ECE661 Fall 2024: Homework 1

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Due Date: Midnight, 28 Aug 2024

Late submissions will be accepted with penalty: -10 points per-late-day, up to 5 days.

Turn in typed solutions via BrightSpace. Additional instructions can be found at BrightSpace.

The following notation conventions are used for representing vector, matrix, and scalar variables. Boldface lowercase letters are used to represent vectors and boldface uppercase letters are used to represent matrices. Lowercase letters (without any special typeface) are used to represent scalars.

1. What are all the points in the representational space $\mathbb{R}^3$ that are the homogeneous coordinates of the origin in the physical space $\mathbb{R}^2$? (10 pts)

[**Answer**]

All the points $\boldsymbol{x} = k * [0, 0, 1] \quad st \quad \{k \neq 0 | k \in \mathbb{R}\}$ (**Points**: 10)

2. Are all points at infinity in the physical plane $\mathbb{R}^2$ the same? Justify your answer. (10 pts)

[**Answer**]

No. Any point at infinity is given by $\boldsymbol{x} = [u, v, 0]$. These are not the same points as the scalars u, v can be different and are indicative of the direction associated with the vanishing point. (**Points**: 10)

3. Prove that the matrix rank of a degenerate conic can never exceed 2. (10 pts)

[**Answer**]

The degenerate conic is given by $\boldsymbol{C} = \boldsymbol{m}\boldsymbol{l}^T + \boldsymbol{l}\boldsymbol{m}^T$. The conic is defined using sum of two outer product terms. We know that the matrix formed using outer product of two vectors has a rank of 1. Therefore $Rank(\boldsymbol{C}) \leq 2$.

Proof of $Rank(\boldsymbol{m}\boldsymbol{l}^T) = 1$:

$$\boldsymbol{m}\boldsymbol{l}^T = \begin{bmatrix} m_1 l_1 & m_1 l_2 & m_1 l_3 \\ m_2 l_1 & m_2 l_2 & m_2 l_3 \\ m_3 l_1 & m_3 l_2 & m_3 l_3 \end{bmatrix}$$

We can write the second and third column in terms of the first column as:

$$\boldsymbol{ml}^T[:,1] = \frac{l_2}{l_1}\boldsymbol{ml}^T[:,0]$$
$$\boldsymbol{ml}^T[:,2] = \frac{l_3}{l_1}\boldsymbol{ml}^T[:,0]$$

Therefore there is only one independent column in the $\boldsymbol{ml}^T$. Thus it's rank is 1. Finally we use the identity: $Rank(A+B) \leq Rank(A) + Rank(B)$, to prove that $Rank(\boldsymbol{C}) \leq 2$.(**Points**: 10)

4. A line in $\mathbb{R}^2$ is defined by two points. That raises the question how many points define a conic and a degenerate conic in $\mathbb{R}^2$? Justify your answer. (10 pts)

[**Answer**]

Regular conic: We need 5 points. Regular conic is a 3x3 symmetric matrix. Therefore we have 6 unknowns. Moreover, the conic is in homogeneous coordinates which means it is invariant to multiplication by a non-zero scalar. Thus, we have 5 unknowns. Now, a single point on the conic gives us one equation. Therefore, we need 5 points to estimate the conic.

(see solution-1 of Fall 2022 for rigorous proof) (**Points**: 5)

Degenrate conic: We need 3 points. We need two intersecting lines to define a degenerate conic. To define a line we need 2 points. In total we need 3 points (1 for the common intersection and 1 each for defining the line directions.). If the two lines are parallel, even then we need 3 points (2 points to define the first line which defines the line direction as well, for the second line we only need 1 point on the line as we already know the line direction) (**Points**: 5)

5. Derive in just 3 steps the intersection of two lines $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$ with $\boldsymbol{l}_1$ passing through the points $(0,0)$ and $(3,4)$, and with $\boldsymbol{l}_2$ passing through the points $(-1,4)$ and $(3/2,-1)$. How many steps would take you if the first line passed through $(-1,2)$ and $(1,-2)$? (15 pts)

[**Answer**]

For the first case we have three steps of computation.

$$\boldsymbol{l}_1 = \begin{bmatrix}0\\0\\1\end{bmatrix} \times \begin{bmatrix}3\\4\\1\end{bmatrix} = \begin{bmatrix}-4\\3\\0\end{bmatrix} \quad \text{(Step-1)}$$
$$\boldsymbol{l}_2 = \begin{bmatrix}-1\\4\\1\end{bmatrix} \times \begin{bmatrix}3/2\\-1\\1\end{bmatrix} = \begin{bmatrix}5\\5/2\\-5\end{bmatrix} = \begin{bmatrix}2\\1\\-2\end{bmatrix} \quad \text{(Step-2)}$$
$$\boldsymbol{l}_1 \times \boldsymbol{l}_2 = \begin{bmatrix}-4\\3\\0\end{bmatrix} \times \begin{bmatrix}2\\1\\-2\end{bmatrix} = \begin{bmatrix}-3\\-4\\-5\end{bmatrix} = \begin{bmatrix}0.6\\0.8\\1\end{bmatrix} \quad \text{(Step-3)}$$

The intersection point for the first case is $(0.6, 0.8)$.(**Points**: 10)

For the second case, the new line is given by:

$$\boldsymbol{l}_1 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \quad \text{(Step-1)}$$

$$\boldsymbol{l}_2 = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} \times \begin{bmatrix} 3/2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5/2 \\ -5 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \quad \text{(Step-2)}$$

We can notice that the two lines have the same slope. Therefore, the lines are parallel and do not intersect in $\mathbb{R}^2$. We took two steps to come to this conclusion.(**Points**: 5)

6. As you know, when a point $\boldsymbol{p}$ is on a conic $\boldsymbol{C}$, the tangent to the conic at that point is given by $\boldsymbol{l} = \boldsymbol{C}\boldsymbol{p}$. That raises the question as to what $\boldsymbol{C}\boldsymbol{p}$ would correspond to when $\boldsymbol{p}$ was outside the conic. As you'll see later in class, when $\boldsymbol{p}$ is outside the conic, $\boldsymbol{C}\boldsymbol{p}$ is the line that joins the two points of contact if you draw tangents to $\boldsymbol{C}$ from the point p. This line is referred to as the polar line. Now let our conic $\boldsymbol{C}$ be an ellipse that is centered at the coordinates (1, 4), with a = 1 and b = 2, where a and b, respectively, are the lengths of semi-minor and semi-major axes. For simplicity, assume that the minor axis is parallel to x-axis and the major axis is parallel to y-axis. Let $\boldsymbol{p}$ be the origin of the $\mathbb{R}^2$ physical plane. Find the intersections points of the polar line with x- and y-axes. (20 pts)

[**Answer**]

The ellipse is given by:

$$\frac{(x-1)^2}{1^2} + \frac{(y-4)^2}{2^2} = 1$$

$$\Rightarrow 4x^2 + y^2 - 8x - 8y + 16 = 0$$

$$\Rightarrow \begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & -4 \\ 0 & 1 & -4 \\ -4 & -4 & 16 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$$

$$\boldsymbol{C} = \begin{bmatrix} 4 & 0 & -4 \\ 0 & 1 & -4 \\ -4 & -4 & 16 \end{bmatrix}$$

(**Points**: 10 till calculation of C)

The x-axis (y=0) is given by $\boldsymbol{l}_1 = [0, 1, 0]^T$. And the y-axis (x=0) is given by $\boldsymbol{l}_2 = [1, 0, 0]^T$.

The polar line is given by $\boldsymbol{l} = \boldsymbol{C}\boldsymbol{p}$. For origin, the point is given by $\boldsymbol{p} = [0, 0, 1]^T$ and the polar line is given by:

$$
\begin{aligned}
\boldsymbol{l} &= \boldsymbol{C}\boldsymbol{p} \\
&= \begin{bmatrix} 4 & 0 & -4 \\ 0 & 1 & -4 \\ -4 & -4 & 16 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ -4 \\ 16 \end{bmatrix} \equiv \begin{bmatrix} -1 \\ -1 \\ 4 \end{bmatrix} \\
\boldsymbol{l} \times \boldsymbol{l}_1 &= \begin{bmatrix} -1 \\ -1 \\ 4 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \\ -1 \end{bmatrix} \equiv \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} \\
\boldsymbol{l} \times \boldsymbol{l}_2 &= \begin{bmatrix} -1 \\ -1 \\ 4 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix}
\end{aligned}
$$

From above equations, we have the following:

(a) Intersection of the polar line from origin to the ellipse with the x-axis is at the point (4, 0)

(b) Intersection of the polar line from origin to the ellipse with the t-axis is at the point (0, 4)

(**Points**: 10)

7. Lets say you are designing an arcade game where the user has to aim at random triangles displayed on the screen as shown in Fig. 1 . The user is sitting at origin and has a laser gun to aim at the incoming triangles. Given the aiming angle chosen by the user α along with the coordinates of the vertices of the triangle $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$, how would you quickly compute if the user's aim is correct? Solve for the instance shown in the figure (25 pts)

   Hint:

   (a) Convert aiming angle to a line representation in homogeneous coordinates.

   (b) Check if the aiming line is intersecting with **any** of the three **line-segments** of the triangle. Here, in addition to computing the intersection point, you will need to check if the intersection point lies between the two points defining the line segment.

[**Answer**]

The aiming angle/line is a line joining the origin and any arbtrary point $(cos(\theta), sin(\theta))$. This line is given by:

$$
\boldsymbol{l} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} cos(\theta) \\ sin(\theta) \\ 1 \end{bmatrix} = \begin{bmatrix} -sin(\theta) \\ cos(\theta) \\ 0 \end{bmatrix}
$$

For $\theta = 45^\circ$

$$
\boldsymbol{l} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \equiv \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}
$$

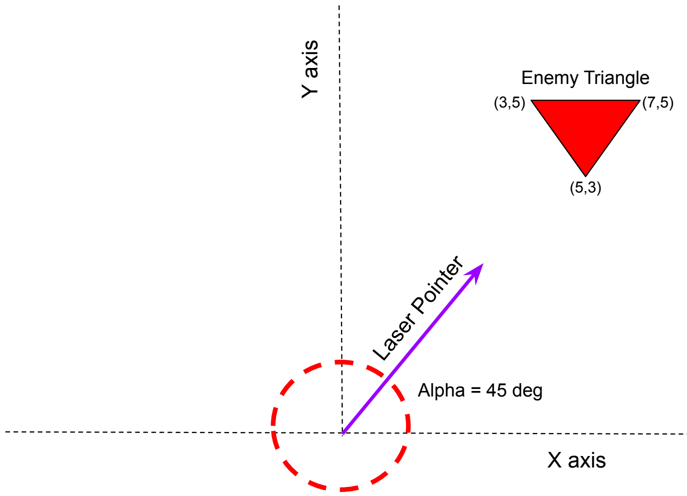

Figure 1: Laser aim game: The enemy triangle is moving downward (i.e. along negative Y-axis) and vanishes after crossing the X-axis.

(**Points**: 5)

The three lines for the triangle are as follows:

$$\boldsymbol{l}_{12} = \boldsymbol{v}_1 \times \boldsymbol{v}_2 = \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix} \times \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -16 \end{bmatrix} \equiv \begin{bmatrix} 1 \\ 1 \\ -8 \end{bmatrix}$$

$$\boldsymbol{l}_{23} = \boldsymbol{v}_2 \times \boldsymbol{v}_3 = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix} \times \begin{bmatrix} 7 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 4 \end{bmatrix} \equiv \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$$

$$\boldsymbol{l}_{31} = \boldsymbol{v}_3 \times \boldsymbol{v}_1 = \begin{bmatrix} 7 \\ 5 \\ 1 \end{bmatrix} \times \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -4 \\ 20 \end{bmatrix} \equiv \begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix}$$

(**Points**: 5)

The intersection of the aiming line with the above three lines is as follows:

$$\boldsymbol{x}_1 = \boldsymbol{l} \times \boldsymbol{l}_{12} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ -8 \end{bmatrix} = \begin{bmatrix} -8 \\ -8 \\ -2 \end{bmatrix} \equiv \begin{bmatrix} 4 \\ 4 \\ 1 \end{bmatrix}$$

$$\boldsymbol{x}_2 = \boldsymbol{l} \times \boldsymbol{l}_{23} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ 0 \end{bmatrix} \equiv \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\boldsymbol{x}_3 = \boldsymbol{l} \times \boldsymbol{l}_{31} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix} = \begin{bmatrix} -5 \\ -5 \\ -1 \end{bmatrix} \equiv \begin{bmatrix} 5 \\ 5 \\ 1 \end{bmatrix}$$

(**Points**: 5)

To check if the intersection point ($\boldsymbol{p}$) lies between the two points ($\boldsymbol{a}, \boldsymbol{b}$) defining a line segment we have two ways to do so:

1. **Checking Bounds**: If the x and y coordinates of the $\boldsymbol{p}$ lie between the x and y coordinates of the two points. More formally:

$$\text{Intersection} = \begin{cases} \text{True} & (min(x_a, x_b) \leq x_p \leq max(x_a, x_b)) \text{ and } (min(y_a, y_b) \leq y_p \leq max(y_a, y_b)) \\ \text{False} & \text{Elsewhere} \end{cases}$$

2. **Checking for partition of unity**: Here we start with assuming that the point $\boldsymbol{p}$ partitions the line segment in the ratio $\lambda : 1 - \lambda$ as shown in Fig. 2. We then calculate λ as follows:

$$x_p = \lambda x_b + (1 - \lambda)x_a \text{ and } y_p = \lambda y_b + (1 - \lambda)y_a$$
$$\Rightarrow \lambda = \frac{x_p - x_a}{x_b - x_a} = \frac{y_p - y_a}{y_b - y_a}$$

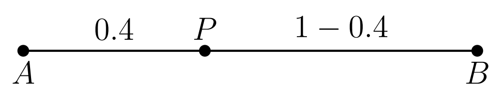

Figure 2: Partition of unity example with $\lambda = 0.4$

If $\lambda \in [0, 1]$ then the point $\boldsymbol{p}$ is on the line segment.

NOTE: **The above two ways are both acceptable solution**

Using the above logic we check for the intersection points as follows:

1. For $\boldsymbol{x}_1 : (3 \leq 4 \leq 5)$ and $(3 \leq 4 \leq 5)$ and $\lambda = 0.5$. Yes it is on the line segment.
2. For $\boldsymbol{x}_2$: This is an ideal point, which means the lines $\boldsymbol{l}$ and $\boldsymbol{l}_{23}$ are parallel. Furthermore, they are not the same lines. Therefore, the intersection point does not lie on the line segment.
3. For $\boldsymbol{x}_3 : (3 \leq 5 \leq 7)$ and $(5 \leq 5 \leq 5)$ and $\lambda = 0.5$. Yes it is on the line segment.

From above, we can say that the instance shown in the figure is a positive instance i.e. a correct aim.

(**Points**: 5)

Extra Credit (20 Pts)

Write python code to solve the problem-7 for any random triangle and random aiming angle. Show plots of a few instances with positive aim and negative aim. Feel free to constrain the triangle size and position as per your choice but the triangle is always pointing down.

[**Answer**]

Judge based on the plots and explanation of their solution. (**Points**: 20)