ECE66100 Homework #1

Ziyue "Alan" Xiang

August 31, 2022

Contents

Question 1

The origin in $\mathbb{R}^2$ is given by $\begin{bmatrix}0 & 0\end{bmatrix}^T$. The corresponding Homogeneous Coordinates (HC) representation is given by $\begin{bmatrix}0 & 0 & 1\end{bmatrix}^T$. We know that $\forall\, k \in \mathbb{R}\ (k \neq 0), k\begin{bmatrix}0 & 0 & 1\end{bmatrix}^T$ corresponds to the same physical point. Therefore, the HC of the origin in physical $\mathbb{R}^2$ can be written as $\begin{bmatrix}0 & 0 & k\end{bmatrix}^T$, where $k \in \mathbb{R}$ and $k \neq 0$.

Question 2

No, not all points at infinity in the physical plane $\mathbb{R}^2$ are the same.

Consider two infinity points $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$, where $\boldsymbol{p}_1$ is the infinity point generated by following the line $y = x, x \to \infty$; and $\boldsymbol{p}_2$ is the infinity point generated by following the line $y = 2x, x \to \infty$, respectively. Intuitively, $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ are on different locations of the $\mathbb{R}^2$ plane, because they are approaching infinity from different directions.

Given any point (x_1, y_1) on the line $y = x$, the corresponding HC representation is given by $\begin{bmatrix} 1 & 1 & 1/x_1 \end{bmatrix}^T$. Given any point (x_2, y_2) on the line $y = 2x$, the corresponding HC representation is given by $\begin{bmatrix} 1 & 2 & 1/x_2 \end{bmatrix}^T$. When we set $x_1 \to \infty$ and $x_2 \to \infty$, the infinity points of $y = x$ and $y = 2x$ are $\begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T$ and $\begin{bmatrix} 1 & 2 & 0 \end{bmatrix}^T$, respectively. It can be seen that they are different infinity points with distinct HC representation.

In general, an infinity point is of the form $\begin{bmatrix} u & v & 0 \end{bmatrix}^T$, where the (u, v) pair determines the direction that the point approaches infinity.

Question 3

Let $\boldsymbol{l} = \begin{bmatrix} l_1 & l_2 & l_3 \end{bmatrix}^T$, $\boldsymbol{m} = \begin{bmatrix} m_1 & m_2 & m_3 \end{bmatrix}^T$, it can be seen that

$$\boldsymbol{lm}^T = \begin{bmatrix} l_1m_1 & l_1m_2 & l_1m_3 \\ l_2m_1 & l_2m_2 & l_2m_3 \\ l_3m_1 & l_3m_2 & l_3m_3 \end{bmatrix}, \tag{1}$$

$$\boldsymbol{ml}^T = \begin{bmatrix} l_1m_1 & l_2m_1 & l_3m_1 \\ l_1m_2 & l_2m_2 & l_3m_2 \\ l_1m_3 & l_2m_3 & l_3m_3 \end{bmatrix}. \tag{2}$$

Clearly, all three columns of $\boldsymbol{lm}^T$ and $\boldsymbol{ml}^T$ are linearly dependent. That is to say, $\text{rank}(\boldsymbol{lm}^T) = \text{rank}(\boldsymbol{ml}^T) = 1$.

Definition 1. *Let U and V be two vector spaces, the sum of two the two vector spaces, denoted by $U + V$, is the set given by*

$$\{\boldsymbol{u} + \boldsymbol{v} \mid \boldsymbol{u} \in U, \boldsymbol{v} \in V\}.$$

Theorem 1. *Let U and V be two vector spaces, then* $\dim(U + V) \leq \dim U + \dim V$.

Proof. Let $\dim U = u$, $\dim V = v$. We can further let $B_1 = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_u\}$ and $B_2 = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_v\}$ to be the basis of U and V, respectively.

From Definition 1, we know that an arbitrary vector $\boldsymbol{q} \in U + V$ can be written in the form of $\boldsymbol{q} = \boldsymbol{u} + \boldsymbol{v}$, where $\boldsymbol{u} \in U$, $\boldsymbol{v} \in V$. With the basis of U and V provided above, we can further write

$$\boldsymbol{u} = k_1\boldsymbol{u}_1 + k_2\boldsymbol{u}_2 + \cdots + k_u\boldsymbol{u}_u, \tag{3}$$

$$\boldsymbol{v} = s_1\boldsymbol{v}_1 + s_2\boldsymbol{v}_2 + \cdots + s_v\boldsymbol{v}_v, \tag{4}$$

where $k_1, \dots, k_u$ and $s_1, \dots, s_v$ are scalars. Since $\boldsymbol{q} = \boldsymbol{u}+\boldsymbol{v}$, we know $\boldsymbol{q} \in \text{span}(\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_u, \boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_v)$. It can be seen that

$$\dim(U + W) \leq \dim \text{span}(\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_u, \boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_v) \leq u + v = \dim(U) + \dim(V). \tag{5}$$

□

Theorem 2. *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be $m \times n$ matrices, then* $\text{rank}(\boldsymbol{A} + \boldsymbol{B}) \leq \text{rank}(\boldsymbol{A}) + \text{rank}(\boldsymbol{B})$.

Proof. Denote the column space of a matrix $\boldsymbol{M}$ by $C(\boldsymbol{M})$. By definition, we know that $\text{rank}(\boldsymbol{M}) = \dim C(\boldsymbol{M})$. If we write $\boldsymbol{A}$ and $\boldsymbol{B}$ in terms of column vectors, that is $\boldsymbol{A} = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \cdots & \boldsymbol{a}_n \end{bmatrix}^T$ and $\boldsymbol{B} = \begin{bmatrix} \boldsymbol{b}_1 & \boldsymbol{b}_2 & \cdots & \boldsymbol{b}_n \end{bmatrix}^T$, then we can express the column spaces of $\boldsymbol{A}$ and $\boldsymbol{B}$ as

$$C(\boldsymbol{A}) = \text{span}(\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n), \tag{6}$$

$$C(\boldsymbol{B}) = \text{span}(\boldsymbol{b}_1, \boldsymbol{b}_2, \cdots, \boldsymbol{b}_n). \tag{7}$$

It is obvious that

$$C(\boldsymbol{A}+\boldsymbol{B}) = \operatorname{span}(\boldsymbol{a}_1+\boldsymbol{b}_1, \boldsymbol{a}_2+\boldsymbol{b}_2, \cdots, \boldsymbol{a}_n+\boldsymbol{b}_n). \tag{8}$$

$\forall \boldsymbol{p} \in C(\boldsymbol{A}+\boldsymbol{B})$, we have

$$\begin{aligned} \boldsymbol{p} &= k_1(\boldsymbol{a}_1+\boldsymbol{b}_1) + k_2(\boldsymbol{a}_2+\boldsymbol{b}_2) + \cdots + k_n(\boldsymbol{a}_n+\boldsymbol{b}_n) && (9)\\ &= (k_1\boldsymbol{a}_1 + k_2\boldsymbol{a}_2 + \cdots + k_n\boldsymbol{a}_n) + (k_1\boldsymbol{b}_1 + k_2\boldsymbol{b}_2 + \cdots + k_n\boldsymbol{b}_n), && (10) \end{aligned}$$

where $k_1, \ldots, k_n$ are scalars. It can be seen that $\boldsymbol{p} \in \operatorname{span}(\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n) + \operatorname{span}(\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n)$. We can derive

$$\begin{aligned} \operatorname{rank}(\boldsymbol{A}+\boldsymbol{B}) &= \dim C(\boldsymbol{A}+\boldsymbol{B}) && (11)\\ &\le \dim\left[\operatorname{span}(\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n) + \operatorname{span}(\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n)\right] && (12)\\ &\le \dim\left[\operatorname{span}(\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n)\right] + \dim\left[\operatorname{span}(\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n)\right] && (13)\\ &= \dim C(\boldsymbol{A}) + \dim C(\boldsymbol{B}) = \operatorname{rank}\boldsymbol{A} + \operatorname{rank}\boldsymbol{B}. && (14) \end{aligned}$$

□

Using Theorem 2, it can be seen that

$$\operatorname{rank}\boldsymbol{C} = \operatorname{rank}(\boldsymbol{l}\boldsymbol{m}^T + \boldsymbol{l}\boldsymbol{m}^T) \le \operatorname{rank}(\boldsymbol{l}\boldsymbol{m}^T) + \operatorname{rank}(\boldsymbol{l}\boldsymbol{m}^T) = 2. \tag{15}$$

Question 4

We know that a conic can be written in the form of

$$\boldsymbol{C} = \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix}. \tag{16}$$

For an HC point $\boldsymbol{x}_1 = \begin{bmatrix} x_1 & y_1 & 1 \end{bmatrix}^T$, it can be seen that

$$\boldsymbol{x}_1^T \boldsymbol{C} \boldsymbol{x}_1 = ax_1^2 + bx_1y_1 + cy_1^2 + dx_1 + ey_1 + f = 0. \tag{17}$$

We know that $\forall\, k \in \mathbb{R}$ $(k \neq 0)$, $k\boldsymbol{C}$ and $\boldsymbol{C}$ represent the same conic. Therefore, the following expression should also hold true:

$$\boldsymbol{x}_1^T (kC) \boldsymbol{x}_1 = kax_1^2 + kbx_1y_1 + kcy_1^2 + kdx_1 + key_1 + kf = 0. \tag{18}$$

In a nontrivial conic, at least one of the coefficients among a, b, c, d, e, f will be nonzero. Without loss of generality, we can assume a is nonzero, and let $k = \frac{1}{a}$. Now Eq. (18) can be written as

$$x_1^2 + \frac{b}{a}x_1y_1 + \frac{c}{a}y_1^2 + \frac{d}{a}x_1 + \frac{e}{a}y_1 + \frac{f}{a} = 0. \tag{19}$$

If we let $b/a = b'$, $c/a = c'$, $d/a = d'$, $e/a = e'$, $f/a = f'$, then this expression becomes

$$x_1^2 + b'x_1y_1 + c'y_1^2 + d'x_1 + e'y_1 + f' = 0. \tag{20}$$

This is an equation with 5 unknowns, which means we need 5 points $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_5$ to solve for the coefficients of the conic. That is to say, a conic is defined with 5 points.

Question 5

(1) The HC representation of $\boldsymbol{l}_1$ is given by

$$\begin{bmatrix}0\\0\\1\end{bmatrix} \times \begin{bmatrix}1\\2\\1\end{bmatrix} = \begin{bmatrix}-2\\1\\0\end{bmatrix}. \tag{21}$$

The HC representation of $\boldsymbol{l}_2$ is given by

$$\begin{bmatrix}3\\4\\1\end{bmatrix} \times \begin{bmatrix}5\\6\\1\end{bmatrix} = \begin{bmatrix}-2\\2\\-2\end{bmatrix}. \tag{22}$$

The intersection between $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$ is

$$\begin{bmatrix}-2\\1\\0\end{bmatrix} \times \begin{bmatrix}-2\\2\\-2\end{bmatrix} = \begin{bmatrix}-2\\-4\\-2\end{bmatrix}. \tag{23}$$

That is, the intersection is given by $(1, 2)$.

(2) The HC representation of line $\boldsymbol{l}_3$ is given by

$$\begin{bmatrix}7\\-8\\1\end{bmatrix} \times \begin{bmatrix}-7\\8\\1\end{bmatrix} = \begin{bmatrix}-16\\-14\\0\end{bmatrix}. \tag{24}$$

Since the third element of $\boldsymbol{l}_1$ and $\boldsymbol{l}_3$ are both zero, we know that the two points pass through the origin. Therefore, they must intersect at the origin. In this case, we only need to compute $\boldsymbol{l}_1$ and $\boldsymbol{l}_3$, which consists of two steps.

Question 6

The HC representation of $\boldsymbol{l}_1$ is given by

$$\begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} -2 \\ 8 \\ 1 \end{bmatrix} = \begin{bmatrix} -8 \\ 2 \\ -32 \end{bmatrix}. \tag{25}$$

The HC representation of $\boldsymbol{l}_2$ is given by

$$\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \times \begin{bmatrix} 4 \\ 14 \\ 1 \end{bmatrix} = \begin{bmatrix} -16 \\ 4 \\ 8 \end{bmatrix}. \tag{26}$$

The intersection between $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$ is given by

$$\begin{bmatrix} -8 \\ 2 \\ 32 \end{bmatrix} \times \begin{bmatrix} -16 \\ 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 144 \\ 576 \\ 0 \end{bmatrix}. \tag{27}$$

Since this is an infinity point, we know that $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$ are parallel.

Question 7

$x = 1$ can be written as $1x + 0y - 1 = 0$, which means its corresponding HC representation is $\begin{bmatrix}1 & 0 & -1\end{bmatrix}^T$. $y = -1$ can be written as $0x + 1y + 1 = 0$, which means its corresponding HC representation is $\begin{bmatrix}0 & 1 & 1\end{bmatrix}^T$. The intersection between them is given by

$$\begin{bmatrix}1\\0\\-1\end{bmatrix} \times \begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}1\\-1\\1\end{bmatrix}. \tag{28}$$

That is, the intersection is at $(1, -1)$.

Question 8

The equation of the ellipse is

$$\frac{(x-2)^2}{\left(\frac{1}{2}\right)^2} + (y-3)^2 = 1. \tag{29}$$

This simplifies to

$$4x^2 + y^2 - 16x - 6y + 24 = 0. \tag{30}$$

This conic can be written as

$$\boldsymbol{C} = \begin{bmatrix} 4 & 0 & -8 \\ 0 & 1 & -3 \\ -8 & -3 & 24. \end{bmatrix} \tag{31}$$

Since $\boldsymbol{p}$ is the origin, the HC representation is given by $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$. Therefore, the polar line is given by $\boldsymbol{l} = \boldsymbol{C}\boldsymbol{p} = \begin{bmatrix} -8 & -3 & 24 \end{bmatrix}^T$.

The HC representation of the x axis is given by $\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$. Therefore, the intersection between $\boldsymbol{l}$ and the x axis is

$$\begin{bmatrix} -8 \\ -3 \\ 24 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -24 \\ 0 \\ -8 \end{bmatrix}. \tag{32}$$

That is to say, the intersection between the polar line and the x axis is $(3, 0)$.

The HC representation of the y axis is given by $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$. Therefore, the intersection between $\boldsymbol{l}$ and the y axis is

$$\begin{bmatrix} -8 \\ -3 \\ 24 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 24 \\ 3 \end{bmatrix}. \tag{33}$$

That is to say, the intersection between the polar line and the y axis is $(0, 8)$.

A. Source Code

A.1. Source Listing of ../alanhm.cls

```
\NeedsTeXFormat{LaTeX2e}

\gdef\AlanHMDate{2022/08/22}
\gdef\AlanHMVersion{0.0.1}

\ProvidesExplClass{alanhm}{\AlanHMDate}{\AlanHMVersion}{Alan~Xiang's~Homework~Template}


\LoadClass[12pt]{scrartcl}

\RequirePackage{fontspec}
\RequirePackage{scrlayer-scrpage}
\RequirePackage[bold-style=TeX]{unicode-math}
\RequirePackage{booktabs}
\RequirePackage{microtype}
\RequirePackage{enumitem}
\RequirePackage{amsthm}
\RequirePackage{caption,float}
\RequirePackage{xcolor}
\RequirePackage{etoolbox}
\RequirePackage{csquotes}
\RequirePackage{nicefrac}
\RequirePackage[skins,breakable,listings,minted]{tcolorbox}
\RequirePackage{metalogo}
\RequirePackage[acronym]{glossaries}
\RequirePackage[english]{babel}
\RequirePackage[colorlinks]{hyperref}
\RequirePackage[capitalise]{cleveref}


\setmainfont{ETbb-Regular}[
    Extension=.otf,
    BoldFont=ETbb-Bold,
    ItalicFont=ETbb-Italic,
    BoldItalicFont=ETbb-BoldItalic
]

\setsansfont{texgyreadventor-regular}[
    Extension=.otf,
    BoldFont=texgyreadventor-bold,
    ItalicFont=texgyreadventor-italic,
    BoldItalicFont=texgyreadventor-bolditalic,
    Scale=0.8
]

\newfontfamily{\FONTqtfloraline}{QTFloraline}[
    Extension=.otf,
    BoldFont=QTFloraline-Bold,
    Scale=0.8
]

\setmonofont{NewCMMono10-Regular}[
  Extension=.otf,
  BoldFont=NewCMMono10-Bold,
  ItalicFont=NewCMMono10-Italic,
  BoldItalicFont=NewCMMono10-BoldOblique,
  NFSSFamily=NewCMMonoFamily,
  Scale=0.85
]

\setmathfont{Asana-Math}[
    Extension=.otf,
    bold-style=TeX
]

% make bold math fonts look heavier
\setmathfont{Asana-Math}[
    Extension=.otf,
    bold-style=TeX,
    FakeBold=1.6,
    range={bfup,bfit}
]

\newfontfamily{\FONTfiramono}{FiraMono-Regular}[
    Extension=.otf,
    BoldFont=FiraMono-Bold,
    ItalicFont=FiraMono-Oblique,
    BoldItalicFont=FiraMono-BoldOblique,
    NFSSFamily=FiraMonoFamily,
    RawFeature={fallback=stdfallback},
    Scale=0.75
]

% set paper size and margin
\KOMAoption{paper}{letter}
\KOMAoption{DIV}{12}
% set title font
\addtokomafont{disposition}{\FONTqtfloraline}

\tl_new:N \l_ahm_now_time_tl
\cs_gset:Npn \ahm_get_time: {
    \sys_get_shell:nnN {date~--iso-8601=ns} {\cctab_select:N \c_document_cctab} \l_ahm_now_time_tl
    \tl_use:N \l_ahm_now_time_tl
}

\tl_new:N \l_ahm_xetex_info_tl
\cs_gset:Npn \ahm_get_xetex_info: {
    \sys_get_shell:nnN {bash~-c~'xelatex~--version|head~-n~1'} {\cctab_select:N \c_document_cctab} \l_ahm_xetex_info_tl
    \tl_use:N \l_ahm_xetex_info_tl
}

\tl_new:N \l_ahm_system_info_tl
\tl_new:N \l_ahm_system_info_tmpa_tl
\cs_gset:Npn \ahm_get_system_info: {
    \tl_clear:N \l_ahm_system_info_tl
    \sys_get_shell:nnN {bash~-c~'echo~$USER@$HOSTNAME'} {\cctab_select:N \c_code_cctab} \l_ahm_system_info_tmpa_tl
    \tl_put_right:NV \l_ahm_system_info_tl \l_ahm_system_info_tmpa_tl
    \tl_put_right:Nn \l_ahm_system_info_tl {;\ }
    \sys_get_shell:nnN {lsb_release~-d~-s} {\cctab_select:N \c_document_cctab} \l_ahm_system_info_tmpa_tl
    \tl_put_right:NV \l_ahm_system_info_tl \l_ahm_system_info_tmpa_tl
    \sys_get_shell:nnN {uname~-r} {\cctab_select:N \c_code_cctab} \l_ahm_system_info_tmpa_tl
    \tl_put_right:Nx \l_ahm_system_info_tl {
        \ (\l_ahm_system_info_tmpa_tl)
    }
    \tl_use:N \l_ahm_system_info_tl
```

```
}

% page margin metadata
\DeclareNewLayer[
  background,
  bottommargin,
  align=c,
  mode=picture,
  everypage,
  contents={%
    \putC{\ahm_generate_bottom:}
  }
]{bottom}

%\definecolor{button-primary}{HTML}{007bff}
%\definecolor{button-primary}{HTML}{007bff}
%\definecolor{button-primary}{rgb}{gray!50}
\colorlet{button-primary}{gray!50}

% hook for going back to the first page
\AtBeginDocument{
    \hypertarget{@ahm@begin@doc}{}
}


\newtcbox{\AHMPrimaryButton}{
    colback=button-primary,
    colframe=white,
    fontupper=\sffamily\scriptsize\color{white},
    boxsep=0ex,
    left=0.8ex,
    right=0.8ex,
    top=0.8ex,
    bottom=0.8ex
}

\newcommand{\AHMBackToTopButton}{
    \hyperlink{@ahm@begin@doc}{\AHMPrimaryButton{BACK~TO~TOP}}
}

% construct button nav bar
\seq_new:N \g_ahm_bottom_nav_seq
\tl_new:N \l_ahm_bottom_nav_tmpa_tl


\DeclareNewLayer[
  background,
  bottommargin,
  align=c,
  mode=picture,
  everypage,
  contents={%
    \putC{
            \parbox{\paperwidth}{
                \centering
                \vspace*{4cm}
                \seq_gclear:N \g_ahm_bottom_nav_seq
                \seq_gput_left:Nn \g_ahm_bottom_nav_seq {\mbox{\AHMBackToTopButton}}
                \clist_map_variable:NNn \g_ahm_questions_aux_clist \l_ahm_bottom_nav_tmpa_tl {
                    \seq_gput_right:Nx \g_ahm_bottom_nav_seq {
                        \exp_not:N \hyperlink {@ahm@question@\exp_not:V \l_ahm_bottom_nav_tmpa_tl}{
                            \exp_not:N \mbox{
                                \exp_not:N \AHMPrimaryButton{Q \exp_not:V \l_ahm_bottom_nav_tmpa_tl}
                            }
                        }
                    }
                }
                \seq_use:Nn \g_ahm_bottom_nav_seq {\hspace*{1pt}}
            }
        }
  }
]{bottom-link}


\AddLayersAtEndOfPageStyle{scrheadings}{bottom}
\AddLayersAtEndOfPageStyle{scrheadings}{bottom-link}
% make sure there is bottom note to the title page
\renewcommand\titlepagestyle{scrheadings}
\pagestyle{scrheadings}


\tl_new:N \l_ahm_gb_tmpa_tl
\tl_new:N \l_ahm_gb_tmpb_tl
\cs_gset:Npn \ahm_generate_bottom: {
    \parbox{\paperwidth}{
        \centering
        \scriptsize
        \sffamily
        \vspace*{6em}
        \color{gray!50}
        \int_compare:nTF {\thepage = 1} {
            Ziyue~``Alan"~Xiang's~Homework~Template~(alanhm.cls);~VERSION=\AlanHMVersion;~TEMPLATE~DATE=\AlanHMDate\\
            \XeLaTeX\ signature:~\ahm_get_xetex_info:\\
            User~and~computer~system~signature:~\ahm_get_system_info:\\
            Page~compile~timestamp:~\ahm_get_time:\\
        }{
            Page~compile~timestamp:~\ahm_get_time:
        }
        \par
    }
}

% change the TOC style
% \let\oldtoc\tableofcontents
% \renewcommand{\tableofcontents}{
%     \bgroup
%     \normalfont\sffamily
%     \oldtoc
%     \egroup
% }
% \clist_map_inline:nn {section,subsection} {
%     \addtokomafont{#1entry}{\normalfont\sffamily}
% }


\setkomafont{sectionentry}{\normalfont\sffamily}
\setkomafont{sectionentrypagenumber}{\normalfont\sffamily}
\RedeclareSectionCommands[
  tocentryformat=\normalfont\sffamily,
  tocentrynumberformat=\normalfont\sffamily,
  tocpagenumberformat=\normalfont\sffamily
]{subsection,subsubsection}
```

```latex

% large appendix for source code
\newcommand{\largeappendix}{
    \clearpage
    \KOMAoptions{paper=a3,DIV=18}
    \recalctypearea
    \appendix
    % \setkomafont{sectionentry}{\normalfont\sffamily}
    % \setkomafont{sectionentrypagenumber}{\normalfont\sffamily}
}



% implementation of quick matrices
% note: nested usage of this command is not allowed!
\seq_new:N \l_ahm_qmat_tmpa_seq
\seq_new:N \l_ahm_qmat_tmpb_seq
\seq_new:N \l_ahm_qmat_tmpc_seq
\tl_new:N \l_ahm_qmat_tmpa_tl
% #1: matrix type
% #2: command name
\newcommand{\DeclareQuickMatrix}[2]{
    \NewDocumentCommand{#2}{sm}{
        \regex_split:nnN {;} {##2} \l_ahm_qmat_tmpa_seq
        \seq_clear:N \l_ahm_qmat_tmpb_seq % store rows
        \seq_map_variable:NNn \l_ahm_qmat_tmpa_seq \l_ahm_qmat_tmpa_tl {
            \exp_args:NNV \seq_set_from_clist:Nn \l_ahm_qmat_tmpc_seq \l_ahm_qmat_tmpa_tl
            \seq_put_right:Nx \l_ahm_qmat_tmpb_seq {
                \seq_use:Nn \l_ahm_qmat_tmpc_seq {&}
            }
        }
        \ensuremath{
            \begin{#1matrix}
            \seq_use:Nn \l_ahm_qmat_tmpb_seq {\\}
            \end{#1matrix}
            \IfBooleanTF{##1}{^T}{}
        }
    }
}

\DeclareQuickMatrix{v}{\qvmat}
\DeclareQuickMatrix{p}{\qpmat}
\DeclareQuickMatrix{b}{\qbmat}


% short hands for common math commands
\KOMAoptions{sectionentrydots=true}
\newcommand{\DefineShortHand}[2]{
    \cs_if_exist:NTF #1 {
        \GenericError{}{\cs_to_str:N #1~already~exists}{}{}
    }{
        \cs_gset_eq:NN #1 #2
    }

}

\DefineShortHand\bit\symbfit
\DefineShortHand\bup\symbfup
\DefineShortHand\bm\symbfit
\DefineShortHand\mca\mathcal
\DefineShortHand\mrm\mathrm
\DefineShortHand\mbb\mathbb


% change enumitem settings
\setlist{
    left=1em,
    itemsep=0ex,
}



\clist_new:N \g_ahm_questions_clist
\clist_new:N \g_ahm_questions_aux_clist

% homework related commands
\newcommand{\question}[1]{
    \clearpage
    \addcontentsline{toc}{section}{Question~#1}
    \hypertarget{@ahm@question@#1}{}
    \section*{Question~#1}
    \clist_gput_right:Nn \g_ahm_questions_clist {#1}
}



\tl_new:N \l_ahm_auxout_tmpa_tl
\AtEndDocument{
    \iow_now:cn {@auxout} {\ExplSyntaxOn}
    \iow_now:cx {@auxout} {
        \exp_not:N \clist_gset:Nn
        \exp_not:N \g_ahm_questions_aux_clist
        {\exp_not:V \g_ahm_questions_clist}
    }
    \iow_now:cn {@auxout} {\ExplSyntaxOff}
}

% amsthm
\newtheorem{theorem}{Theorem}
\newtheorem{definition}{Definition}


\makeatletter
% metadata
\AtBeginDocument{
    \hypersetup{
        pdftitle={\@title},
        pdfauthor={\@author}
    }
}
\makeatother


% change line number style
\renewcommand{\theFancyVerbLine}{
    \textcolor{gray!80}{
        \fontsize{7}{7}\selectfont\ttfamily
            %\oldstylenums{
                \arabic{FancyVerbLine}
            %}
        }
}
```

```
% source code listings
\clist_new:N \l_ahm_src_listing_clist
\tl_new:N \l_ahm_src_listing_tmpa_tl
\tl_new:N \l_ahm_src_listing_tmpb_tl
\tl_new:N \l_ahm_src_listing_tmpc_tl
\tl_new:N \l_ahm_src_listing_tmpd_tl
\newcommand{\SourceListing}[1]{
    \clist_set:Nn \l_ahm_src_listing_clist {#1}
    \section{Source~Code}
    \bool_do_until:nn {\clist_if_empty_p:N \l_ahm_src_listing_clist} {
        \clist_pop:NN \l_ahm_src_listing_clist \l_ahm_src_listing_tmpa_tl % filename
        \clist_pop:NN \l_ahm_src_listing_clist \l_ahm_src_listing_tmpb_tl % language
        \str_set:NV \l_ahm_src_listing_tmpa_tl \l_ahm_src_listing_tmpa_tl
        \str_set:NV \l_ahm_src_listing_tmpb_tl \l_ahm_src_listing_tmpb_tl

        \tl_set:Nx \l_ahm_src_listing_tmpd_tl {
            \exp_not:N \subsection[
                Source~Listing~of~
                \exp_not:N \texorpdfstring{
                    \exp_not:N \texttt {
                        \c_backslash_str detokenize{\exp_not:V \l_ahm_src_listing_tmpa_tl}
                    }
                }{
                    \exp_not:V \l_ahm_src_listing_tmpa_tl
                    }
            ]{
                Source~Listing~of~\exp_not:N \texttt{\exp_not:V \l_ahm_src_listing_tmpa_tl}
            }
        }
        \tl_use:N \l_ahm_src_listing_tmpd_tl
        \exp_args:Nx
        \tcbinputlisting{
            enhanced,
            listing~file=\exp_not:V \l_ahm_src_listing_tmpa_tl,
            listing~engine=minted,
            listing~only,
            breakable,
            left=2.6em,
            minted~options={
                autogobble,
                fontfamily=FiraMonoFamily,
                fontsize=\exp_not:N\fontsize{8}{8}\exp_not:N\selectfont,
                obeytabs,
                breaklines,
                linenos,
                numbersep=1em
            }
        }
    }
}
```

A.2. *Source Listing of* `1_ZiyueXiang.tex`

```
% !TeX TS-program = xelatex

\documentclass{../alanhm}
\usepackage{svg}

\title{ECE66100 Homework \#1}
\author{Ziyue ``Alan" Xiang}
\date{\today}


% acronyms

\newacronym{hc}{HC}{Homogeneous Coordinates}
\newcommand{\rank}{\operatorname{rank}}
\newcommand{\algspan}{\operatorname{span}}

\begin{document}
\maketitle
\tableofcontents

\question{1}

The origin in $\mbb R^2$ is given by \qbmat*{0,0}.
The corresponding \gls{hc} representation is given by \qbmat*{0,0,1}.
We know that $\forall k \in \mbb R~(k\neq 0)$, $k\qbmat*{0,0,1}$ corresponds to the same physical point.
Therefore, the \gls{hc} of the origin in physical $\mbb R^2$ can be written as $\qbmat*{0,0,k}$, where  $k\in\mbb R$ and $k \neq 0$.

\question{2}

No, not all points at infinity in the physical plane $\mbb R^2$ are the same.

Consider two infinity points $\bm p_1$ and $\bm p_2$, where $\bm p_1$ is the infinity point generated by following the line $y=x, x\rightarrow \infty$; and $\bm p_2$ is the infinity point generated by following the line $y=2x, x\rightarrow \infty$, respectively.
Intuitively, $\bm p_1$ and $\bm p_2$ are on different locations of the $\mbb R^2$ plane, because they are approaching infinity from different directions.

Given any point $(x_1,y_1)$ on the line $y=x$, the corresponding \gls{hc} representation is given by \qbmat*{1, 1, \nicefrac{1}{x_1}}.
Given any point $(x_2,y_2)$ on the line $y=2x$, the corresponding \gls{hc} representation is given by \qbmat*{1, 2, \nicefrac{1}{x_2}}. When we set $x_1 \rightarrow \infty$ and $x_2 \rightarrow \infty$, the infinity points of $y=x$ and $y=2x$ are \qbmat*{1,1,0} and \qbmat*{1,2,0}, respectively. It can be seen that they are different infinity points with distinct \gls{hc} representation.

In general, an infinity point is of the form \qbmat*{u,v,0}, where the $(u,v)$ pair determines the direction that the point approaches infinity.


\question{3}

Let $\bm l = \qbmat*{l_1,l_2,l_3}$, $\bm m=\qbmat*{m_1,m_2,m_3}$, it can be seen that
\begin{align}
    \bm l \bm m^T &= \left[\begin{matrix}l_{1} m_{1} & l_{1} m_{2} & l_{1} m_{3}\\l_{2} m_{1} & l_{2} m_{2} & l_{2} m_{3}\\l_{3} m_{1} & l_{3} m_{2} & l_{3} m_{3}\end{matrix}\right],\\
    \bm m \bm l^T &= \left[\begin{matrix}l_{1} m_{1} & l_{2} m_{1} & l_{3} m_{1}\\l_{1} m_{2} & l_{2} m_{2} & l_{3} m_{2}\\l_{1} m_{3} & l_{2} m_{3} & l_{3} m_{3}\end{matrix}\right].
\end{align}
Clearly, all three columns of $\bm l \bm m^T$ and $\bm m \bm l^T$ are linearly dependent. That is to say, $\rank (\bm l \bm m^T) = \rank (\bm m \bm l^T) = 1$.

\begin{definition}\label{def:q3-1}
    Let $U$ and $V$ be two vector spaces, the sum of two the two vector spaces, denoted by $U + V$, is the set given by
    \begin{align*}
        \{\bm u + \bm v \mid \bm u \in U, \bm v \in V\}.
    \end{align*}
\end{definition}

\begin{theorem}\label{thm:q3-1}
    Let $U$ and $V$ be two vector spaces, then $\dim (U+V) \leq \dim U + \dim V$.
\end{theorem}

```

```latex
\begin{proof}
    Let $\dim U = u$, $\dim V = v$. We can further let
        $B_1 = \{\bm u_1, \bm u_2, \ldots, \bm u_u\}$ and
        $B_2 = \{\bm v_1, \bm v_2, \ldots, \bm v_v\}$
    to be the basis of $U$ and $V$, respectively.

    From \cref{def:q3-1}, we know that an arbitrary vector $\bm q \in U+V$ can be written in the form of $\bm q = \bm u + \bm v$, where $\bm u \in U,~ \bm v \in V$.
    With the basis of $U$ and $V$ provided above, we can further write
    \begin{align}
        \bm u &= k_1\bm u_1 +  k_2\bm u_2 + \cdots + k_u\bm u_u,\\
        \bm v &= s_1\bm v_1 +  s_2\bm v_2 + \cdots + s_v\bm v_v,
    \end{align}
    where $k_1, \ldots, k_u$ and $s_1, \ldots, s_v$ are scalars.
    Since $\bm q = \bm u + \bm v$, we know $\bm q \in \algspan (\bm u_1, \bm u_2, \ldots, \bm u_u, \bm v_1, \bm v_2, \ldots, \bm v_v)$.
    It can be seen that
    \begin{align}
        \dim(U + W) \leq \dim \span \algspan (\bm u_1, \bm u_2, \ldots, \bm u_u, \bm v_1, \bm v_2, \ldots, \bm v_v) \leq u + v = \dim(U) + \dim (V).
    \end{align}
\end{proof}

\begin{theorem}\label{thm:q3-2}
Let $\bm A$ and $\bm B$ be $m \times n$ matrices, then $\rank(\bm A + \bm B) \leq \rank(\bm A) + \rank (\bm B)$.
\end{theorem}

\begin{proof}
    Denote the column space of a matrix $\bm M$ by $C(\bm M)$. By definition, we know that $\rank (\bm M) = \dim C(\bm M)$.
    If we write $\bm A$ and $\bm B$ in terms of column vectors, that is $\bm A = \qbmat*{\bm a_1, \bm a_2, \cdots, \bm a_n}$ and $\bm B = \qbmat*{\bm b_1, \bm b_2, \cdots, \bm b_n}$, then we can express the column spaces of $\bm A$ and $\bm B$ as
    \begin{align}
        C(\bm A) &= \algspan(\bm a_1, \bm a_2, \cdots, \bm a_n),\\
        C(\bm B) &= \algspan(\bm b_1, \bm b_2, \cdots, \bm b_n).
    \end{align}
    It is obvious that
    \begin{align}
        C(\bm A + \bm B) &= \algspan(\bm a_1 + \bm b_1, \bm a_2 + \bm b_2, \cdots, \bm a_n + \bm b_n).
    \end{align}
    $\forall \bm p \in C(\bm A + \bm B)$, we have
    \begin{align}
        \bm p &= k_1 (\bm a_1 + \bm b_1) + k_2 (\bm a_2 + \bm b_2) + \cdots + k_n (\bm a_n + \bm b_n)\\
        &= (k_1 \bm a_1 + k_2 \bm a_2 + \cdots + k_n \bm a_n) + (k_1 \bm b_1 + k_2 \bm b_2 + \cdots + k_n \bm b_n),
    \end{align}
    where $k_1, \ldots, k_n$ are scalars.
    It can be seen that $\bm p \in \algspan(\bm a_1, \bm a_2, \cdots, \bm a_n) + \algspan(\bm a_1, \bm a_2, \cdots, \bm a_n)$.
    We can derive
    \begin{align}
        \rank (\bm A+ \bm B) &= \dim C(\bm A + \bm B) \\
        &\leq \dim \left[ \algspan(\bm a_1, \bm a_2, \cdots, \bm a_n) + \algspan(\bm a_1, \bm a_2, \cdots, \bm a_n) \right]\\
        &\leq \dim \left[\algspan(\bm a_1, \bm a_2, \cdots, \bm a_n)\right] + \dim\left[\algspan(\bm a_1, \bm a_2, \cdots, \bm a_n)\right]\\
        &= \dim C(\bm A) + \dim C(\bm B) = \rank \bm A + \rank \bm B.
    \end{align}

\end{proof}

Using \cref{thm:q3-2}, it can be seen that
\begin{align}
    \rank \bm C = \rank (\bm l \bm m^T + \bm l \bm m^T) \leq \rank (\bm l \bm m^T) + \rank (\bm l \bm m^T) = 2.
\end{align}

\question{4}

We know that a conic can be written in the form of
\begin{align}
    \bm C =
    \qbmat{
        a , \nicefrac{b}{2}, \nicefrac{d}{2};
        \nicefrac{b}{2}, c, \nicefrac{e}{2};
        \nicefrac{d}{2}, \nicefrac{e}{2}, f
    }.
\end{align}

For an \gls{hc} point $\bm x_1 = \qbmat*{x_1, y_1, 1}$, it can be seen that
\begin{align}\label{eqn:q4-1}
    \bm x_1^T \bm C \bm x_1 = a x_{1}^{2} + b x_{1} y_{1} + c y_{1}^{2} + d x_{1} + e y_{1} + f = 0.
\end{align}
We know that $\forall k \in \mbb R~(k\neq 0)$, $k\bm C$ and $\bm C$ represent the same conic. Therefore, the following expression should also hold true:
\begin{align}\label{eqn:q4-2}
    \bm x_1^T \bm (kC) \bm x_1 = ka x_{1}^{2} + kb x_{1} y_{1} + kc y_{1}^{2} + kd x_{1} + ke y_{1} + kf = 0.
\end{align}
In a nontrivial conic, at least one of the coefficients among $a,b,c,d,e,f$ will be nonzero. Without loss of generality, we can assume $a$ is nonzero, and let $k=\frac{1}{a}$. Now \cref{eqn:q4-2} can be written as
\begin{align}\label{eqn:q4-3}
    x_{1}^{2} + \frac{b}{a} x_{1} y_{1} + \frac{c}{a} y_{1}^{2} + \frac{d}{a} x_{1} + \frac{e}{a}y_{1} + \frac{f}{a} = 0.
\end{align}
If we let $\nicefrac{b}{a}=b',~\nicefrac{c}{a}=c',~\nicefrac{d}{a}=d',~\nicefrac{e}{a}=e',~\nicefrac{f}{a}=f'$, then this expression becomes
\begin{align}\label{eqn:q4-4}
    x_{1}^{2} + b' x_{1} y_{1} + c' y_{1}^{2} + d' x_{1} + e' y_{1} + f' = 0.
\end{align}
This is an equation with 5 unknowns, which means we need 5 points $\bm x_1, \ldots, \bm x_5$ to solve for the coefficients of the conic. That is to say, a conic is defined with 5 points.

\question{5}



\begin{enumerate}[label=(\arabic*)]
    \item The \gls{hc} representation of $\bm l_1$ is given by
    \begin{align}
        \qbmat{0;0;1} \times \qbmat{1;2;1} = \qbmat{-2;1;0}.
    \end{align}
    The \gls{hc} representation of $\bm l_2$ is given by
    \begin{align}
        \qbmat{3;4;1} \times \qbmat{5;6;1} = \qbmat{-2;2;-2}.
    \end{align}
    The intersection between $\bm l_1$ and $\bm l_2$ is
    \begin{align}
        \qbmat{-2;1;0} \times \qbmat{-2;2;-2} = \qbmat{-2;-4;-2}.
    \end{align}
    That is, the intersection is given by $\left(1, 2 \right)$.
    \item The \gls{hc} representation of line $\bm l_3$ is given by
    \begin{align}
        \qbmat{7;-8;1} \times \qbmat{-7;8;1} = \qbmat{-16;-14;0}.
    \end{align}
    Since the third element of $\bm l_1$ and $\bm l_3$ are both zero, we know that the two points pass through the origin. Therefore, they must intersect at the origin. In this case, we only need to compute $\bm l_1$ and $\bm l_3$, which consists of two steps.
\end{enumerate}

\question{6}

The \gls{hc} representation of $\bm l_1$ is given by
\begin{align}
    \qbmat{-4;0;1} \times \qbmat{-2;8;1} = \qbmat{-8;2;-32}.
\end{align}
```

```latex
The \gls{hc} representation of $\bm l_2$ is given by
\begin{align}
    \qbmat{0;-2;1} \times \qbmat{4;14;1} = \qbmat{-16;4;8}.
\end{align}
The intersection between $\bm l_1$ and $\bm l_2$ is given by
\begin{align}
    \qbmat{-8;2;32} \times \qbmat{-16;4;8} = \qbmat{144;576;0}.
\end{align}
Since this is an infinity point, we know that $\bm l_1$ and $\bm l_2$ are parallel.

\question{7}

$x=1$ can be written as $1x + 0y -1 = 0$, which means its corresponding \gls{hc} representation is \qbmat*{1,0,-1}. $y=-1$ can be written as $0x+1y+1=0$, which means its corresponding \gls{hc} representation is \qbmat*{0,1,1}. The intersection between them is given by
\begin{align}
    \qbmat{1;0;-1}\times \qbmat{0;1;1} = \qbmat{1;-1;1}.
\end{align}
That is, the intersection is at $(1,-1)$.

\question{8}

The equation of the ellipse is
\begin{align}
    \frac{(x-2)^2}{\left(\frac{1}{2}\right)^2} + (y-3)^2 = 1.
\end{align}
This simplifies to
\begin{align}
    4x^2 + y^2 - 16x - 6y + 24 = 0.
\end{align}
This conic can be written as
\begin{align}
    \bm C = \qbmat{
        4, 0, -8;
        0, 1, -3;
        -8, -3, 24.
    }
\end{align}
Since $\bm p$ is the origin, the \gls{hc} representation is given by \qbmat*{0,0,1}. Therefore, the polar line is given by $\bm l = \bm C \bm p = \qbmat*{-8,-3,24}$.

The \gls{hc} representation of the $x$ axis is given by \qbmat*{0,1,0}. Therefore, the intersection between $\bm l$ and the $x$ axis is
\begin{align}
    \qbmat{-8;-3;24} \times \qbmat{0;1;0} = \qbmat{-24;0;-8}.
\end{align}
That is to say, the intersection between the polar line and the $x$ axis is $(3, 0)$.

The \gls{hc} representation of the $y$ axis is given by \qbmat*{1,0,0}. Therefore, the intersection between $\bm l$ and the $y$ axis is
\begin{align}
    \qbmat{-8;-3;24} \times \qbmat{1;0;0} = \qbmat{0;24;3}.
\end{align}
That is to say, the intersection between the polar line and the $y$ axis is $(0, 8)$.


\largeappendix
\SourceListing{
    ../alanhm.cls,latex,
    1_ZiyueXiang.tex,latex
}

\end{document}
```