Fast Computation of the Dynamic Allan Variance

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***Abstract*— The dynamic Allan variance (DAVAR) is a tool that allows to understand if the stability of an atomic clock is changing with time. Since an anomaly in the clock behavior generates a change in its stability, the DAVAR can be used to monitor the performances of a clock. In this paper we present a fast algorithm for the computation of the DAVAR, which outperforms the classical computational method.**

I. Introduction

The performances of an atomic clock change throughout its life, due to aging, cyclostationary effects such as temperature and humidity, as well as sudden breakdowns. Therefore, also its stability changes with time. The dynamic Allan variance (DAVAR) [1], [2], [3] can be used to represent the change in stability of an atomic clock, and, in general, of any precise oscillator. Consequently, the DAVAR can be used to detect and identify clock anomalies [4], [5].

A critical point of the DAVAR is its computational cost. The DAVAR is, in fact, a sliding version of the Allan variance [6]. To compute the DAVAR, at every time instant, the signal representing the clock phase deviation is truncated, and its Allan variance represents the clock stability at the given time instant. Therefore, if a signal has N samples, the computation of the DAVAR requires, in general, the evaluation of N Allan variances, which can turn in a computational burden if N is large.

By using a recursive formulation of the Allan variance it is possible to formulate the DAVAR in a recursive way [7], [8]. Such formulation can be used to develop a fast computational algorithm, whose performances are far superior to those of the classical DAVAR algorithm. In this paper we give the recursive formulation of the DAVAR, and we show the performances of the fast algorithm.

II. The Dynamic Allan Variance

The standard definition of stability for an atomic clock is the Allan variance [6], [9]

$$\sigma_y^2(\tau) = \frac{1}{2}\left\langle \left(\bar{y}(t+\tau) - \bar{y}(t)\right)^2 \right\rangle \tag{1}$$

where τ is the observation interval, the average frequency deviation is given by

$$\bar{y}(t) = \frac{1}{\tau}\int_{t-\tau}^{t} y(t')dt' \tag{2}$$

and $y(t)$ is the normalized frequency deviation. By using the phase deviation $x(t)$, defined as,

$$y(t) = \frac{dx(t)}{dt} \tag{3}$$

and Eq. (2), we can write the Allan variance as

$$\sigma_y^2(\tau) = \frac{1}{2\tau^2}\left\langle \left(x(t+\tau) - 2x(t) + x(t-\tau)\right)^2 \right\rangle \tag{4}$$

The Allan variance is usually evaluated by using the estimator

$$\hat{\sigma}_y^2[k] = \frac{1}{2k^2\tau_0^2}\frac{1}{N-2k} \tag{5}$$

$$\times \sum_{n=0}^{N-2k-1} \left(x[n+2k] - 2x[n+k] + x[n]\right)^2 \tag{6}$$

where τ_0 is the sampling time, $k = \tau/\tau_0$ is the discrete-time observation interval, and N is the available number of samples of $x[n]$. This estimator can be evaluated at the observation intervals

$$k = 1,\ 2, \ldots,\ \frac{N}{2} - 1 \tag{7}$$

where we assume N to be even.

In the bilogarithmic representation, the Allan variance of the typical clock noise is made by straight lines [10]. For a white frequency noise, for example, the Allan variance is proportional to $\tau^{-1} = (k\tau_0)^{-1}$. To prove this fact, we show in Fig. 1 a white frequency noise $y[n]$, and, in Fig. 2, the corresponding Allan deviation $\hat{\sigma}_y[k]$. The slope $-1/2$ of the Allan deviation indicates the presence of the white frequency noise. In Fig. 4 we instead show a white frequency noise whose variance increases with time. Figure 3 shows the corresponding Allan deviation. As it can be seen, the slope of the Allan deviation is still $-1/2$, and from this plot we might erroneously conclude that the clock noise is a stationary white frequency noise.

To overcome this problem, we use the dynamic Allan variance

$$\sigma_y^2[n,k] = \frac{1}{2k^2\tau_0^2}\frac{1}{N_w - 2k} \times \sum_{m=n-\frac{N_w}{2}}^{n+\frac{N_w}{2}-2k-1} E\left[\left(x[m+2k] - 2x[m+k] + x[m]\right)^2\right] \tag{8}$$

where N_w is the analysis window and E is the expected value. The DAVAR can be evaluated with the estimator

$$\hat{\sigma}_y^2[n,k] = \frac{1}{2k^2\tau_0^2}\frac{1}{N_w - 2k} \times \sum_{m=n-\frac{N_w}{2}}^{n+\frac{N_w}{2}-2k-1} \left(x[m+2k] - 2x[m+k] + x[m]\right)^2 \tag{9}$$

In Fig. 5 we show the dynamic Allan deviation (DADEV) $\hat{\sigma}_y[n,k]$ of the frequency deviation in Fig. 1. It can be seen,

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that, aside from the fluctuations due to the estimation process, the DADEV is a stationary surface, with a slope of $-1/2$ which indicates a white frequency noise. In Fig. 6 we instead show the DADEV of the frequency deviation in Fig. 3. We see that the DADEV is an increasing surface which reveals the increase in the variance of the frequency deviation. The shape of the DAVAR changes with respect to the clock anomaly, and it can hence be used as a diagnostic tool [4].

III. Fast Computational Algorithm

By using a known recursive property of the Allan variance [7], [8], it is possible to write the DAVAR at time $n+1$ as a function of the DAVAR at time n. We obtain such recursive formulation for the DAVAR estimator, Eq. (9). An identical formulation exists for the theoretical DAVAR, Eq. (8).

We start by writing the DAVAR estimator as

$$\hat{\sigma}_y^2[n,k] = \frac{1}{2k^2\tau_0^2}\frac{1}{N_w-2k}\sum_{m=n-\frac{N_w}{2}}^{n+\frac{N_w}{2}-2k-1}\Delta_k^2[m] \tag{10}$$

where $\Delta_k[m]$ is the discrete second order difference defined by

$$\Delta_k[m] = x[m+2k] - 2x[m+k] + x[m] \tag{11}$$

The DAVAR estimator at time $n+1$ is hence given by

$$\hat{\sigma}_y^2[n+1,k] = \frac{1}{2k^2\tau_0^2}\frac{1}{N_w-2k}\sum_{m=n+1-\frac{N_w}{2}}^{n+1+\frac{N_w}{2}-2k-1}\Delta_k^2[m] \tag{12}$$

and we can rewrite as

$$\hat{\sigma}_y^2[n+1,k] = \frac{1}{2k^2\tau_0^2}\frac{1}{N_w-2k}\sum_{m=n-\frac{N_w}{2}}^{n+\frac{N_w}{2}-2k-1}\Delta_k^2[m]$$
$$+\frac{1}{2k^2\tau_0^2}\frac{1}{N_w-2k}\left(\Delta_k^2[n+N_w/2-2k]-\Delta_k^2[n-N_w/2]\right)$$

We recognize the first term to be $\hat{\sigma}_y^2[n,k]$, therefore

$$\hat{\sigma}_y^2[n+1,k] = \hat{\sigma}_y^2[n,k] + \frac{1}{2k^2\tau_0^2}\frac{1}{N_w-2k} \times \left(\Delta_k^2[n+N_w/2-2k]-\Delta_k^2[n-N_w/2]\right) \tag{13}$$

which holds for $k=1,\ldots,\ N/2-1$. This relationship represents the desired recursive formulation of the DAVAR. We note that the term $\Delta_k^2[n+N_w/2-2k]$ represents the discrete difference we must add when the new sample $x[n+N_w/2+1]$ is available. Similarly, the term $\Delta_k^2[n-N_w/2]$ is the discrete difference we must subtract when the sample $x[n-N_w/2]$ is discarded. At the time n_0 at which the DAVAR computation begins, we must evaluate the Allan variance

$$\hat{\sigma}_y^2[n_0,k] = \frac{1}{2k^2\tau_0^2}\frac{1}{N_w-2k}\sum_{m=n_0-\frac{N_w}{2}}^{n_0+\frac{N_w}{2}-2k-1}\Delta_k^2[m] \tag{14}$$

This Allan variance represents the initial condition of the recursive procedure given in Eq. (13).

In Tab. 1 we show a comparison between the speed of the classical algorithm, Eq. (9), and the speed of the fast one, Eq. (13) and Eq. (14). We consider a time series with an increasing length N, and a DAVAR with a fixed window length $N_w = N/10$. The table shows the computational time obtained on an Intel Core 2 Duo processor, with a 2.53 GHz clock. It can be seen that, as N grows, the fast algorithm outperforms the classical one.

N	DAVAR [s]	Fast DAVAR [s]
10^2	2.9×10^{-3}	3.2×10^{-4}
10^3	0.28	4.8×10^{-3}
10^4	117	0.29

IV. Conclusion

The dynamic Allan variance, or DAVAR, is a tool that can be used to monitor the behavior of an atomic clock. When an anomaly occurs, in fact, the DAVAR surface changes as a function of the type of anomaly occurred. A critical point is the computational time. In this paper we develop a fast algorithm for the computation of the DAVAR. The performances of the proposed algorithm are far superior to those of the classical one. The fast algorithm can be extended to the case of missing data [11]. We will present this extension in a forthcoming paper.

The fast algorithm can be useful in a variety of situations. First, when long time series must be analyzed. Secondly, when a large number of clock must be monitored, as happens, for instance, in satellite navigation systems, where dozens of space and ground clocks must be observed and analyzed in real time. Finally, the fast algorithm can be employed when little computational power is available, such as onboard a satellite.

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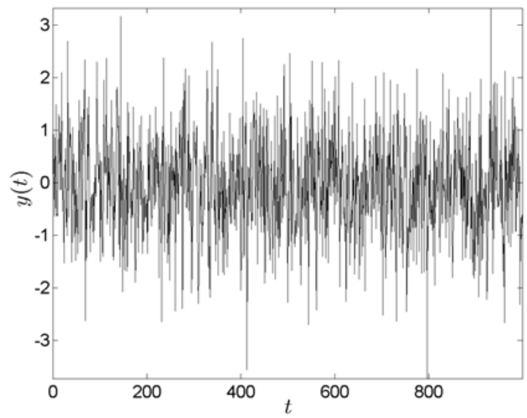

Fig. 1. Simulated white frequency noise $y[n]$. The noise is stationary.

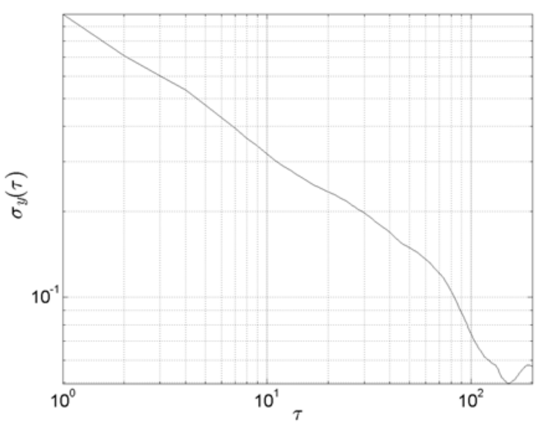

Fig. 2. Allan deviation of the frequency deviation shown in Fig. 1. The slope of the Allan deviation is $-1/2$, which correctly indicates a white frequency noise.

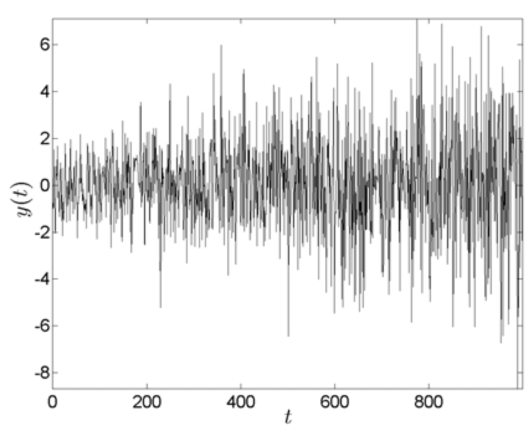

Fig. 3. A simulated white frequency noise with a variance that increases with time.

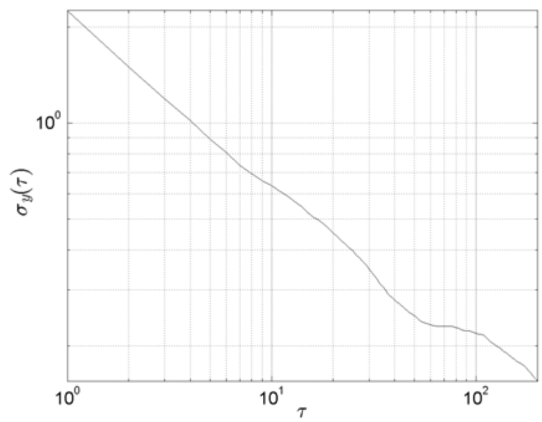

Fig. 4. Allan deviation of the clock noise shown in Fig. 3. The slope of the Allan deviation is still $-1/2$. From this plot we might erroneously conclude that the frequency deviation is a white frequency noise.

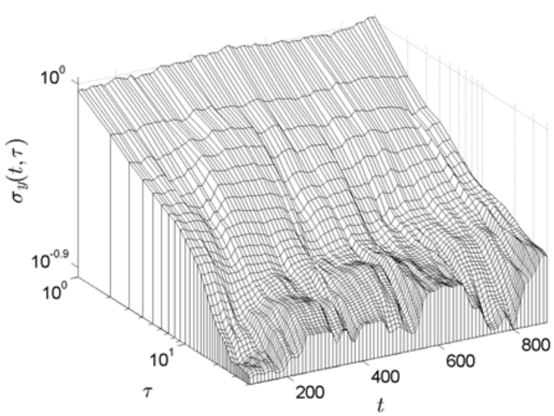

Fig. 5. Dynamic Allan deviation of the clock noise shown in Fig. 1.

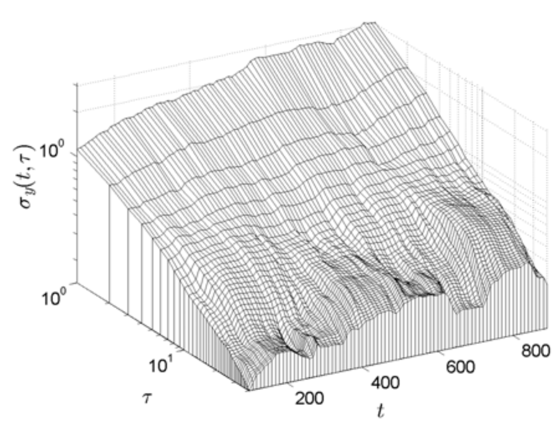

Fig. 6. Dynamic Allan deviation of the clock noise shown in Fig. 3.