}\right) \operatorname{sgn}\bar{a}\right] \right. \\ & \left.\left. + \frac{\overline{ad}}{|\bar{a}|} \cos\left[\left(\frac{1}{2}\bar{a}t^2 + \bar{b}t + \frac{\bar{b}^2}{2\bar{a}}\right) \operatorname{sgn}\bar{a}\right] \right\} \right] \qquad (18) \end{aligned}$$

where

$$\tilde{C} = \int_{\frac{\bar{b}^2}{2\bar{a}} \operatorname{sgn}\bar{a}}^{\left(\frac{1}{2}\bar{a}t^2 + \bar{b}t + \frac{\bar{b}^2}{2\bar{a}}\right)\operatorname{sgn}\bar{a}} \frac{\cos s}{\sqrt{s}}\, ds ,$$

$$\tilde{S} = \int_{\frac{\bar{b}^2}{2\bar{a}} \operatorname{sgn}\bar{a}}^{\left(\frac{1}{2}\bar{a}t^2 + \bar{b}t + \frac{\bar{b}^2}{2\bar{a}}\right)\operatorname{sgn}\bar{a}} \frac{\sin s}{\sqrt{s}}\, ds$$

$$\lambda_1 = \frac{I_z - I_y}{I_x}, \quad \lambda_2 = \frac{I_z - I_x}{I_y}$$

$$\bar{a} = \sqrt{\lambda_1 \lambda_2}\, \frac{M_z}{I_z}, \quad \bar{b} = \sqrt{\lambda_1 \lambda_2}\, \omega_{z0}$$

$$c = \frac{M_x}{I_x}, \quad d = \frac{M_y}{I_y}, \quad x = \omega_x, \quad y = \omega_y$$

and

$u = 1$ for pure spin up ($\bar{a}$ and $\bar{b}$ same sign)

$= -1$ for pure spin down ($\bar{a}$ and $\bar{b}$ opposite sign and $0 \le t \le -b/\bar{a}$)

The integrals $\tilde{S}$ and $\tilde{C}$ in Eqs. (18) are closely related to the Fresnel Integrals[2]:

$$C(z) = \int_o^z \cos\left(\frac{\pi}{2} t^2\right) dt$$

$$S(z) = \int_o^z \sin\left(\frac{\pi}{2} t^2\right) dt \qquad (19)$$

Other functions that are in use are

$$C_1(x) = \sqrt{\frac{2}{\pi}} \int_o^x \cos t^2\, dt ,$$

$$C_2(x) = \frac{1}{\sqrt{2\pi}} \int_o^x \frac{\cos t}{\sqrt{t}}\, dt$$

$$S_1(x) = \sqrt{\frac{2}{\pi}} \int_o^x \sin t^2\, dt , \qquad (20)$$

$$S_2(x) = \frac{1}{\sqrt{2\pi}} \int_o^x \frac{\sin t}{\sqrt{t}}\, dt$$

These functions are interrelated through a change of the argument:

$$C(x) = C_1\left(x\sqrt{\frac{\pi}{2}}\right) = C_2\left(\frac{\pi}{2} x^2\right)$$

$$S(x) = S_1\left(x\sqrt{\frac{\pi}{2}}\right) = S_2\left(\frac{\pi}{2} x^2\right) \qquad (21)$$

From the definition of $\tilde{S}$ and $\tilde{C}$ in Eqs. (18) it is apparent that

$$\tilde{C} = \sqrt{2\pi}\left[C_2\left(\frac{\bar{a}t^2}{2} + \bar{b}t + \frac{\bar{b}^2}{2\bar{a}}\right) - C_2\left(\frac{\bar{b}^2}{2\bar{a}}\right)\right]$$

$$= \sqrt{2\pi}\left[C\left(\sqrt{\frac{2}{\pi}\left(\frac{\bar{a}t^2}{2} + \bar{b}t + \frac{\bar{b}^2}{2\bar{a}}\right)}\right) - C\left(\sqrt{\frac{\bar{b}^2}{\pi\bar{a}}}\right)\right]$$

$$\tilde{S} = \sqrt{2\pi}\left[S_2\left(\frac{\bar{a}t^2}{2} + \bar{b}t + \frac{\bar{b}^2}{2\bar{a}}\right) - S_2\left(\frac{\bar{b}^2}{2\bar{a}}\right)\right]$$

$$= \sqrt{2\pi}\left[S\left(\sqrt{\frac{2}{\pi}\left(\frac{\bar{a}t^2}{2} + \bar{b}t + \frac{\bar{b}^2}{2\bar{a}}\right)}\right) - S\left(\sqrt{\frac{\bar{b}^2}{\pi\bar{a}}}\right)\right] \qquad (22)$$

There are many ways of obtaining accurate expressions for the Fresnel Integrals. For the case of the Galileo Spacecraft, for maneuvers in which the spacecraft acts as a single spinner, the initial spin rate parameter, $\bar{b}^2$, is large compared to the rotational acceleration parameter, $2\bar{a}$, so that the following asymptotic expansions are adequate for the maneuver analysis

$$C_2(x) = 0.5 + \left(0.3989423 - \frac{0.3}{x^2}\right)\frac{\sin x}{\sqrt{x}} - \left(0.19947 - \frac{0.748}{x^2}\right)\frac{\cos x}{x\sqrt{x}} + \varepsilon(x)$$

$$S_2(x) = 0.5 - \left(0.3989423 - \frac{0.3}{x^2}\right)\frac{\cos x}{\sqrt{x}} - \left(0.19947 - \frac{0.748}{x^2}\right)\frac{\sin x}{x\sqrt{x}} + \varepsilon(x) \qquad (23)$$

where

$$|\varepsilon(x)| < 3 \times 10^{-7} \text{ for } x > 39$$

Thus Eqs. (2), (18), (22) and (23) completely specify the heuristic solutions for $\omega_x(t)$, $\omega_y(t)$ and $\omega_z(t)$ for the spin up and spin down maneuver analysis of the Galileo Spacecraft.

For the sake of completeness, the Fresnel Integrals will be evaluated for small arguments. The Fresnel Integrals can be expressed in terms of the auxiliary functions f(x) and g(x)

$$C(x) = \frac{1}{2} + f(x)\sin\left(\frac{\pi}{2}x^2\right) - g(x)\cos\left(\frac{\pi}{2}x^2\right)$$

$$S(x) = \frac{1}{2} - f(x)\cos\left(\frac{\pi}{2}x^2\right) - g(x)\sin\left(\frac{\pi}{2}x^2\right) \qquad (24)$$

The following rational approximations for f(x) and g(x) can be used[2]

$$f(x) = \frac{1 + 0.926x}{2 + 1.792x + 3.104x^2} + \varepsilon(x)$$

$$g(x) = \frac{1}{2 + 4.142x + 3.492x^2 + 6.670x^3} + \varepsilon(x)$$

$$|\varepsilon(x)| \le 2 \times 10^{-3} \text{ for } (0 \le x \le \infty) \qquad (25)$$

Note that the approximations apply to all positive values of x. For large values of x, however, Eqs. (23) are recommended since the asymptotic expansions are much more accurate while the fixed error in Eqs. (25) allows a fairly large fractional error to occur.

IV. Numerical Results for the Solution of Euler's Equations of Motion

Eqs. (18) and (2) provide the heuristic solutions for $\omega_x(t)$, $\omega_y(t)$ and $\omega_z(t)$ for near symmetric rigid bodies. When $I_x = I_y$, the solutions become the exact analytic expressions for the symmetric rigid body. The accuracy of the heuristic solutions is tested by comparison with the numerical integration of Eqs. (1) by ACSL (Advanced Continuous Simulation Language).

In order to test the solution for the spin up maneuver, the following parameters were chosen from the fully loaded, fully deployed configuration of the Galileo Spacecraft.

$$I_x = 2985 \text{ kilogram - meters}^2$$

$$I_y = 2729$$

$$I_z = 4183$$

$$M_x = -1.253 \text{ newton - meters}$$

$$M_y = -1.494$$

$$M_z = 13.5 \qquad (26)$$

In the case of the spin down maneuver two thrusters are used, forming a couple, so that

$$c = \frac{M_x}{I_x} = d = \frac{M_y}{I_y} = 0 \qquad (27)$$

and only the homogeneous solutions remain. Since the particular solutions are the main area of interest, the case for Eq. (27) was not simulated. Rather, the hypothetical case of spin down with a single thruster was simulated with reverse signs on the moments in Eqs. (26). All four combinations of positive and negative signs for a and b were tested to ensure the correctness of the heuristic solutions (Eqs. (2) and (18)).

For the sake of brevity, only the spin up maneuver simulation results will be included in this report with the assumption that $\omega_x(0) = \omega_y(0) = 0$. It should be noted, however, that the heuristic solutions given by Eqs. (2) and (18) have been verified and that the general conclusions for the spin up maneuver apply to the spin down maneuvers as well.

Case 1 - Spin Up from 3.15 to 10 RPM

This case directly applies to the spin up maneuver of the Galileo Spacecraft from 3.15 rpm to 10 rpm. The heuristic solution is taken from Eqs. (18), (22) and the asymptotic expansions of the Fresnel Integrals of Eqs. (23). The exact solution is obtained by the numerical integration of Eqs. (1) by ACSL. The discrepancy between the heuristic and exact solutions is indiscernible in Figs. 1, 3 and 5. The accuracy of the heuristic solution is shown in Figs. 2, 4 and 6 in which the differences between the exact and the heuristic solutions have been plotted. From these plots it is clear that the heuristic solutions for ω_x and ω_y deviate from the exact solution by only 0.1 percent. The heuristic

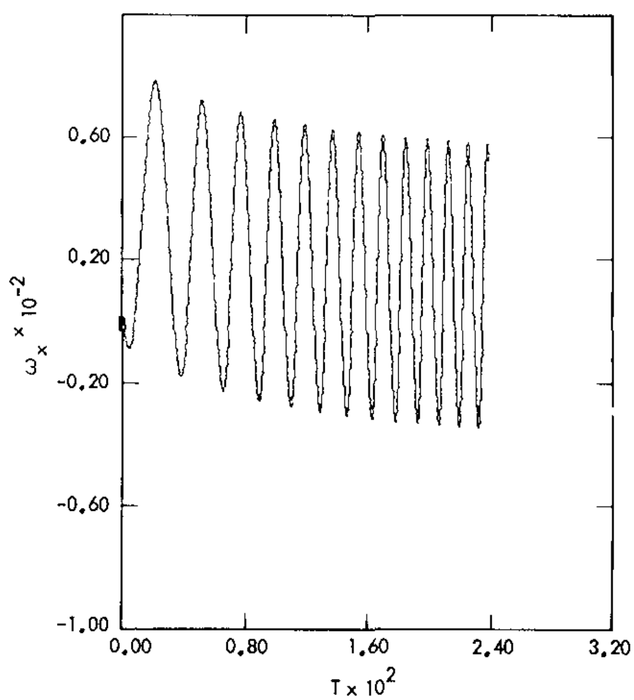

Fig 1 Case 1: Heuristic and Exact Solutions of $\omega_x(t)$ for the Near Symmetric Case with Spin Up from 3.15 rpm to 10 rpm. The heuristic solution for $\omega_x(t)$ is determined from Eqs. 18, 22 and 23 while the exact solution is found by integrating Eqs. 1 with ACSL. Note that in each figure the ω's are given in radians per second and the time in seconds.

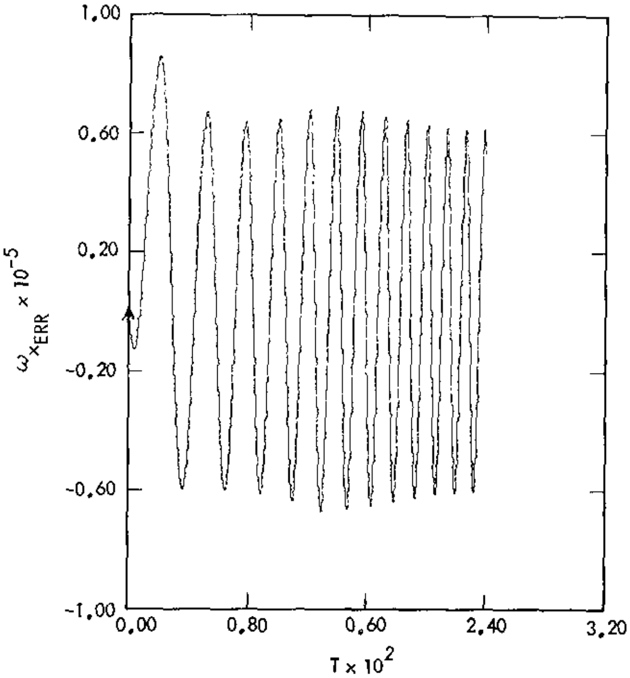

Fig 2 Case 1: $\omega_{x_{err}} = \omega_{x_{exact}} - \omega_{x_{heuristic}}$ (See Fig. 1).

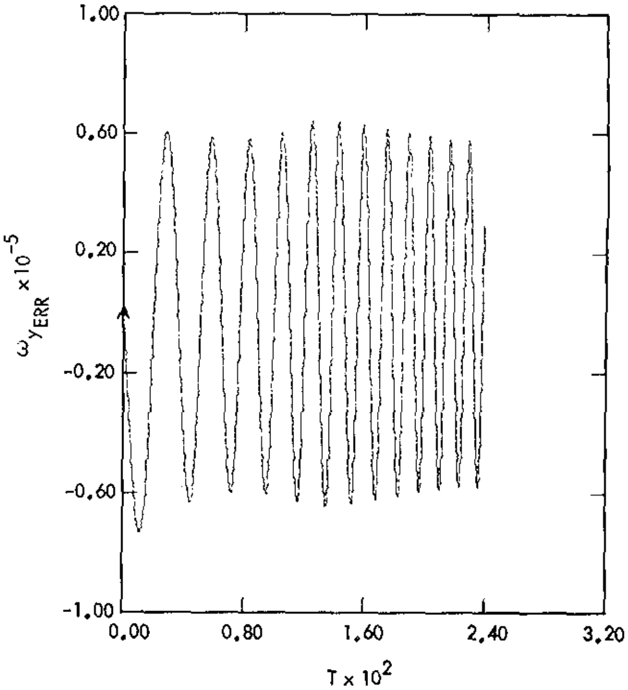

Fig 4 Case 1: $\omega_{y_{err}} = \omega_{y_{exact}} - \omega_{y_{heuristic}}$ (See Fig. 3).

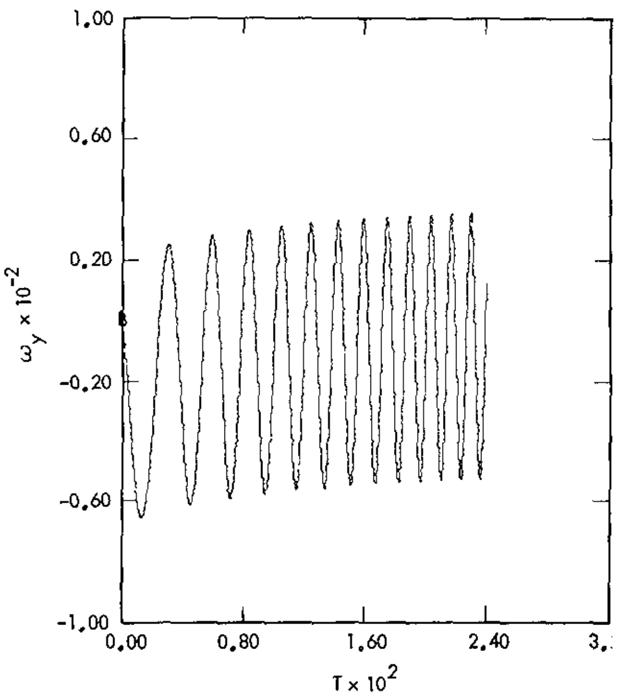

Fig 3 Case 1: Heuristic and Exact Solutions of $\omega_y(t)$ *for the Near Symmetric Case with* Spin Up from 3.15 rpm to 10 rpm. The heuristic solution for $\omega_y(t)$ is determined from Eqs. 18, 22 and 13 while the exact solution is found by integrating Eqs. 1 with ACSL.

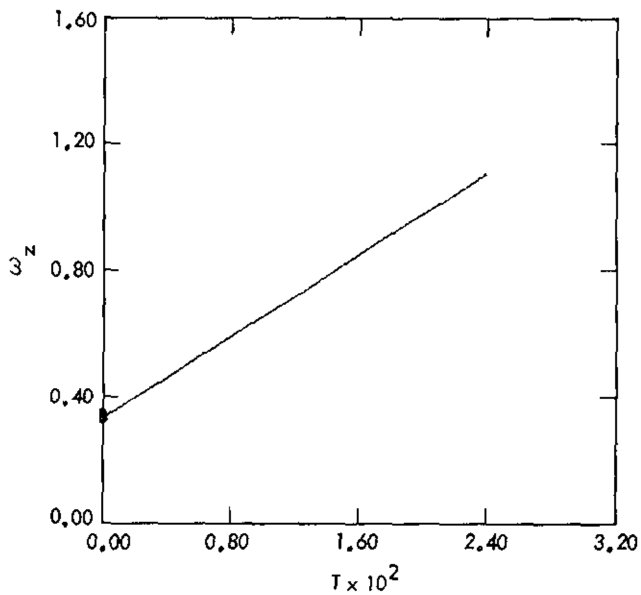

Fig 5 *Case 1: Heuristic and Exact Solutions for* the Near Symmetric Case with Spin Up from 3.15 rpm to 10 rpm. The heuristic solution for $\omega_z(t)$ is from Eq. 2 while the exact solution is found by integrating Eqs. 1 with ACSL.

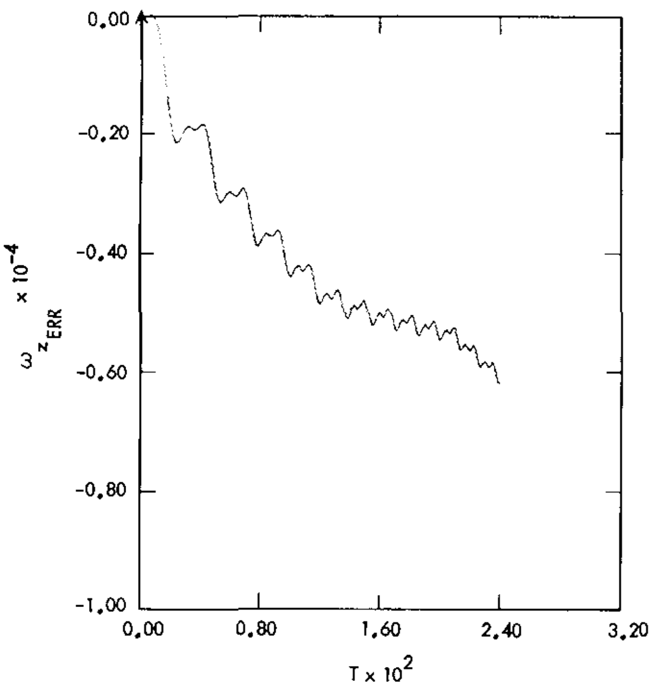

Fig 6 Case 1: $\omega_{z_{err}} = \omega_{z_{exact}} - \omega_{z_{heuristic}}$ (See Fig. 5).

assumption that ω_z varies linearly with time is time is justified in Fig. 6 which indicates a discrepancy of only 0.01 percent from the exact solution.

Case 2 - Spin up from 0 to 3.15 RPM

While this case does not apply to the Galileo Spacecraft in the present plan, it is included for the sake of completeness. Here it is emphasized that the asymptotic expansions for the Fresnel Integrals cannot be used since there is a singularity when $\omega_{z0} = 0$. Instead, the rational approximations for f(x) and g(x) are used (Eqs. 25) along with Eqs. (24), (22) and (18) to provide the heuristic solutions. Figs. 7 through 12 display the simulation results. All the remarks of Case 1 apply to Case 2 except that the latter is less accurate than the former. This is probably due to the less accurate expressions used for the Fresnel Integrals.

V. Euler Angle Representation

A Type 1: 3-1-2 Euler Angle Rotation is used for the kinematic equations of motion.[3] This means that the Eulerian Angles (ϕ_x, ϕ_y, ϕ_z) are defined by successive rotations by angles ϕ_z, ϕ_x and ϕ_y about the Z, X' and Y" coordinate axes. The resulting kinematic equations of motion are

$$\dot{\phi}_x = \omega_x \cos\phi_y + \omega_z \sin\phi_y$$

$$\dot{\phi}_y = \omega_y - (\omega_z \cos\phi_y - \omega_x \sin\phi_y) \tan\phi_x$$

$$\dot{\phi}_z = (\omega_z \cos\phi_y - \omega_x \sin\phi_y) \sec\phi_x \qquad (28)$$

VI. Solution of Eulerian Angles for a Near Symmetric Rigid Body Subject to Constant Moments

The solution of Eqs. (28) provides the attitude of the body as a function of time. Unfortunately, the equations are nonlinear in general and are very difficult (if not impossible) to solve. However, for many spacecraft applications, small angle approximations for ϕ_x and ϕ_y are appropriate and

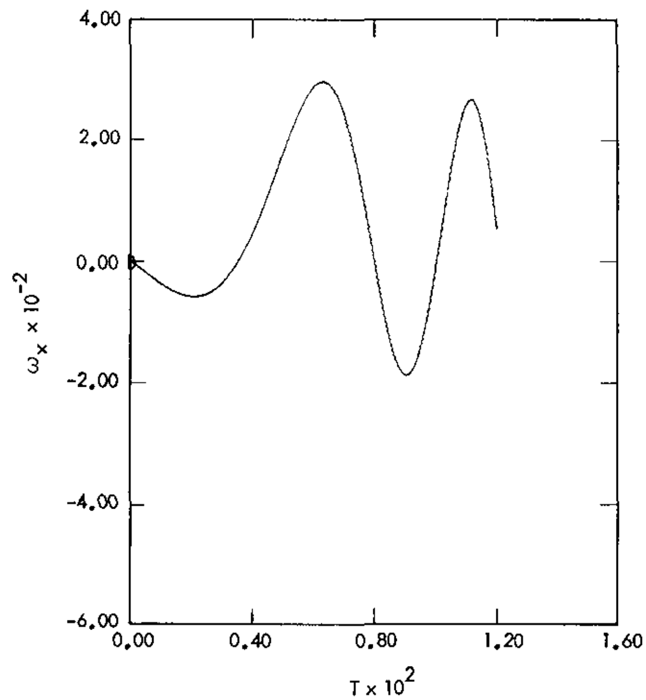

Fig 7 Case 2: Heuristic and Exact Solutions of $\omega_x(t)$ for the Near Symmetric Case with Spin Up from 0 rpm to 3.15 rpm. The heuristic solution for $\omega_x(t)$ is determined from Eqs. 18, 22, 24 and 25 while the exact solution is found by integrating Eqs. 1 with ACSL.

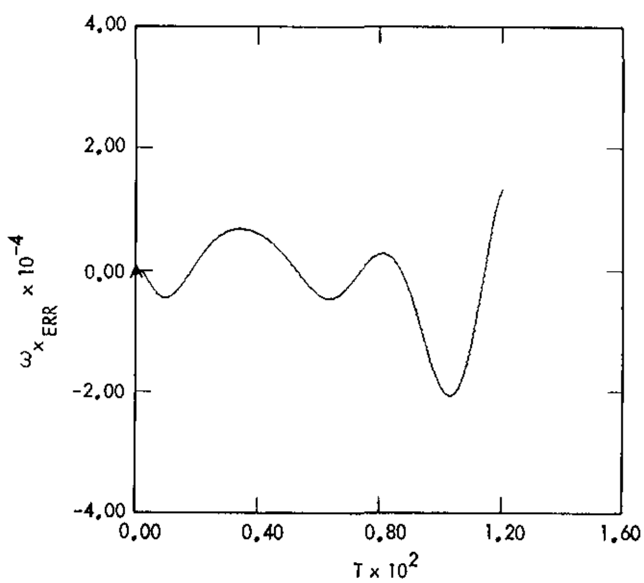

Fig 8 Case 2: $\omega_{x_{err}} = \omega_{x_{exact}} - \omega_{x_{heuristic}}$ (See Fig. 7).

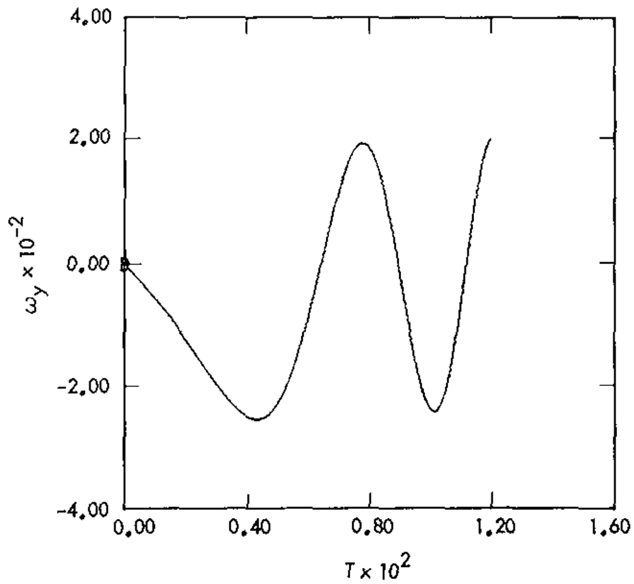

Fig 9 Case 2: Heuristic and Exact Solutions of $\omega_y(t)$ for the Near Symmetric Case and Spin Up from 0 rpm to 3.15 rpm. The heuristic solution for $\omega_y(t)$ is determined from Eqs. 18, 22, 24 and 15 while the exact solution is found by integrating Eqs. 1 with ACSL.

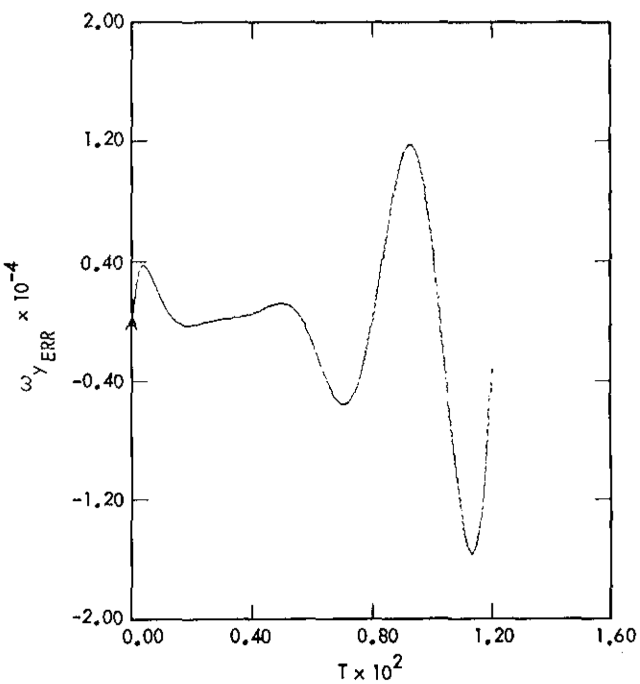

Fig 10 Case 2: $\omega_{y_{err}} = \omega_{y_{exact}} - \omega_{y_{heuristic}}$ (See Fig. 9).

useful assumptions since they apply to rigid bodies with high initial spin rates or with a large torque about a single axis. Thus, solution of the linearized form of Eqs. (28) can be used in the analysis of spin-up and spin-down maneuvers of spacecraft and in the error analysis of thruster misalignments in spinning spacecraft.

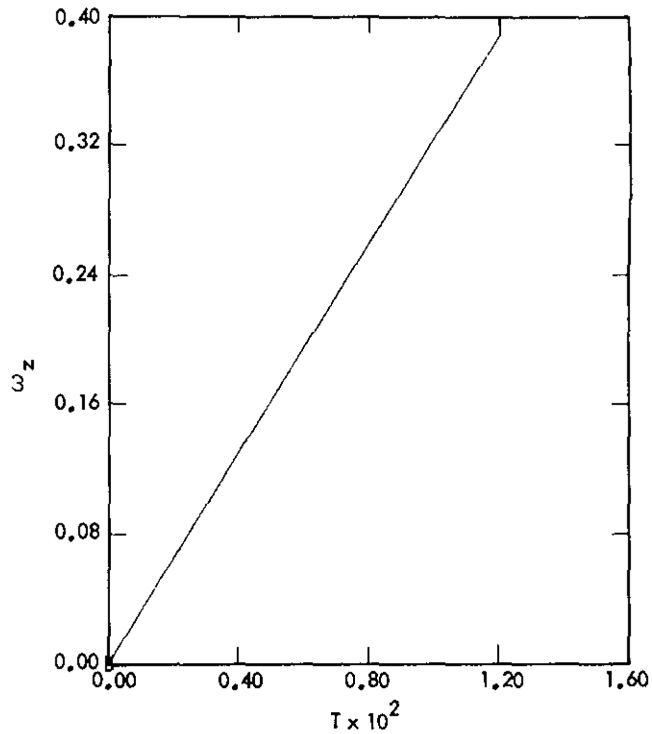

Fig 11 Case 2: Heuristic and Exact Solutions for the Near Symmetric Case with Spin Up from 0 rpm to 3.15 rpm. The heuristic solution for $\omega_z(t)$ is from Eq. 2 while the exact solution is found by integrating Eqs. 1 with ACSL.

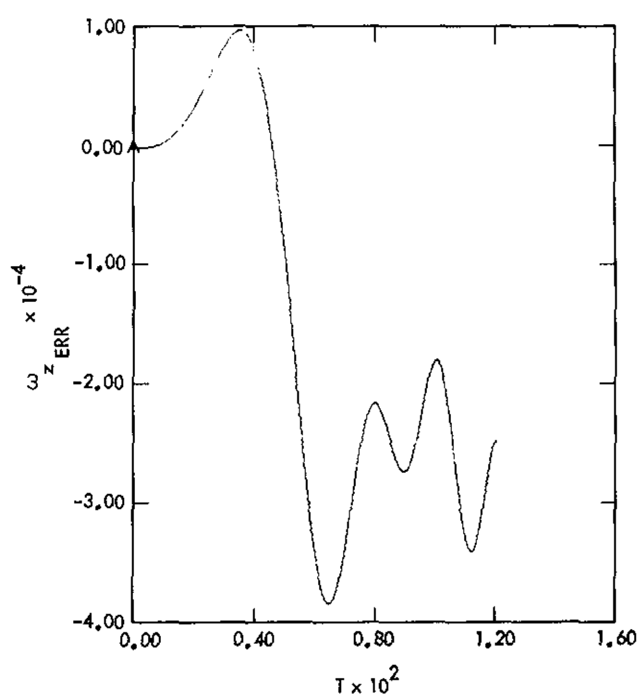

Fig 12 Case 2: $\omega_{z_{err}} = \omega_{z_{exact}} - \omega_{z_{heuristic}}$ (See Fig. 11).

Assuming ϕ_x and ϕ_y are small we obtain from Eqs. (28)

$$\dot{\phi}_x = \omega_x + \phi_y \omega_z$$

$$\dot{\phi}_y = \omega_y - \phi_x \omega_z$$

$$\dot{\phi}_z = \omega_z - \phi_y \omega_x \tag{29}$$

It is also appropriate to assume that $\phi_y \omega_x$ is small compared to ω_z so that

$$\dot{\phi}_z = \omega_z = \frac{M_z}{I_z} t + \omega_{z0} \tag{30}$$

The solution for ω_z is given in Eq. (2). Integrating

$$\phi_z = \frac{M_z}{2I_z} t^2 + \omega_{z0}t + \phi_{z0}$$

$$= \frac{1}{2} at^2 + bt + \phi_{z0}$$

$$= \tau + \phi_{z0}$$

$$= \frac{\bar{\tau}}{\sqrt{\lambda_1\lambda_2}} + \phi_{z0} \tag{31}$$

Following the method of Section III, the independent variable is changed to $\bar{\tau}$ as given by Eqs. (7) and (13), so that the first two equations of Eqs. (29) become (after differentiating and rearranging):

$$\frac{d^2\phi_\lambda}{d\bar{\tau}^2} + \alpha^2\phi_x = u \operatorname{sgn}\bar{a} \left[\frac{\alpha\omega_y(\bar{\tau})}{\sqrt{\bar{b}^2 + 2\alpha\bar{\tau}}} + \frac{d}{d\bar{\tau}}\left\{\frac{\omega_x(\bar{\tau})}{\sqrt{\bar{b}^2 + 2\alpha\bar{\tau}}}\right\}\right]$$

$$\frac{d^2\phi_\lambda}{d\bar{\tau}^2} + \alpha^2\phi_y = u \operatorname{sgn}\bar{a} \left[\frac{-\alpha\omega_x(\bar{\tau})}{\sqrt{\bar{b}^2 + 2\bar{a}\bar{\tau}}} + \frac{d}{d\bar{\tau}}\left\{\frac{\omega_y(\bar{\tau})}{\sqrt{\bar{b}^2 + 2\alpha\bar{\tau}}}\right\}\right] \tag{32}$$

where

$$\alpha = \frac{1}{\sqrt{\lambda_1\lambda_2}}$$

Solving Eqs. (32) as functions of $\bar{\tau}$, then applying initial conditions and putting the result in terms of t gives the form of the solutions $\phi_x(t)$ and $\phi_y(t)$:

$$\phi_x(t) = \phi_{xo} \cos\left[\alpha\left(\frac{1}{2}\bar{a}t^2 + \bar{b}t\right)\right] + \phi_{yo} \sin\left[\alpha\left(\frac{1}{2}\bar{a}t^2 + \bar{b}t\right)\right]$$

$$+ u \operatorname{sgn}\bar{a} \int_0^{\frac{1}{2}\bar{a}t^2 + \bar{b}t} \cdot \frac{\omega_y(\xi) \sin\left[\alpha\left(\frac{1}{2}\bar{a}t^2 + \bar{b}t\right) - \alpha\xi\right]}{\sqrt{\bar{b}^2 + 2\bar{a}\xi}} d\xi$$

Equation continued

$$+ u \operatorname{sgn}\bar{a} \int_0^{\frac{1}{2}\bar{a}t^2 + \bar{b}t} \frac{\omega_x(\xi) \cos\left[\alpha\left(\frac{1}{2}\bar{a}t^2 + \bar{b}t\right) - \alpha\xi\right]}{\sqrt{\bar{b}^2 + 2\bar{a}\xi}} d\xi$$

$$\phi_y(t) = \phi_{yo} \cos\left[\alpha\left(\frac{1}{2}\bar{a}t^2 + \bar{b}t\right)\right] - \phi_{xo} \sin\left[\alpha\left(\frac{1}{2}\bar{a}t^2 + \bar{b}t\right)\right]$$

$$- u \operatorname{sgn}\bar{a} \int_0^{\frac{1}{2}\bar{a}t^2 + \bar{b}t} \cdot \frac{\omega_x(\xi) \sin\left[\alpha\left(\frac{1}{2}\bar{a}t^2 + \bar{b}t\right) - \alpha\xi\right]}{\sqrt{\bar{b}^2 + 2\bar{a}\xi}} d\xi$$

$$+ u \operatorname{sgn}\bar{a} \int_0^{\frac{1}{2}\bar{a}t^2 + \bar{b}t} \cdot \frac{\omega_y(\xi) \cos\left[\alpha\left(\frac{1}{2}\bar{a}t^2 + \bar{b}t\right) - \alpha\xi\right]}{\sqrt{\bar{b}^2 + 2\bar{a}\xi}} d\xi \tag{33}$$

To obtain the solutions of $\phi_x(t)$ and $\phi_y(t)$ the integrals of Eqs. (33) must be evaluated. Unfortunately, certain terms appear which cannot be directly integrated and so they are evaluated by asymptotic expansions. In order to make the labor systematic the problem is divided into those integrals that are known and those that are unknown. This is seen most clearly by replacing the Fresnel Integrals of the solutions of $\omega_x(\xi)$ and $\omega_y(\xi)$ in Eqs. (18) with the f and g functions of Eqs. (24):

$$\omega_x(\xi) = k_{x1}c_1 + k_{x2}s_1 + k_{x3}c_2$$

$$+ k_{x4}s_2 + k_{x5}f + k_{x6}g$$

$$\omega_y(\xi) = k_{y1}c_1 + k_{y2}s_1 + k_{y3}c_2 + k_{y4}s_2 + k_{y5}f + k_{y6}g$$

where

$$k_{x1} = \omega_{x0},\ k_{x2} = -\sqrt{\frac{\lambda_1}{\lambda_2}}\,\omega_{y0}$$

$$k_{x3} = u\ \mathrm{sgn}\bar{a}\sqrt{\frac{\pi}{|\bar{a}|}} \cdot \left[\frac{\lambda_1 d}{\sqrt{\lambda_1\lambda_2}}\left(\frac{1}{2} - S_0\right) + \frac{\bar{a}c}{|\bar{a}|}\left(\frac{1}{2} - C_0\right)\right]$$

$$k_{x4} = u\ \mathrm{sgn}\bar{a}\sqrt{\frac{\pi}{|\bar{a}|}}\left[\frac{\bar{a}c}{|\bar{a}|}\left(\frac{1}{2} - S_0\right) - \frac{\lambda_1 d}{\sqrt{\lambda_1\lambda_2}}\left(\frac{1}{2} - C_0\right)\right]$$

$$k_{x5} = u\ \mathrm{sgn}\bar{a}\sqrt{\frac{\pi}{|\bar{a}|}}\left[-\frac{\lambda_1 d}{\sqrt{\lambda_1\lambda_2}}\right],$$

$$k_{x6} = u\ \mathrm{sgn}\bar{a}\sqrt{\frac{\pi}{|\bar{a}|}}\left[-\frac{\bar{a}c}{|\bar{a}|}\right]$$

$$k_{y1} = \omega_{y0},$$

$$k_{y2} = \sqrt{\frac{\lambda_2}{\lambda_1}}\,\omega_{xo}$$

$$k_{y3} = u\ \mathrm{sgn}\bar{a}\sqrt{\frac{\pi}{|\bar{a}|}} \cdot \left[-\frac{\lambda_2 c}{\sqrt{\lambda_1\lambda_2}}\left(\frac{1}{2} - S_0\right) + \frac{\bar{a}d}{|\bar{a}|}\left(\frac{1}{2} - C_0\right)\right]$$

$$k_{y4} = u\ \mathrm{sgn}\bar{a}\sqrt{\frac{\pi}{|\bar{a}|}} \cdot \left[\frac{\bar{a}d}{|\bar{a}|}\left(\frac{1}{2} - S_0\right) + \frac{\lambda_2 c}{\sqrt{\lambda_1\lambda_2}}\left(\frac{1}{2} - C_0\right)\right]$$

$$k_{y5} = u\ \mathrm{sgn}\bar{a}\sqrt{\frac{\pi}{|\bar{a}|}}\left[\frac{\lambda_2 c}{\sqrt{\lambda_1\lambda_2}}\right],$$

$$k_{y6} = -u\ \mathrm{sgn}\bar{a}\sqrt{\frac{\pi}{|\bar{a}|}}\left[\frac{\bar{a}d}{|\bar{a}|}\right]$$

$$c_1 = \cos\xi,\quad s_1 = \sin\xi$$

$$c_2 = \cos\left[\left(\xi + \frac{\bar{b}^2}{2\bar{a}}\right)\mathrm{sgn}\bar{a}\right]$$

$$s_2 = \sin\left[\left(\xi + \frac{\bar{b}^2}{2\bar{a}}\right)\mathrm{sgn}\bar{a}\right]$$

$$C_0 = C\left(\sqrt{\frac{\bar{b}^2}{\pi\bar{a}}\ \mathrm{sgn}\bar{a}}\right) \tag{34}$$

$$S_0 = S\left(\sqrt{\frac{\bar{b}^2}{\pi\bar{a}}\ \mathrm{sgn}\bar{a}}\right)$$

$$g = g\left(\sqrt{\frac{2}{\pi}\left[\xi + \frac{\bar{b}^2}{2\bar{a}}\right]\mathrm{sgn}\bar{a}}\right)$$

$$f = f\left(\sqrt{\frac{2}{\pi}\left[\xi + \frac{\bar{b}^2}{2\bar{a}}\right]\mathrm{sgn}\bar{a}}\right)$$

It is convenient to work with the independent variable $\bar{\tau}$. Let the integrals to be evaluated be defined as follows.

$$W_{ys}(\bar{\tau}) = \int_o^{\bar{\tau}} \frac{\omega_y(\xi)\ \sin(\alpha\bar{\tau} - \alpha\xi)}{\sqrt{\bar{b}^2 + 2\bar{a}\,\xi}}\,d\xi$$

$$W_{xc}(\bar{\tau}) = \int_o^{\bar{\tau}} \frac{\omega_x(\xi)\ \cos(\alpha\bar{\tau} - \alpha\xi)}{\sqrt{\bar{b}^2 + 2\bar{a}\,\xi}}\,d\xi$$

$$W_{yc}(\bar{\tau}) = \int_o^{\bar{\tau}} \frac{\omega_y(\xi)\ \cos(\alpha\bar{\tau} - \alpha\xi)}{\sqrt{\bar{b}^2 + 2\bar{a}\,\xi}}\,d\xi \tag{35}$$

$$W_{xs}(\bar{\tau}) = \int_o^{\bar{\tau}} \frac{\omega_x(\xi)\ \sin(\alpha\bar{\tau} - \alpha\xi)}{\sqrt{\bar{b}^2 + 2\bar{a}\,\xi}}\,d\xi$$

Inspection of Eqs. (34) and (35) reveals that the following integrals must be solved:

$$J_{cc}(\bar{\tau}, k_1, k_2, k_3, k_4) = \int_o^{\bar{\tau}} \frac{\cos(k_1\xi + k_2)\cos(k_3\xi + k_4)}{\sqrt{\bar{b}^2 + 2\bar{a}\,\xi}}\,d\xi$$

$J_{cs}(\bar{\tau}, k_1, k_2, k_3, k_4) =$

$$\int_o^{\bar{\tau}} \frac{\cos(k_1\xi + k_2)\sin(k_3\xi + k_4)}{\sqrt{\bar{b}^2 + 2\bar{a}\,\xi}}\, d\xi$$

$J_{ss}(\bar{\tau}, k_1, k_2, k_3, k_4) =$

$$\int_o^{\tau} \frac{\sin(k_1\xi + k_2)\sin(k_3\xi + k_4)}{\sqrt{\bar{b}^2 + 2\bar{a}\,\xi}}\, d\xi$$

$$F_c(\bar{\tau}) = \int_o^{\bar{\tau}} \frac{f\left(\sqrt{\frac{2}{\pi}\left[\xi + \frac{\bar{b}^2}{2\bar{a}}\right]\operatorname{sgn}\bar{a}}\right)\cos(\alpha\bar{\tau} - \alpha\xi)}{\sqrt{\bar{b}^2 + 2\bar{a}\,\xi}}\, d\xi$$

$$F_s(\bar{\tau}) = \int_o^{\bar{\tau}} \frac{f\left(\sqrt{\frac{2}{\pi}\left[\xi + \frac{\bar{b}^2}{2\bar{a}}\right]\operatorname{sgn}\bar{a}}\right)\sin(\alpha\bar{\tau} - \alpha\xi)}{\sqrt{\bar{b}^2 + 2\bar{a}\,\xi}}\, d\xi$$

$$G_c(\bar{\tau}) = \int_o^{\bar{\tau}} \frac{g\left(\sqrt{\frac{2}{\pi}\left[\xi + \frac{\bar{b}^2}{2\bar{a}}\right]\operatorname{sgn}\bar{a}}\right)\cos(\alpha\bar{\tau} - \alpha\xi)}{\sqrt{\bar{b}^2 + 2\bar{a}\,\xi}}\, d\xi$$

$$G_s(\bar{\tau}) = \int_o^{\bar{\tau}} \frac{g\left(\sqrt{\frac{2}{\pi}\left[\xi + \frac{\bar{b}^2}{2\bar{a}}\right]\operatorname{sgn}\bar{a}}\right)\sin(\alpha\bar{\tau} - \alpha\xi)}{\sqrt{\bar{b}^2 + 2\bar{a}\,\xi}}\, d\xi \tag{36}$$

From Eqs. (34), (35) and (36), the W_{ys}, W_{xc}, W_{yc} and W_{xs} integrals can be written

$$\begin{aligned} W_{ys}(\bar{\tau}) = {} & k_{y1}J_{cs}(\bar{\tau}, 1, 0, -\alpha, \alpha\bar{\tau}) + k_{y2}J_{ss}(\bar{\tau}, 1, 0, -\alpha, \alpha\bar{\tau}) \\ & + k_{y3}J_{cs}\left(\bar{\tau}, \operatorname{sgn}\bar{a}, \frac{\bar{b}^2}{2\bar{a}}\operatorname{sgn}\bar{a}, -\alpha, \alpha\bar{\tau}\right) \\ & + k_{y4}J_{ss}\left(\bar{\tau}, \operatorname{sgn}\bar{a}, \frac{\bar{b}^2}{2\bar{a}}\operatorname{sgn}\bar{a}, -\alpha, \alpha\bar{\tau}\right) \\ & + k_{y5}F_s(\bar{\tau}) + k_{y6}G_s(\bar{\tau}) \end{aligned}$$

$$\begin{aligned} W_{xc}(\bar{\tau}) = {} & k_{x1}J_{cc}(\bar{\tau}, 1, 0, -\alpha, \alpha\bar{\tau}) + k_{x2}J_{cs}(\bar{\tau}, -\alpha, \alpha\bar{\tau}, 1, 0) \\ & + k_{x3}J_{cc}\left(\bar{\tau}, \operatorname{sgn}\bar{a}, \frac{\bar{b}^2}{2\bar{a}}\operatorname{sgn}\bar{a}, -\alpha, \alpha\bar{\tau}\right) \\ & + k_{x4}J_{cs}\left(\bar{\tau}, -\alpha, \alpha\bar{\tau}, \operatorname{sgn}\bar{a}, \frac{\bar{b}^2}{2\bar{a}}\operatorname{sgn}\bar{a}\right) \\ & + k_{x5}F_s(\bar{\tau}) + k_{x6}G_s(\bar{\tau}) \end{aligned}$$

$$\begin{aligned} W_{yc}(\bar{\tau}) = {} & k_{y1}J_{cc}(\bar{\tau}, 1, 0, -\alpha, \alpha\bar{\tau}) + k_{y2}J_{cs}(\bar{\tau}, -\alpha, \alpha\bar{\tau}, 1, 0) \\ & + k_{y3}J_{cc}\left(\bar{\tau}, \operatorname{sgn}\bar{a}, \frac{\bar{b}^2}{2\bar{a}}\operatorname{sgn}\bar{a}, -\alpha, \alpha\bar{\tau}\right) \\ & + k_{y4}J_{cs}\left(\bar{\tau}, -\alpha, \alpha\bar{\tau}, \operatorname{sgn}\bar{a}, \frac{\bar{b}^2}{2\bar{a}}\operatorname{sgn}\bar{a}\right) \\ & + k_{y5}F_c(\bar{\tau}) + k_{y6}G_c(\bar{\tau}) \end{aligned}$$

$$\begin{aligned} W_{xs}(\bar{\tau}) = {} & k_{x1}J_{cs}(\bar{\tau}, 1, 0, -\alpha, \alpha\bar{\tau}) + k_{x2}J_{ss}(\bar{\tau}, 1, 0, -\alpha, \alpha\bar{\tau}) \\ & + k_{x3}J_{cs}\left(\bar{\tau}, \operatorname{sgn}\bar{a}, \frac{\bar{b}^2}{2\bar{a}}\operatorname{sgn}\bar{a}, -\alpha, \alpha\bar{\tau}\right) \\ & + k_{x4}J_{ss}\left(\bar{\tau}, \operatorname{sgn}\bar{a}, \frac{\bar{b}^2}{2\bar{a}}\operatorname{sgn}\bar{a}, -\alpha, \alpha\bar{\tau}\right) \\ & + k_{x5}F_s(\bar{\tau}) + k_{x6}G_s(\bar{\tau}) \end{aligned} \tag{37}$$

Define the integrals L_c and L_s:

$$L_c(\bar{\tau}, h_1, h_2) = \frac{1}{2}\int_o^{\bar{\tau}} \frac{\cos(h_1\xi + h_2)}{\sqrt{\bar{b}^2 + 2\bar{a}\,\xi}}\, d\xi$$

$$L_s(\bar{\tau}, h_1, h_2) = \frac{1}{2}\int_o^{\bar{\tau}} \frac{\sin(h_1\xi + h_2)}{\sqrt{\bar{b}^2 + 2\bar{a}\,\xi}}\, d\xi \tag{38}$$

Then, by well-known trigonometric identities:

$$\begin{aligned} J_{cc}(\bar{\tau}, k_1, k_2, k_3, k_4) = {} & L_c(\bar{\tau}, k_1 - k_3, k_2 - k_4) \\ & + L_c(\bar{\tau}, k_1 + k_3, k_2 + k_4) \end{aligned}$$

$$\begin{aligned} J_{cs}(\bar{\tau}, k_1, k_2, k_3, k_4) = {} & L_s(\bar{\tau}, k_1 + k_3, k_2 + k_4) \\ & - L_s(\bar{\tau}, k_1 - k_3, k_2 - k_4) \end{aligned}$$

$$J_{ss}(\bar{\tau}, k_1, k_2, k_3, k_4) = L_c(\bar{\tau}, k_1 - k_3, k_2 - k_4) - L_c(\bar{\tau}, k_1 + k_3, k_2 + k_4) \tag{39}$$

The integrals L_c and L_s can be expressed solely in terms of Fresnel Integrals C_2 and S_2

$$L_c(\bar{\tau}, h_1, h_2) = \frac{1}{2h_1}\sqrt{\pi\left|\frac{h_1}{\bar{a}}\right|} \cdot \operatorname{sgn}\frac{\bar{a}}{h_1}\left\{\cos t_2\left[C_2(t_1) - C_2(t_0)\right] - \sin t_2\left[S_2(t_1) - S_2(t_0)\right]\right\}$$

$$L_s(\bar{\tau}, h_1, h_2) = \frac{\sqrt{2\pi}}{2h_1}\sqrt{\left|\frac{h_1}{2\bar{a}}\right|} \cdot \left\{\cos t_2\left[S_2(t_1) - S_2(t_0)\right] + \sin t_2\left[C_2(t) - C_2(t_0)\right]\right\} \tag{40}$$

where

$$t_0 = \frac{h_1\bar{b}^2}{2\bar{a}}\operatorname{sgn}\frac{\bar{a}}{h_1}, \quad t_1 = h_1\left(\bar{\tau} + \frac{\bar{b}^2}{2\bar{a}}\right)\operatorname{sgn}\frac{\bar{a}}{h_1},$$

$$t_2 = \left(h_2 - \frac{h_1\bar{b}^2}{2\bar{a}}\right)\operatorname{sgn}\frac{\bar{a}}{h_1}$$

Thus the integrals J_{cc}, J_{cs} and J_{ss} are known integrals since they are explicit functions of the Fresnel Integrals.

Next, the unknown integrals F_c, F_s, G_c and G_s must be evaluated by asymptotic expansion since they cannot be expressed explicitly in terms of known functions. The asymptotic expansions of the f and g functions are

$$\pi z f(z) \sim 1 + \sum_{m=1}^{\infty} (-1)^m \frac{1 \cdot 3 \cdot \ldots (4m-1)}{(\pi z^2)^{2m}}$$

$$\pi z g(z) \sim \sum_{m=0}^{\infty} (-1)^m \frac{1 \cdot 3 \cdot \ldots (4m+1)}{(\pi z^2)^{2m+1}} \tag{41}$$

Only two terms of the expansions will be used

$$f(z) \approx \frac{1}{\pi z} - \frac{3}{\pi^3 z^5}$$

$$g(z) \approx \frac{1}{\pi^2 z^3} - \frac{15}{\pi^4 z^7} \tag{42}$$

Substitution of Eqs. (42) into the last four equations of Eqs. (36) yields the approximations for F_c, F_s, G_c and G_s:

$$F_c(\bar{\tau}) \approx \frac{1}{2\sqrt{\pi|\bar{a}|}}\left\{\left[\cos u_1 \cos_1(u_0, u_1) + \sin u_1 \sin_1(u_0, u_1)\right] - 3\left(\frac{\alpha}{2}\right)^2 \cdot \left[\cos u_1 \cos_3(u_0, u_1) = \sin u_1 \sin_3(u_0, u_1)\right]\right\}$$

$$F_s(\bar{\tau}) \approx \frac{1}{2\sqrt{\pi|\bar{a}|}}\left\{\left[\sin u_1 \cos_1(u_0, u_1) - \cos u_1 \sin_1(u_0, u_1)\right] - 3\left(\frac{\alpha}{2}\right)^2 \cdot \left[\sin u_1 \cos_3(u_0, u_1) - \cos u_1 \sin_3(u_0, u_1)\right]\right\}$$

$$G_c(\bar{\tau}) \approx \frac{\alpha}{4\sqrt{\pi|\bar{a}|}}\left\{\left[\cos u_1 \cos_2(u_0, u_1) + \sin u_1 \sin_2(u_0, u_1)\right] - 15\left(\frac{\alpha}{2}\right)^2 \cdot \left[\cos u_1 \cos_4(u_0, u_1) + \sin u_1 \sin_4(u_0, u_1)\right]\right\}$$

$$G_s(\bar{\tau}) \approx \frac{\alpha}{4\sqrt{\pi|\bar{a}|}}\left\{\left[\sin u_1 \cos_2(u_0, u_1) - \cos u_1 \sin_2(u_0, u_1)\right] - 15\left(\frac{\alpha}{2}\right)^2 \cdot \left[\sin u_1 \cos_4(u_0, u_1) - \cos u_1 \sin_4(u_0, u_1)\right]\right\}$$

where

$$u_1 = \alpha\bar{\tau} + \frac{\alpha\bar{b}^2}{2\bar{a}}, \quad u_0 = \frac{\alpha\bar{b}^2}{2\bar{a}}, \quad \alpha = \frac{1}{\sqrt{\lambda_1\lambda_2}}$$

and the definitions have been used

$$\sin_m(t_0, t_1) \equiv \int_{t_0}^{t_1} \frac{\sin s}{s^m}\, ds$$

$$\cos_m(t_0, t_1) \equiv \int_{t_0}^{t_1} \frac{\cos s}{s^m} ds \quad (43)$$

$\sin_m(t_0, t_1)$ and $\cos_m(t_0, t_1)$ can be reduced to $\sin_1(t_0, t_1)$ and $\cos_1(t_0, t_1)$ by repeated application of the formulas:

$$\cos_m(u_0, u_1) = \frac{1}{1-m}\left[u_1^{1-m} \cos u_1 - u_0^{1-m} \cos u_0\right] + \frac{1}{1-m} \sin_{m-1}(u_0, u_1)$$

$$\sin_m(u_0, u_1) = \frac{1}{1-m}\left[u_1^{1-m} \sin u_1 - u_0^{1-m} \sin u_0\right] - \frac{1}{1-m} \cos_{m-1}(u_0, u_1) \quad (44)$$

Thus

$$\cos_4(u_0, u_1) = -\frac{1}{3}\left[\frac{\cos u_1}{u_1^3} - \frac{\cos u_0}{u_0^3}\right] + \frac{1}{6}\left[\frac{\sin u_1}{u_1^2} - \frac{\sin u_0}{u_0^2}\right] + \frac{1}{6}\left[\frac{\cos u_1}{u_1} - \frac{\cos u_0}{u_0}\right] + \frac{1}{6} \sin_1(u_0, u_1)$$

$$\sin_4(u_0, u_1) = -\frac{1}{3}\left[\frac{\sin u_1}{u_1^3} - \frac{\sin u_0}{u_0^3}\right] + \frac{1}{6}\left[\frac{\cos u_1}{u_1^2} - \frac{\cos u_0}{u_0^2}\right] + \frac{1}{6}\left[\frac{\sin u_1}{u_1} - \frac{\sin u_0}{u_0}\right] - \frac{1}{6} \cos_1(u_0, u_1)$$

$$\cos_3(u_0, u_1) = -\frac{1}{2}\left[\frac{\cos u_1}{u_1^2} - \frac{\cos u_0}{u_0^2}\right] + \frac{1}{2}\left[\frac{\sin u_1}{u_1} - \frac{\sin u_0}{u_0}\right] - \frac{1}{2} \cos_1(u_0, u_1)$$

$$\sin_3(u_0, u_1) = -\frac{1}{2}\left[\frac{\sin u_1}{u_1^2} - \frac{\sin u_0}{u_0^2}\right] - \frac{1}{2}\left[\frac{\cos u_1}{u_1} - \frac{\cos u_0}{u_0}\right] - \frac{1}{2} \sin_1(u_0, u_1)$$

$$\cos_2(u_0, u_1) = -\left[\frac{\cos u_1}{u_1} - \frac{\cos u_0}{u_0}\right] - \sin_1(u_0, u_1)$$

$$\sin_2(u_0, u_1) = -\left[\frac{\sin u_1}{u_1} - \frac{\sin u_0}{u_0}\right] + \cos_1(u_0, u_1) \quad (45)$$

$\cos_1(u_0, u_1)$ and $\sin_1(u_0, u_1)$ are expressible in terms of the sine and cosine integrals

$$\cos_1(u_0, u_1) = C_i(u_1) - C_i(u_0)$$

$$\sin_1(u_0, u_1) = S_i(u_1) - S_i(u_0)$$

where

$$C_i(z) \equiv \gamma + \ln z + \int_0^z \frac{\cos t - 1}{t} dt$$

$$S_i(z) \equiv \int_0^z \frac{\sin t}{t} dt \quad (46)$$

The Sine and Cosine Integrals can be written in terms of the auxiliary functions $\bar{f}$ and $\bar{g}$:

$$C_i(z) = \bar{f}(z) \sin z - \bar{g}(z) \cos z$$

$$S_i(z) = \frac{\pi}{2} - \bar{f}(z) \cos z - \bar{g}(z) \sin z \quad (47)$$

Rational approximations for $\bar{f}$ and $\bar{g}$ are:[2]

$$\bar{f}(x) = \frac{1}{x}\left(\frac{x^8 + a_1x^6 + a_2x^4 + a_3x^2 + a_4}{x^8 + b_1x^6 + b_2x^4 + b_3x^2 + b_4}\right) + \varepsilon(x)$$

$$|\varepsilon(x)| < 5 \times 10^{-7} \quad \text{for } 1 \le x < \infty$$

$a_1 = 38.027264$ $\quad b_1 = 40.021433$

$a_2 = 265.187033$ $\quad b_2 = 322.624911$

$a_3 = 335.677320$ $\quad b_3 = 570.236280$

$a_4 = 38.102495$ $\quad b_4 = 157.105423$

$$\bar{g}(x) = \frac{1}{x^2}\left(\frac{x^8 + a_1x^6 + a_2x^4 + a_3x^2 + a_4}{x^8 + b_1x^6 + b_2x^4 + b_3x^2 + b_4}\right) + \varepsilon(x)$$

$$|\varepsilon(x)| < 3 \times 10^{-7} \quad 1 \le x < \infty$$

$$a_1 = 42.242855 \qquad b_1 = 48.196927$$

$$a_2 = 302.757865 \qquad b_2 = 482.485984$$

$$a_3 = 352.018498 \qquad b_3 = 1114.978885$$

$$a_4 = 21.821899 \qquad b_4 = 449.690326 \tag{48}$$

When the argument is large (>10) the asymptotic expansions are accurate

$$\bar{f}(z) \sim \frac{1}{z}\left(1 - \frac{2!}{z^2} + \frac{4!}{z^4} - \frac{6!}{z^6} + \dots\right)$$

$$\bar{g}(z) \sim \frac{1}{z^2}\left(1 - \frac{3!}{z^2} + \frac{5!}{z^4} - \frac{7!}{z^6} + \dots\right) \tag{49}$$

Thus, the approximate analytic solution of Eqs. (28) when ϕ_x, ϕ_y and $\phi_y\omega_x$ are small is

$$\phi_x(t) = \phi_{x0} \cos\left[\alpha\left(\frac{1}{2}\bar{a}t^2 + \bar{b}t\right)\right] + \phi_{y0} \sin\left[\alpha\left(\frac{1}{2}\bar{a}t^2 + \bar{b}t\right)\right] + u \operatorname{sgn}\bar{a}\left[W_{ys}\left(\frac{1}{2}\bar{a}t^2 + \bar{b}t\right) + W_{xc}\left(\frac{1}{2}\bar{a}t^2 + \bar{b}t\right)\right]$$

$$\phi_y(t) = \phi_{y0} \cos\left[\alpha\left(\frac{1}{2}\bar{a}t^2 + \bar{b}t\right)\right] - \phi_{x0} \sin\left[\alpha\left(\frac{1}{2}\bar{a}t^2 + \bar{b}t\right)\right] + u \operatorname{sgn}\bar{a}\left[W_{yc}\left(\frac{1}{2}\bar{a}t^2 + \bar{b}t\right) - W_{xs}\left(\frac{1}{2}\bar{a}t^2 + \bar{b}t\right)\right]$$

$$\phi_z(t) = \frac{M_z}{2I_z} t^2 + \omega_{z0}t + \phi_{z0} \tag{50}$$

where

$$\alpha = \frac{1}{\sqrt{\lambda_1\lambda_2}}$$

$$\lambda_1 = \frac{I_z - I_y}{I_x}$$

$$\lambda_2 = \frac{I_z - I_x}{I_y}$$

$$\bar{a} = \sqrt{\lambda_1\lambda_2}\,\frac{M_z}{I_z}$$

$$\bar{b} = \sqrt{\lambda_1\lambda_2}\,\omega_{z0}$$

and the symbols W_{ys}, W_{xs}, W_{yc} and W_{xs} are defined in Eqs. (37), (38), (39), (40), (43), (45), (46), (47), and (20). Useful approximations for the Fresnel, Sine and Cosine Integrals are given in Eqs. (23) - (25) and (47) - (49).

VII. Conclusion

Analytic expressions have been found for Euler's Equations of Motion and for the Eulerian Angles for both symmetric and near symmetric rigid bodies under the influence of arbitrary constant body-fixed torques. When the body is symmetric the solution of Euler's Equations of Motion is exact. Numerical studies have shown that for a typical near symmetric spacecraft the near symmetric solution is very accurate. The solution of the Eulerian Angles is much more restricted. In this case the two angles describing the orientation of the spin axis must be small. Further research will be done to determine the accuracy and range of application of these solutions.

VIII. References

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