ECE661 Fall 2024: Homework 1

Alexandre Olivé Pellicer

aolivepe@purdue.edu

1. **What are all the points in the representational space R^3 that are the homogeneous coordinates of the origin in the physical space R^2?**

   The origin in R^2 is $\begin{pmatrix}0\\0\end{pmatrix}$. Its Homogeneous Coordinates (HC) are $\begin{pmatrix}0\\0\\1\end{pmatrix}$. Since $k \cdot \begin{pmatrix}0\\0\\1\end{pmatrix}$ $with\ k \in R\ k \neq 0$ corresponds to the same point in the physical space R^2, the homogeneous coordinates of the origin in the physical space R^2 can be written as $\begin{pmatrix}0\\0\\k\end{pmatrix}$ $with\ k \in R\ k \neq 0$.

2. **Are all points at infinity in the physical plane R^2 the same? Justify your answer.**
   No, not all points at infinity in the physical plane R^2 are the same. Points at infinity, also known as Ideal Points, are of the form $\begin{pmatrix}u\\v\\0\end{pmatrix}$. Two ideal points $x_1 = \begin{pmatrix}u_1\\v_1\\0\end{pmatrix}$ and $x_2 = \begin{pmatrix}u_2\\v_2\\0\end{pmatrix}$ approach infinity in different directions controlled by the values of u and v. In fact, if $\frac{u_1}{v_1} \neq \frac{u_2}{v_2}$, the directions towards infinity will be different and, therefore, even though they are points at infinity in the physical space R^2, x_1 and x_2 will not be the same.

   Alternatively, all ideal points form a straight line in R^2. Therefore, they are not the same. Given any two ideal points $\begin{pmatrix}u_1\\v_1\\0\end{pmatrix}$ and $\begin{pmatrix}u_2\\v_2\\0\end{pmatrix}$. The line that passes through these two points is

   $$l = \begin{pmatrix}u_1\\v_1\\0\end{pmatrix} X \begin{pmatrix}u_2\\v_2\\0\end{pmatrix} = k \cdot \begin{pmatrix}0\\0\\1\end{pmatrix} \ with\ k \in R\ k \neq 0$$

   by keeping in mind the notation of equivalence classes in the representation space R^3. We can see that being independent of (u_1, v_1) and (u_2, v_2) parameters, this line remains the same for all pair of ideal points.

3. **Prove that the matrix rank of a degenerate conic can never exceed 2.**
   A degenerate conic C can be expressed as $C = l \cdot m^T + m \cdot l^T$ where l and m are the HC representation of the two intersecting lines that we get by slicing the

double cones with a plane that passes through the axis of the double cones. See that $l \cdot m^T$ and $m \cdot l^T$ are outer products and that the rank of outer product matrixes is always 1 since every column of the resulting outer product matrix is a constant times the first column.

By the subadditivity property of matrix ranks we know that $\text{rank(A + B)} \leq \text{rank(A)} + \text{rank(B)}$. Thus, $\text{rank(C)} \leq \text{rank}(\, l \cdot m^T) + \text{rank}(m \cdot l^T) = 1 + 1 = 2$

4. **A line in R^2 is defined by two points. That raises the question how many points define a conic and a degenerate conic in R^2 ? Justify your answer.**

A conic is defined by 5 points in R^2. The implicit form for a conic is:

$$a \cdot x^2 + b \cdot x \cdot y + c \cdot y^2 + d \cdot x + e \cdot y + f = 0$$

Following the argument presented in Hartley & Zisserman's "Multiple View Geometry in Computer Vision" (2nd edition, page 5), we can see that a conic is defined by 5 points because if we count the number of coefficients of x and y terms in the implicit form for a conic we get 5.

We could do the same for the implicit form of a line:

$$a \cdot x + b \cdot y + c = 0$$

where we can count 2 coefficients of x and y terms. Therefore, a line in R^2 is defined by two points.

A degenerate conic C can be expressed as $C = l \cdot m^T + m \cdot l^T$ where l and m are the HC representation of the two intersecting lines that we get by slicing the double cones with a plane that passes through the axis of the double cones. Since a degenerate conic can be expressed as two intersecting lines and lines are defined with 2 points, degenerate conics can be defined with 4 points in R^2. Note that in this case the intersection point wouldn't be one of the points provided to define the degenerate conic. In case the intersection point is given, we would only need two more points (one corresponding to each line) to define the degenerate conic resulting in a total of 3 points in R^2.

5. **Derive in just 3 steps the intersection of two lines l1 and l2 with l1 passing through the points (0, 0) and (3, 4), and with l2 passing through the points (−1, 4) and (3/2, −1). How many steps would take you if the first line passed through (−1, 2) and (1, −2)?**

$$l_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} X \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \\ 0 \end{pmatrix}$$

$$l_2 = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} X \begin{pmatrix} 3/2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 5/2 \\ -5 \end{pmatrix}$$

$$Intersection = \begin{pmatrix} -4 \\ 3 \\ 0 \end{pmatrix} X \begin{pmatrix} 5 \\ 5/2 \\ -5 \end{pmatrix} = \begin{pmatrix} -15 \\ -20 \\ -25 \end{pmatrix} \rightarrow \begin{pmatrix} \frac{-15}{-25} \\ \frac{-20}{-25} \\ \frac{-25}{-25} \end{pmatrix} = \begin{pmatrix} 0.6 \\ 0.8 \\ 1 \end{pmatrix}$$

Therefore, the intersection point is $\begin{pmatrix} 0.6 \\ 0.8 \end{pmatrix}$.

If the first line passed through (−1, 2) and (1, −2):

$$l_1 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} X \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix}$$

$$l_2 = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} X \begin{pmatrix} 3/2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 5/2 \\ -5 \end{pmatrix}$$

Knowing that the implicit representation of a line is of the form:

$$a \cdot x + b \cdot y + c = 0$$

And its explicit representation is of the form:

$$y = -\frac{a}{b} \cdot x - \frac{c}{b} = m \cdot x + n$$

Where m indicates the slope of the lines. We see that the slope of l_1 ($m = -\frac{4}{2} = 2$) and l_2 ($m = -\frac{5}{2.5} = 2$) are the same. Therefore, l_1 and l_2 are parallel meaning that the intersection point between l_1 and l_2 will be at infinity. Nevertheless, not all points at infinity are the same so we still need to compute the intersection point between l_1 and l_2 using the cross product:

$$Intersection = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} X \begin{pmatrix} 5 \\ 5/2 \\ -5 \end{pmatrix} = \begin{pmatrix} -10 \\ 20 \\ 0 \end{pmatrix}$$

To sum up, if the first line passed through (−1, 2) and (1, −2) we still need 3 steps to compute the intersection between l_1 and l_2.

6. **As you know, when a point p is on a conic C, the tangent to the conic at that point is given by l = Cp. That raises the question as to what Cp would correspond to when p was outside the conic. As you'll see later in class, when p is outside the conic, Cp is the line that joins the two points of contact if you draw tangents to C from the point p. This line is referred to as the polar line. Now let our conic C be an ellipse that is centered at the coordinates (1, 4), with a = 1 and b = 2, where a and b, respectively, are the lengths of semi-minor and semi-major axes. For simplicity, assume that the minor axis is parallel to x-axis and the major axis is parallel to y-axis. Let p be the origin of the R^2 physical plane. Find the intersections points of the polar line with x- and y-axes.**

   The equation of the ellipse can be written as:

$$\frac{(x-1)^2}{(1)^2} + \frac{(y-4)^2}{2^2} = 1$$

Which can be written as:

$$x^2 + \frac{1}{4} \cdot y^2 - 2x - 2y + 4 = 0$$

Which can be written as:

$$C = \begin{bmatrix} 1 & 0 & -1 \\ 0 & \frac{1}{4} & -1 \\ -1 & -1 & 4 \end{bmatrix}$$

Since p is the origin of the R^2 physical plane, the HC representation is given by $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Thus, we can compute the polar line by

$$l = Cp = \begin{bmatrix} 1 & 0 & -1 \\ 0 & \frac{1}{4} & -1 \\ -1 & -1 & 4 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 4 \end{pmatrix}$$

The HC representation of the x axis is given by $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and the HC representation of the y axis is given by $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Therefore, we can compute:

Intersection between l and x-axis:

$$\begin{pmatrix} -1 \\ -1 \\ 4 \end{pmatrix} X \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ 0 \\ -1 \end{pmatrix} \rightarrow \text{In 2D space: } \begin{pmatrix} \frac{-4}{-1} \\ \frac{0}{-1} \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

Intersection between l and y-axis:

$$\begin{pmatrix} -1 \\ -1 \\ 4 \end{pmatrix} X \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix} \rightarrow \text{In 2D space: } \begin{pmatrix} \frac{0}{1} \\ \frac{4}{1} \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$$

To sum up, the intersection between the polar line with the x-axis is at $\begin{pmatrix} 4 \\ 0 \end{pmatrix}$ and the intersection between the polar line with the y-axis is at $\begin{pmatrix} 0 \\ 4 \end{pmatrix}$

7. **Lets say you are designing an arcade game where the user has to aim at random triangles displayed on the screen as shown in Fig. 1 . The user is sitting at origin and has a laser gun to aim at the incoming triangles. Given the aiming angle chosen by the user α along with the coordinates of the vertices of the triangle v1, v2, v3, how would you quickly compute if the user's aim is correct? Solve for the instance shown in the figure**

The explicit representation of a line is:

$$y = m \cdot x + c$$

We will use this expression to represent the laser pointer. Since the user is sitting in the origin, we can set c = 0. Since the information we have is the angle (α) of the laser pointer with respect to the x axis, m can be computed as $tan(\alpha)$. Thus, the resulting explicit representation looks like:

$$y = tan(\alpha) \cdot x + 0$$

And the implicit representation would be:

$$tan(\alpha) \cdot x - y + 0 = 0$$

From this implicit representation we can write the parameter vector. Since in our case $\alpha = 45$:

$$l = \begin{pmatrix} tan(45) \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

Now we will want to compute the line-segments between the vertices of the triangle. We do it by computing the cross product of the HC of the vertices:

$$ls_1 = \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} X \begin{pmatrix} 5 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -16 \end{pmatrix}$$

$$ls_2 = \begin{pmatrix} 5 \\ 3 \\ 1 \end{pmatrix} X \begin{pmatrix} 7 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 4 \end{pmatrix}$$

$$ls_3 = \begin{pmatrix} 7 \\ 5 \\ 1 \end{pmatrix} X \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -4 \\ 20 \end{pmatrix}$$

Now we will compute the intersection points between ls_1, ls_2, ls_3 and l:

$$Intersection\ ls_1 and\ l = \begin{pmatrix} 2 \\ 2 \\ -16 \end{pmatrix} X \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -16 \\ -16 \\ -4 \end{pmatrix}$$

$$Intersection\ ls_2 and\ l = \begin{pmatrix} -2 \\ 2 \\ 4 \end{pmatrix} X \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 0 \end{pmatrix}$$

$$Intersection\ ls_3 and\ l = \begin{pmatrix} 0 \\ -4 \\ 20 \end{pmatrix} X \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 20 \\ 20 \\ 4 \end{pmatrix}$$

We convert the HC from the 3D space to the 2D space. We get:

$$Intersection\ ls_1 andl = \begin{pmatrix} \frac{-16}{-4} \\ \frac{-16}{-4} \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

$$Intersection\ ls_2 andl = \begin{pmatrix} \frac{4}{0} \\ \frac{4}{0} \end{pmatrix} \rightarrow \text{Intersection at infinite (parallel lines)}$$

$$Intersection\ ls_3 and l = \begin{pmatrix} \frac{20}{4} \\ \frac{20}{4} \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$$

Now we have to check if the intersection points fall in between the two vertices. If one of the intersection points fall in between its corresponding vertices, the triangle will be touched.

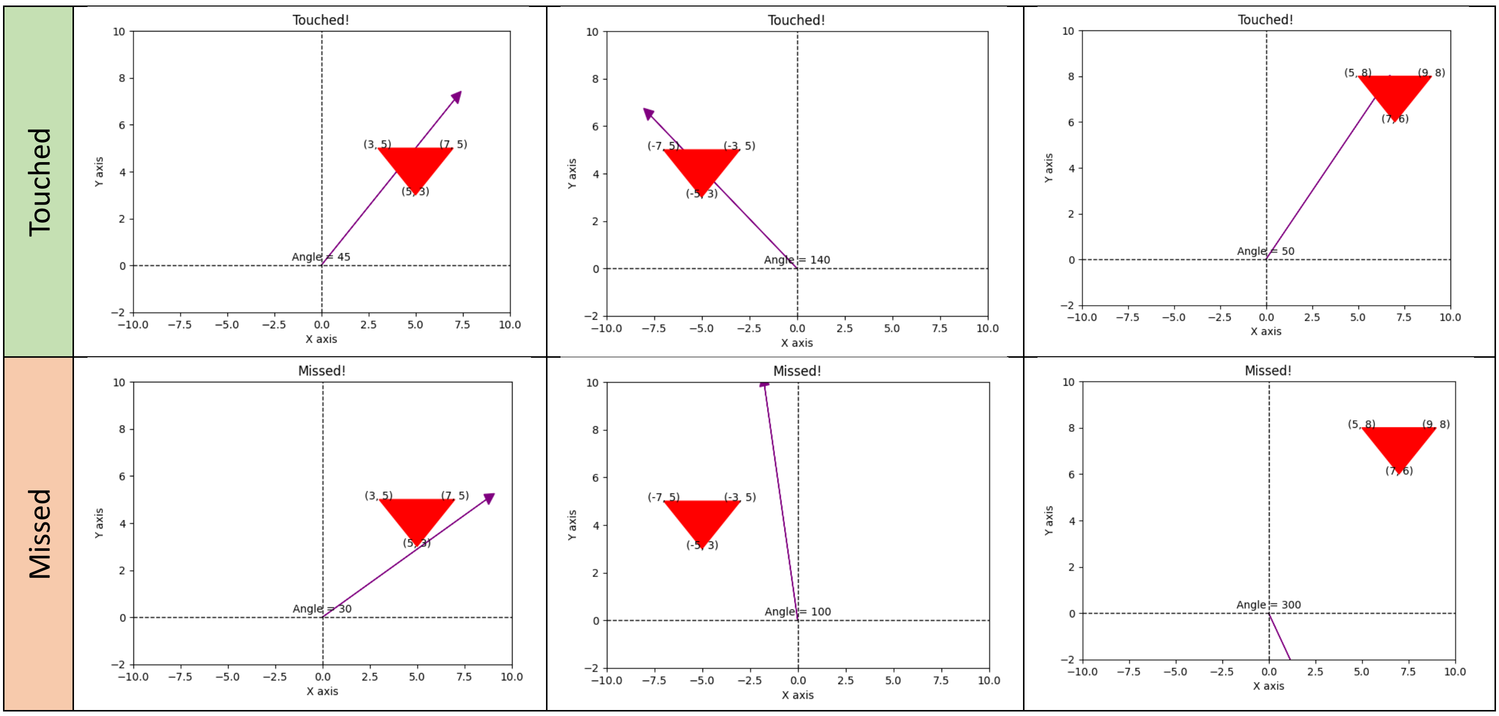
Given 3 points in the same line A(x_1, y_1), B(x_2, y_2) and C(x_3, y_3), C will be in between A and B if:

$$\min(x_1, x_2) \leq x_3 \leq \max(x_1, x_2) \text{ and } \min(y_1, y_2) \leq y_3 \leq \max(y_1, y_2)$$

In our case, just by checking the intersection point between $ls_1 and l$ we see that the triangle is touched since the above condition is true:

$$\min(3, 5) \leq 4 \leq \max(3, 5) \text{ and } \min(5, 3) \leq 4 \leq \max(5, 3)$$

8. **Extra credit: Write python code to solve problem 7 for any random triangle and random aiming angle. Show plots of a few instances with positive aim and negative aim. Feel free to constrain the triangle size and position as per your choice but the triangle is always pointing down.**



Python code:

```
import random
import math
import matplotlib.pyplot as plt
import numpy as np

#Set angle
# angle  = math.radians(random.randint(0, 180))
angle = 50

# Parameter vector of laser pointer
l = [math.tan(math.radians(angle)), -1, 0]

#Set vertices
v_1 = (5, 8)
v_2 = (9, 8)
v_3 = (7, 6)

triangle = [[v_1[0], v_1[1], 1], [v_2[0], v_2[1], 1], [v_3[0], v_3[1], 1]]
status = "Missed!"

for i in range(3):
    j = (i + 1)%3
    #Calculate line-segment between 2 vertices of the triangle
    ls_1 = np.cross(triangle[i], triangle[j])

    #Calculate intersection point between laser pointer and line-segment
    p_1 = np.cross(l, ls_1)

    #Get the intersection point in the 2D physical plain
    p_1_2D = [p_1[0]/p_1[2], p_1[1]/p_1[2]]

    #Check if the intersection point is in between the two vertices of the
triangle
         if    min(triangle[i][0],    triangle[j][0])<=    p_1_2D[0]    <=
max(triangle[i][0],      triangle[j][0])      and      min(triangle[i][1],
triangle[j][1])<= p_1_2D[1] <= max(triangle[i][1], triangle[j][1]):
        status = "Touched!"

# Create a figure and axis
fig, ax = plt.subplots()

# Draw the X and Y axes
ax.axhline(0, color='black', linewidth=1, linestyle='--')  # X axis
ax.axvline(0, color='black', linewidth=1, linestyle='--')  # Y axis

# Draw the laser pointer line (at 45 degrees)
length = 10  # length of the laser pointer
```

```
x_end = length * np.cos(np.radians(angle))
y_end = length * np.sin(np.radians(angle))
ax.arrow(0, 0, x_end, y_end, head_width=0.5, head_length=0.5, fc='purple',
ec='purple')

# Draw the enemy triangle
triangle = plt.Polygon([v_1, v_2, v_3], color='red')
ax.text(v_1[0], v_1[1], f'{v_1}', fontsize=10, color='black', ha='center')
ax.text(v_2[0], v_2[1], f'{v_2}', fontsize=10, color='black', ha='center')
ax.text(v_3[0], v_3[1], f'{v_3}', fontsize=10, color='black', ha='center')
ax.add_patch(triangle)

# Label angle
ax.text(0, 0.2, f'Angle = {angle}', fontsize=10, color='black',
ha='center')

# Set the limits and labels
ax.set_xlim(-10, 10)
ax.set_ylim(-2, 10)
ax.set_xlabel('X axis')
ax.set_ylabel('Y axis')
ax.set_title(f'{status}')

# Show the plot
plt.show()
```