Student: Carlos A. S. Oliveira (*Sousa de Oliveira, Carlos Augusto*)

Homework 1

August 31, 2022

1.

As seen in class and the lecture 2 handout, the homogeneous coordinate representation of a physical 2D point $(x, y) \in \mathbb{R}^2$ is the 3D vector $\mathbf{x} = (u, v, 1)^\top \in \mathbb{R}^3$, where $x = u/1$ and $y = v/1$. We also know that for any multiple $k \in \mathbb{R},\ \ k \neq 0,\ \ k \cdot (u, v, 1)^\top$ is the same physical point in $\mathbb{R}^2$ as that represented by $(u, v, 1)^\top$. Therefore, for the origin, that is, the point $(0, 0) \in \mathbb{R}^2$, we have the homogeneous coordinate representation $(0, 0, 1)^\top$ and all the points in the representational space $\mathbb{R}^3$ that are homogeneous coordinates of the origin are given by $k \cdot (0, 0, 1)^\top = (0, 0, k)^\top$.

2.

Points at infinity in the physical plane $\mathbb{R}^2$ have homogeneous coordinate representations of the form $(u, v, 0)^\top$ (ideal points) and are not all the same. As an arbitrary point $(x, y) \in \mathbb{R}^2$ approaches infinity its homogeneous coordinate representation (u, v, w) will be such that $x = u/w$ and $y = v/w$ with $w \to 0$, with the direction of the approximation to infinity determined by the values of u and v. Thus, points at infinity differ from each other according to the direction dictated by their respective u and v coordinates. These differing points at infinity, however, form a straight line in $\mathbb{R}^2$, the line whose representation is $\mathbf{l}_\infty = (0, 0, 1)^\top$.

3.

We know that a degenerate conic $\mathbf{C}$ manifests as a pair of intersecting straight lines, with $\mathbf{C} = \mathbf{l}\mathbf{m}^\top + \mathbf{m}\mathbf{l}^\top$. As noted in the lecture notes, we can see that the two terms that constitute $\mathbf{C}$ are vector outer products and we know from linear algebra that for nonzero vectors $\mathbf{m}$ and $\mathbf{l}$ the outer product is of matrix rank 1 (that is, all columns are linearly dependent on the first column), so we conclude that $\mathbf{C}$, the sum of two vector outer products, can have at most matrix rank 2 (which will be the case when the first column vectors of each of the rank 1 matrices being summed are linearly independent from each other). While the previous statements are mathematically sound, we note, as pointed out in class, that the case where $\mathbf{C}$ has matrix rank 1 is an extreme case where the degenerate conic is two identical, superimposed lines (in effect actually a single line), rather than two intersecting lines, meaning that other than this extreme case, the matrix rank of $\mathbf{C}$ will be precisely 2.

4.

A conic in $\mathbb{R}^2$ is defined by 5 points. As pointed out on a post on Piazza, we can see this by noticing that if we express the equation for a conic as the dot product $(x^2, xy, y^2, x, y, 1)(a, b, c, d, e, f)^\top = 0$ we create a system of linear equations with 6 equations and 6 unknowns and we know that a system of equations with n equations and n unknowns can be solved with $n-1$ of the equations. Therefore, given that our $n = 6$, the system of equations that expresses the conic can be solved with 5 of those equations, meaning that a conic is defined by 5 points. Alternatively, following the argument sketched in Hartley & Zisserman's "Multiple View Geometry in Computer Vision" (2nd edition, page 5), another way we can see that a conic is defined by 5 points is by counting the number of coefficients of x and y terms in the implicit form equation of a conic, which is $ax^2 + bxy + cy^2 + dx + ey + f = 0$ (totalling 5 coefficients of x and y terms), much like we see that a line in $\mathbb{R}^2$ is defined by two points by counting the coefficients of x and y terms in the implicit form equation of a line ($ax+by+c = 0$, with 2 coefficients of x and y terms).

5.

Step 1: We find the line $\mathbf{l}_1 = \begin{bmatrix}0\\0\\1\end{bmatrix} \times \begin{bmatrix}1\\2\\1\end{bmatrix} = \begin{bmatrix}0\cdot 1 - 1\cdot 2\\1\cdot 1 - 0\cdot 1\\0\cdot 2 - 0\cdot 1\end{bmatrix} = \begin{bmatrix}-2\\1\\0\end{bmatrix}$

We note that this line passes through the origin (one of the points through which it passes is the origin $(0, 0)$ and the w coordinate of the line is equal to 0).

Step 2: We find the line $\mathbf{l}_2 = \begin{bmatrix}3\\4\\1\end{bmatrix} \times \begin{bmatrix}5\\6\\1\end{bmatrix} = \begin{bmatrix}4\cdot 1 - 1\cdot 6\\1\cdot 5 - 3\cdot 1\\3\cdot 6 - 4\cdot 5\end{bmatrix} = \begin{bmatrix}-2\\2\\-2\end{bmatrix}$

Step 3: We find the intersection of $\mathbf{l}_1$ and $\mathbf{l}_2$, given by

$$\mathbf{l}_1 \times \mathbf{l}_2 = \begin{bmatrix}-2\\1\\0\end{bmatrix} \times \begin{bmatrix}-2\\2\\-2\end{bmatrix} = \begin{bmatrix}1\cdot -2 - 0\cdot 2\\0\cdot -2 - (-2\cdot -2)\\-2\cdot 2 - 1\cdot -2\end{bmatrix} = \begin{bmatrix}-2\\-4\\-2\end{bmatrix}$$

Now, for a line passing through the points $(7, -8)$ and $(-7, 8)$ we would do, in two steps:

Step 1: We find the line $\mathbf{l}_{2'} = \begin{bmatrix}7\\-8\\1\end{bmatrix} \times \begin{bmatrix}-7\\8\\1\end{bmatrix} = \begin{bmatrix}-8\cdot 1 - 1\cdot 8\\1\cdot -7 - 7\cdot 1\\7\cdot 8 - (-8\cdot -7)\end{bmatrix} = \begin{bmatrix}-16\\-14\\0\end{bmatrix}$

We note that this line passes through the origin (the w coordinate of the line is equal to 0).

Step 2: Since we found that both line $\mathbf{l}_1$ and line $\mathbf{l}_{2'}$ pass through the origin, we conclude that the origin (a point of the form $(0, 0, k)^\top$) is where they intersect and, indeed, $\mathbf{l}_1 \times \mathbf{l}_{2'} = (0, 0, 44)^\top$.

6.

We find the line $\mathbf{l}_1 = \begin{bmatrix}-4\\0\\1\end{bmatrix} \times \begin{bmatrix}-2\\8\\1\end{bmatrix} = \begin{bmatrix}-8\\2\\-32\end{bmatrix}$

We find the line $\mathbf{l}_2 = \begin{bmatrix}0\\-2\\1\end{bmatrix} \times \begin{bmatrix}4\\14\\1\end{bmatrix} = \begin{bmatrix}-16\\4\\8\end{bmatrix}$

We find the intersection of $\mathbf{l}_1$ and $\mathbf{l}_2$, given by $\mathbf{l}_1 \times \mathbf{l}_2 = \begin{bmatrix}-8\\2\\-32\end{bmatrix} \times \begin{bmatrix}-16\\4\\8\end{bmatrix} = \begin{bmatrix}144\\576\\0\end{bmatrix}$

Since we found that the intersection of the lines is of the form $(u, v, 0)^\top$, we conclude that the lines intersect at an ideal point, thus at infinity, meaning that they are parallel.

7.

Knowing that given the algebraic form, $ax + by + c = 0$, the parameter vector $\mathbf{l} = (a, b, c)^\top$ can serve as the homogeneous coordinates representation of the line, we find that, with the equations given (that is, $x = 1 \ \Rightarrow \ x - 1 = 0$ and $y = -1 \ \Rightarrow \ y + 1 = 0$) the two lines are $\mathbf{l}_1 = (1, 0, -1)^\top$ and $\mathbf{l}_2 = (0, 1, 1)^\top$.

Their intersection is thus found at $\mathbf{l}_1 \times \mathbf{l}_2 = \begin{bmatrix}1\\0\\-1\end{bmatrix} \times \begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}1\\-1\\1\end{bmatrix}$

8.

We know from analitic geometry that the equation of the ellipse (with axes parallel to the plane's x and y axes) is $\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1$ (where the point (x_0, y_0) is the center of the ellipse).

The ellipse described in the question prompt thus has the following equation:

$$\frac{(x-2)^2}{(1/2)^2} + \frac{(y-3)^2}{1^2} = 1 \;\Rightarrow\; 4x^2 - 16x + 16 + y^2 - 6y + 9 - 1 = 0 \;\Rightarrow\; 4x^2 + y^2 - 16x - 6y + 24 = 0$$

We thus get $\mathbf{C} = \begin{bmatrix} 4 & 0 & -8 \\ 0 & 1 & -3 \\ -8 & -3 & 24 \end{bmatrix}$ and, with $\mathbf{p}$ being the origin of the plane, we have that $\mathbf{p} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

The polar line is thus given by $\mathbf{Cp} = \begin{bmatrix} 4 & 0 & -8 \\ 0 & 1 & -3 \\ -8 & -3 & 24 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -8 \\ -3 \\ 24 \end{bmatrix}$.

Since the x-axis line $(y = 0)$ is $(0, 1, 0)$ and the y-axis line $(x = 0)$ is $(1, 0, 0)$, we have that their intersections with the polar line are, respectively: $\begin{bmatrix} -8 \\ -3 \\ 24 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -24 \\ 0 \\ -8 \end{bmatrix}$ and $\begin{bmatrix} -8 \\ -3 \\ 24 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 24 \\ 3 \end{bmatrix}$.

That is, we find that in the physical plane $\mathbb{R}^2$ the intersection of the polar line with the x-axis is the point $(-24/-8, 0) = (3, 0)$ and the intersection of the polar line with the y-axis is the point $(0, 24/3) = (0, 8)$.