Theory of one-dimensional acoustic wave phase conjugation

Alain Merlen and Qi Zhang
Joint European Laboratory in Magneto-acoustics (LEMAC),
Laboratoire de Mécanique de Lille, UMR CNRS 8107, USTL, cité scientifique, bât. M3,
59655 Villeneuve d'Ascq, Cedex, France
Email: alain.merlen@univ-lille1.fr

***Abstract*— This paper briefly presents the practical interest of general analytical solutions for the resonant modes in 1D active magneto acoustic phase conjugators in contact with passive media of arbitrary impedance. All results are successfully compared with direct numerical simulations. Application to conjugator optimization is underlined.**

I. INTRODUCTION

The Wave Phase Conjugation (WPC) in ultrasonics is a non-linear coupling between an incident acoustic wave and an oscillating source of energy. Recently, new technical possibilities of WPC were demonstrated particularly in acoustic imaging thanks to magnetostrictive solid materials [1]. In such experiments one spectral component of the incident wave is amplified exponentially and, by conservation of momentum, gives rise to a conjugate wave time-reversed with the same amplification. The potential applications are numerous not only in acoustic imaging but also in non destructive testing, and for higher energy, in non-intrusive surgery, hyperthermia or other medical applications. In practice, a magneto-acoustic wave conjugator is generally a cylinder of magnetostrictive material submitted to a magnetic field. The incident wave travels approximately along the axis of the conjugator, therefore the most simple theoretical model is the one-dimensional situation presented in Fig. 1. Numerical simulations [2], [3] performed with this model gave qualitative results in agreement with experiments [4]. In ultrasonics, the saturation process has been shown to be the feedback from the elastic energy toward electric energy [3]. This happens only when the stress becomes very high, therefore, the linear pumping theory hold for most of the duration of the phenomenon, that is why this theory reduces the pumping to a small harmonic oscillation around a constant value of the sound velocity of the medium. This paper focuses on the use of analytical solutions for optimization of impedance ratios.

II. THEORETICAL BACKGROUND

A. Governing equations

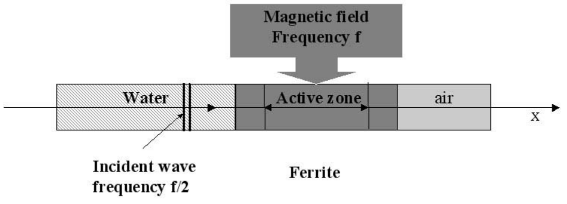

Fig. 1. Physical principle in one-dimension

The magnetostrictive material is supposed to be linearly elastic and isotropic. The physical problem is then ruled by the Navier equation. Under the assumption of one directional stress and displacement along the x axis, the only non zero stress coefficient σ_{xx} can be replaced by : $\theta = -\frac{\sigma_{xx}}{\rho_0 c}$ with ρ_0 the density of the solid and c the instantaneous longitudinal sound speed which depends on time because of the magnetostrictive excitation. Therefore Navier equation simplifies in

$$\frac{\partial v}{\partial t} + c\frac{\partial \theta}{\partial x} = 0 \tag{1}$$

where v is the displacement velocity along x axis. The time derivation of the elastic constitutive law gives the evolution of θ [2]: $\frac{\partial \theta}{\partial t} + c\frac{\partial v}{\partial x} = \frac{\theta}{c}\frac{\partial c}{\partial t}$.

In the frame of the linear pumping assumption, the active properties of the medium are given by $c^2 = c_0^2[1 + m\cos(\Omega t + \varphi)]$, where m is a small parameter ($m \ll 1$) referred as the "modulation depth" and c_0, the constant sound velocity of the material when no magnetic field is applied. The phase φ is an arbitrary constant. Introducing new variables $w_1 = v + \theta$, $w_2 = v - \theta$ yields:

$$\begin{aligned} \frac{\partial w_1}{\partial t} + c\frac{\partial w_1}{\partial x} &= -m\frac{\Omega}{4}(w_1 - w_2)\sin(\Omega t + \varphi), \\ \frac{\partial w_2}{\partial t} - c\frac{\partial w_2}{\partial x} &= m\frac{\Omega}{4}(w_1 - w_2)\sin(\Omega t + \varphi). \end{aligned} \tag{2}$$

Of course outside of the active medium $m = 0$.

B. Resonant solutions

The active zone of length L is in contact on the left and on the right with media of different acoustic impedance (Fig 1). Let $\tau_l = \frac{\rho_l c_l}{\rho_0 c_0}$ be the impedance ratio between the left (l) medium and the active zone and $\tau_r = \frac{\rho_r c_r}{\rho_0 c_0}$ the same for the right (r) edge of the active zone.

Neglecting terms of order m, guess the resonant solutions for w_1 and w_2 inside the active zone, are written:

$$w_1 = -Ae^{\Gamma t}\sin(\frac{k\alpha}{4}x + \xi_1)\sin(-\frac{k}{2}x + \frac{\Omega}{2}t + \varphi_1) \tag{3}$$

and

$$w_2 = Ae^{\Gamma t}\sin(\frac{k\alpha}{4}x + \xi_2)\sin(\frac{k}{2}x + \frac{\Omega}{2}t + \varphi_2) \tag{4}$$

where ξ_1 and ξ_2 are unknowns to be determined by the boundary conditions at $x = \pm L/2$. It can be noticed that the selected angular frequency of the incident wave is half of the pumping one. This is the parametric resonance condition. At $x \leq -L/2$, only the conjugate wave w_2 can radiate towards the left, and at $x \geq L/2$ only the direct wave w_1 exists. Taking into account the stress and velocity continuity at $x = \pm L/2$, the boundary conditions become:

$$w_1(-\frac{L}{2}, t) = \frac{(1 - \tau_l)}{(1 + \tau_l)} w_2(-\frac{L}{2}, t) \tag{5}$$

$$w_1(\frac{L}{2}, t) = \frac{(1 + \tau_r)}{(1 - \tau_r)} w_2(\frac{L}{2}, t) \tag{6}$$

Using (3) and (4) and orthogonality of trigonometric function, this yields:

$$\sin(-\frac{kL}{2} + \varphi_2 - \varphi_1) = 0, \tag{7}$$

$$\sin(-\frac{k\alpha L}{8} + \xi_1) \pm \frac{1 - \tau_l}{1 + \tau_l} \sin(-\frac{k\alpha L}{8} + \xi_2) = 0 \tag{8}$$

$$\sin(\frac{k\alpha L}{8} + \xi_1) \pm \frac{1 + \tau_r}{1 - \tau_r} \sin(\frac{k\alpha L}{8} + \xi_2) = 0 \tag{9}$$

Equations (8) and (9) provide relations between ξ_1, ξ_2 and α, one more equation is needed to solve the problem. Using again formula (3) and (4) in system (2)shows that (3) and (4) are solutions only if:

$$\gamma = \xi_2 - \xi_1 \pmod{2\pi} \tag{10}$$

where $\cos(\gamma) = \frac{8\Gamma}{m\Omega}$ and $\sin(\gamma) = \frac{2\alpha}{m}$ and Γ is the amplification rate.

Relations (10) in (8) and (9) yields:

$$\tan \xi_1 = \frac{\sin(\frac{k\alpha L}{8}) + \frac{1-\tau_l}{1+\tau_l} \sin(-\frac{k\alpha L}{8} + \gamma)}{\cos(\frac{k\alpha L}{8}) - \frac{1-\tau_l}{1+\tau_l} \cos(-\frac{k\alpha L}{8} + \gamma)}$$

and

$$\tan \xi_1 = \frac{-\sin(\frac{k\alpha L}{8}) + \frac{1+\tau_r}{1-\tau_r} \sin(\frac{k\alpha L}{8} + \gamma)}{\cos(\frac{k\alpha L}{8}) - \frac{1+\tau_r}{1-\tau_r} \cos(\frac{k\alpha L}{8} + \gamma)}.$$

Elimination of ξ_1 leads to:

$$\left(\tan(\frac{\gamma}{2})\right)^2 - (\tau_l + \tau_r) \cot(\frac{k\alpha L}{4}) \tan(\frac{\gamma}{2}) - \tau_l \tau_r = 0.$$

Solving this equation and according to the definition of γ, the value $\alpha = \alpha_s$ selected by the system is given by:

$$G(\alpha) = \frac{\sqrt{m^2 - (2\alpha)^2}}{2\alpha} - \cot\left(2 \arctan G_0(\alpha)\right) = 0 \tag{11}$$

with

$$G_0(\alpha) = \frac{1}{2}\left[(\tau_l + \tau_r) \cot \frac{k\alpha L}{4} \pm \sqrt{\left((\tau_l + \tau_r) \cot(\frac{k\alpha L}{4})\right)^2 + 4\tau_l \tau_r}\right] \tag{12}$$

Of course (11) can have multiple solutions for each set of parameters but, since the lowest α_s corresponds to the highest value of Γ, only this most amplified mode is observable.

Fig. 2 shows the total agreement between computation and predicted value by (11) for $\tau_l = 0.3$ and $\tau_r = 0.1$, with a pumping duration of 19 μs, $c_0 = 4\ 10^3 m/s$, $\Omega = 4\pi\ 10^7$, $\rho_0 = 9000 kg/m^3$.

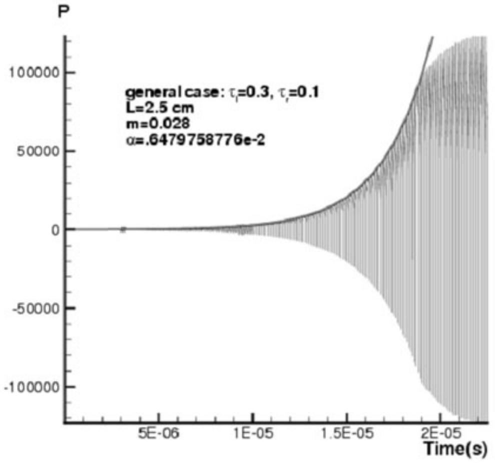

Fig. 2. Comparison theory (bold line) - numerical simulation for the stress at the left edge of the sample

III. IMPEDANCE OPTIMIZATION

Analytical solutions easily allow many analysis that would need a huge amont of computations or experiments. For example, high energy WPC applications are limited by a very bad ratio between the energy flux of the conjugate wave and the pumping energy. The analytical solution for a high value of m gives the explanation of this phenomenon. Fig. 3 shows the repartition of w_2 inside the sample in this situation for the homogeneous case. It is obvious that the wave amplitude (here W_1) has almost a node at the exit edge of the sample. This is because solution of (11) is near $\alpha_s = \frac{4\pi c_0}{\Omega L}$ leading to an amplitude modulation of wave length near $2L$. This situation can be improved adapting the impedance ratio τ_l. Fig. 4 shows the effect of such variation on the energy flux amplitude ratio. i.e: the amplitude of the energy flux at the end of the pumping and at the edge of the sample, normalized by the same quantity in the incident wave.

It is obvious that the optimum case is not the homogeneous one and that the impedance ratio can modify dramatically the energy of the conjugate wave. Of course the optimization criteria can be changed but the analytical solutions allow all the parametric studies without much effort.

An other very important result is the fact that the optimum varies with the frequency (Fig. 5) which means that the pumping frequency can be adapted at each value of τ_l (and τ_r) in order to give the best efficiency.

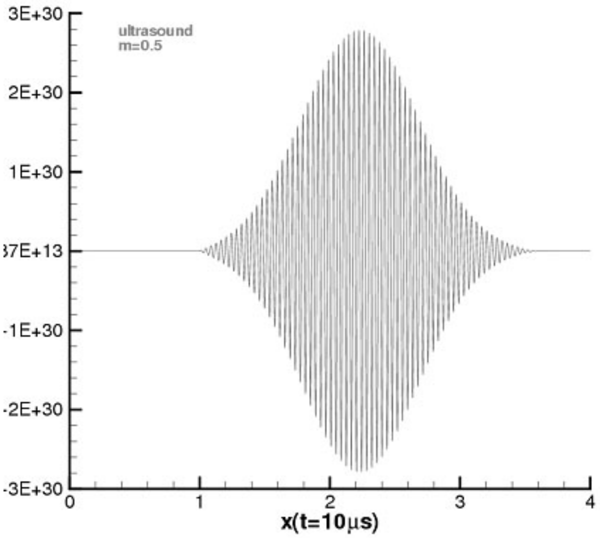

Fig. 3. Spatial distribution of W_1 for $m = 0.5$

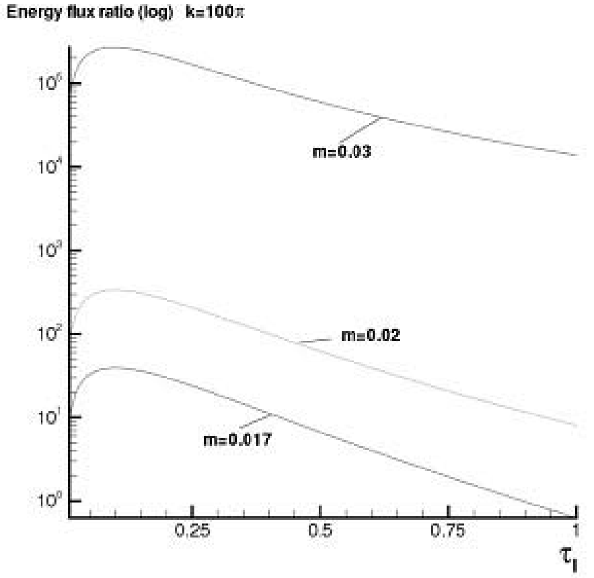

Fig. 4. Energy flux ratio versus τ_l for wave number $k = 100\pi$ and $\tau_r = 1$

IV. CONCLUSION

On the basis of the solution provided here, many properties of the WPC in active media can be deduced. The homogenous case, the only referred in the literature [5], becomes a particular case of these solutions ($\tau_l = \tau_r = 1$). Optimization of the sample becomes possible particularly by impedance adaptation or frequency tuning. Since the one dimensional model has already proven to be a good representation of real situations [2], [3]the explicit knowledge of the influence of each parameter reached here, is a great improvement for further development in WPC.

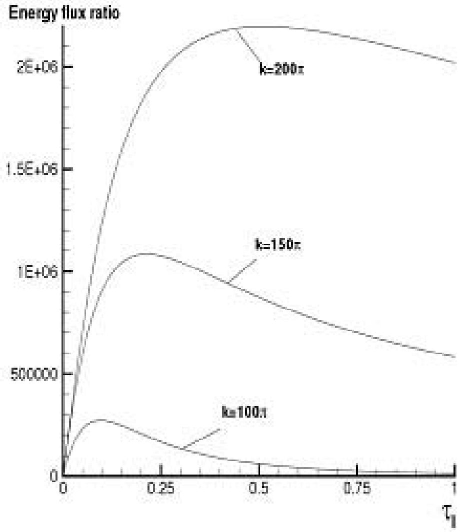

Fig. 5. Energy flux ratio versus τ_l for $m = 0.03$ and $\tau_r = 1$

REFERENCES

- [1] A. Brysev, L. Krutyansky, P. Pernod, V. Preobrazhensky, Acoustic microscope based on magneto-elastic wave phase conjugator, *Appl. Phys. Lett.* **76(21)**, 2000, pp. 3133–3135
- [2] S.Ben Khelil, A.Merlen, and al., “Numerical simulation of acoustic wave phase conjugation in active media” *J. Acoust. Soc. Am.* **109(1)**, 75–83 (2001)
- [3] Merlen, A. and Preobrazhenski, V. L. and Pernod, P. , “Supercritical parametric phase conjugation of ultrasound. Numerical simulation of non linear and non stationary modes,”*J. Acoust. Soc. Am.*, **112**, 2656-2665, (2002)
- [4] P.Pernod, V. Preobrazhensky, Parametric phase conjugation of a wide-band acoustic pulse in supercritical mode, *Appl. Phys. Lett.* **76(3)**, 2000, pp. 387–389
- [5] Bobroff, D.L. and Haus, H.A. , “Impulse response of active coupled wave systems,” *J. of Appl. Phys.*, **38**, 390-403, (1967)