ECE661 Fall 2024: Homework 1

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Question 1:

Given an arbitrary point in the representational space $\mathbb{R}^3$, $(x_1, x_2, x_3)^T$, the homogeneous coordinate representation in the physical space $\mathbb{R}^2$ is $(\frac{x_1}{x_3}, \frac{x_2}{x_3})^T$. At the origin, the point in the physical space can be expressed as $(\frac{0}{x_3}, \frac{0}{x_3})^T$. Thus, the point in the representational space $\mathbb{R}^3$ that are homogeneous coordinates of the origin is $(0, 0, x_3)^T$, or $(0, 0, k)^T$ where $k \in \mathbb{R}$, $k \neq 0$

Question 2:

Points at infinity are also called ideal points, where the last coordinate is equal to 0. $(u_1, u_2, 0)^T$ and $(w_1, w_2, 0)^T$ are both points at infinity and they approach infinity along a specific direction controlled by the pair $(u_1, u_2)^T$ and $(w_1, w_2)^T$. Values in these pairs are arbitrary and do not have to be equal. Thus, not all points at infinity in the physical plane are the same.

Question 3:

The degenerate conic can be represented by the equation: $C = lm^T + ml^T$. lm^T and ml^T are both outer products and the rank of an outer product matrix is always equal to 1. When two rank 1 matrix are summed, the resulting rank cannot exceed 2. When the degenerate conic has a rank of 2, it is represented as two intersecting lines. In the case where the two intersecting lines are superimposed, the rank of C will be 1.

Question 4:

The algebraic form of a line is $ax + by + c = 0$. This equation has two coefficients corresponding to the x y terms. Thus, a line in $\mathbb{R}^2$ can be represented by two points. Now, if we look at the implicit equation for a conic, $ax^2 + bxy + cy^2 + dx + ey + f = 0$, it has five coefficients corresponding to the x y terms. Thus, a conic is defined by five points. A degenerate conic in $\mathbb{R}^2$ is two intersecting lines that share a common point at the intersection. Thus, the degenerate conic can be defined by three points.

Question 5:

Part 1

Step 1: Determine l_1

$$l_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix}$$

Step 2: Determine l_2

$$l_2 = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} \times \begin{bmatrix} 3/2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5/2 \\ -5 \end{bmatrix}$$

Step 3: Determine the intersection of l_1 and l_2

$$P_{1,2} \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix} \times \begin{bmatrix} 5 \\ 5/2 \\ -5 \end{bmatrix} = \begin{bmatrix} -15 \\ -20 \\ -25 \end{bmatrix}$$

Part 2:

Step 1: Determine the line passing through $(-1, 2)$ and $(1, -2)$

$$l_1' = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}$$

Step 2: Determine the intersection of the line passing through $(-1, 2)$ and $(1, -2)$ and l_2:

$$P_{1',2} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} \times \begin{bmatrix} 5 \\ 5/2 \\ -5 \end{bmatrix} = \begin{bmatrix} -10 \\ 20 \\ 0 \end{bmatrix}$$

Question 6:

The algebraic representation of an ellipse is:

$$(x-1)^2 + \frac{(y-4)^2}{4} = 1$$

Rewriting the equation in the implicit form for a conic:

$$x^2 + \frac{y^2}{4} - 2x - 2y + 4 = 0$$

Rewriting as a vector-matrix product:

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & \frac{1}{4} & -1 \\ -1 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

The polar line can be found using the coefficient matrix:

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & \frac{1}{4} & -1 \\ -1 & -1 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 4 \end{bmatrix}$$

Intersection with the x-axis:

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$$

Intersection with the y-axis:

$$\begin{bmatrix} -1 \\ -1 \\ 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix}$$

Question 7:

(a) Convert aiming angle to line:

$$\begin{bmatrix} cos(\alpha) \\ sin(\alpha) \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} sin(\alpha) \\ -cos(\alpha) \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \\ 0 \end{bmatrix} \equiv \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

Line of segment 1:

$$\begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix} \times \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ 16 \end{bmatrix} \equiv \begin{bmatrix} -1 \\ -1 \\ 8 \end{bmatrix}$$

Line of segment 2:

$$\begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix} \times \begin{bmatrix} 7 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 4 \end{bmatrix} \equiv \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

Line of segment 3:

$$\begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix} \times \begin{bmatrix} 7 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ -20 \end{bmatrix} \equiv \begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix}$$

(b) Intersection of aiming line with line segments:

Line segment 1:

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \times \begin{bmatrix} -1 \\ -1 \\ 8 \end{bmatrix} = \begin{bmatrix} -8 \\ -8 \\ -2 \end{bmatrix} \equiv \begin{bmatrix} 4 \\ 4 \\ 1 \end{bmatrix}$$

The aiming line intersects the line of segment 1 at point $(4, 4)$, which falls on the line segment connecting the points $(5, 3)$ and $(3, 5)$.

Line segment 2:

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \times \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ 0 \end{bmatrix} \equiv \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

The zero in the last term means that the aiming line and segment two are parallel. Thus, the aiming line will intersect segment 2 only if all points on segment 2 have equal x and y coordinates, which is not the case.

Line segment 3:

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 1 \end{bmatrix}$$

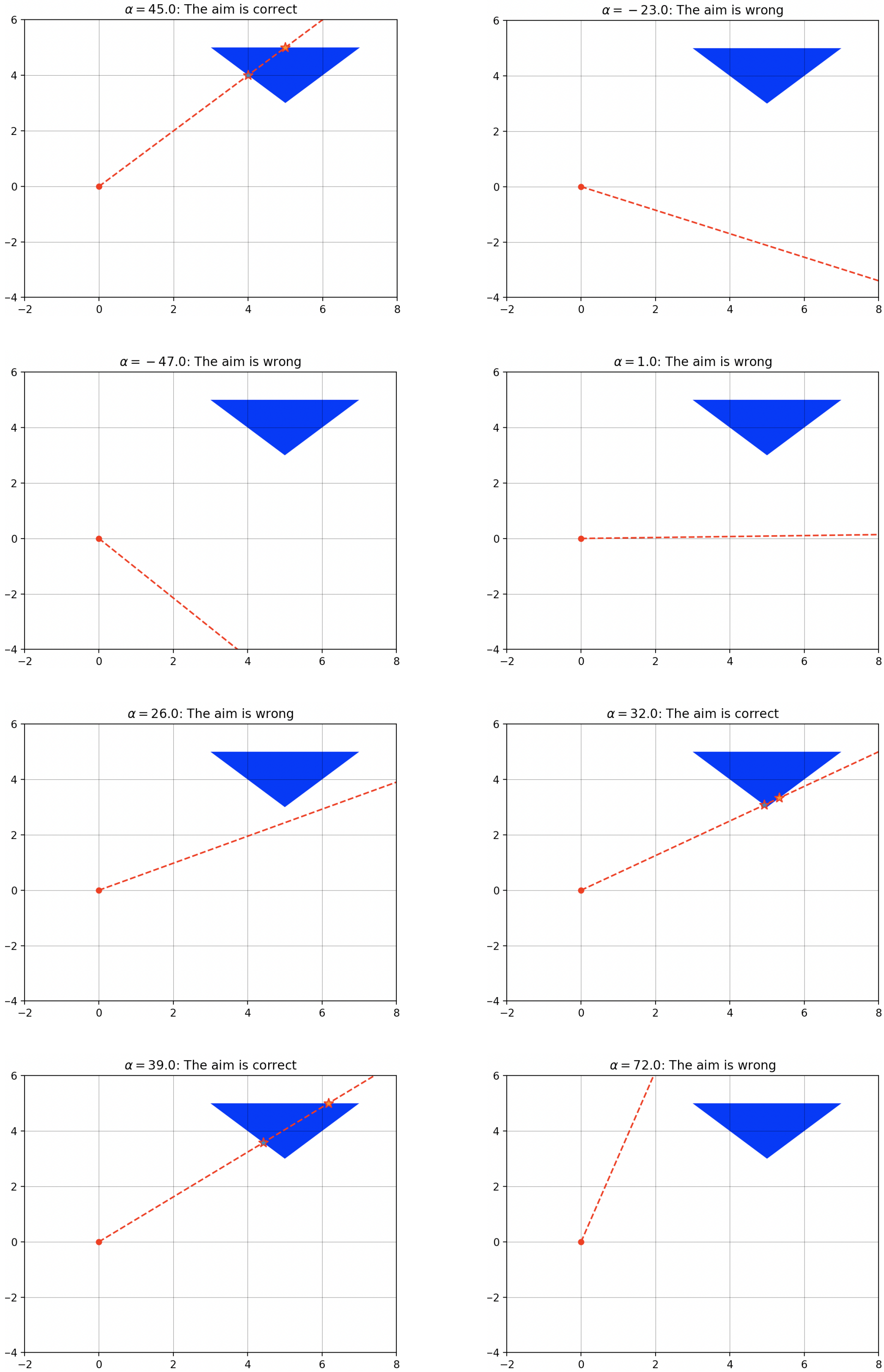
The aiming line intersects the line of segment 3 at point (5,5), which calls on the line segment connecting the points $(3, 5)$ and $(7, 5)$.

Overall, if the user's aim is correct, the aiming line will intersect exactly two line segments, which is the case in this scenario.

Question Extra Credit:

```
import numpy as np
import random
import math
import matplotlib.pyplot as plt

p1 = ([3,5,1])
p2 = ([5,3,1])
p3 = ([7,5,1])

alph = math.radians(random.randint(-90,90))
aim_p1 = np.array([0,0,1])
aim_p2 = np.array([math.cos(alph),math.sin(alph),1])
aim_l = np.cross(aim_p1,aim_p2)

l1 = np.cross(p1,p2)
l2 = np.cross(p2,p3)
l3 = np.cross(p3,p1)

int1 = np.cross(l1,aim_l)
int2 = np.cross(l2,aim_l)
int3 = np.cross(l3,aim_l)

int_points = np.zeros((2,3))
count = 0
result = 'wrong'
if int1[2] != 0:
    int1 = int1/int1[2]
    if int1[0] <= max(p1[0],p2[0]) and int1[0] >= min(p1[0],p2[0]):
        result = 'correct'
        int_points[count][:] = int1
        count = count +1
else:
    if int1[0]/int1[1] == p1[0]/p1[1]:
        result = 'correct'
        int_points[count][:] = p1
        count = count +1

if int2[2] != 0:
```

```
    int2 = int2/int2[2]
    if int2[0] <= max(p2[0],p3[0]) and int2[0] >= min(p2[0],p3[0]):
        result = 'correct'
        int_points[count][:] = int2
        count = count +1
else:
    if int2[0]/int2[1] == p2[0]/p2[1]:
        result = 'correct'
        int_points[count][:] = p2
        count = count + 1

if result == 'correct' and count == 1:
    if int3[2] != 0:
        int3 = int3/int3[2]
        int_points[1][:] = int3
    else:
        int_points[1][:] = p3
print(f'The aim is {result}')

plt.plot(0,0,'r.', markersize = 10,)
plt.fill((p1[0],p2[0],p3[0],p1[0]),(p1[1],p2[1],p3[1],p1[1]),'b')
plt.plot((0,10*aim_p2[0]),(0,10*aim_p2[1]),'r--')
print(int_points)
if result == 'correct':
    plt.plot(int_points[0][0],int_points[0][1], marker = '*', markersize = 10,me
    plt.plot(int_points[1][0],int_points[1][1], marker = '*', markersize = 10,me
plt.xlim((-2,8))
plt.ylim((-4,6))
plt.grid(color='k', linestyle='-', linewidth=0.2)
plt.title(rf'$\alpha = {math.degrees(alph)}$: The aim is {result}')
plt.show()
```


α = 45.0: The aim is correct
α = −23.0: The aim is wrong
α = −47.0: The aim is wrong
α = 1.0: The aim is wrong
α = 26.0: The aim is wrong
α = 32.0: The aim is correct
α = 39.0: The aim is correct
α = 72.0: The aim is wrong