Computer Vision
ECE 661 Homework 1

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August 30, 2016

1. Question 1 Solution

Let us consider the points in the representational space $\mathbb{R}^3$ to be of the form $\begin{pmatrix} u \\ v \\ w \end{pmatrix}$.

Now, $x = \frac{u}{w} = 0$ and $y = \frac{v}{w} = 0 \implies u = 0$, $v = 0$ and $w \in \mathbb{R} \setminus 0$ or $w \neq 0$

$\implies$ So all the points forming the equivalence class represented by $\begin{pmatrix} 0 \\ 0 \\ w \end{pmatrix} = w \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}^3$

where $w \in \mathbb{R} \setminus 0$ or $w \neq 0$, represent the origin $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \mathbb{R}^2$

2. Question 2 Solution

No, all points at infinity in the physical plane $\mathbb{R}^2$ are not the same. Let us consider the point in the physical plane $\mathbb{R}^2$. This point can be represented in the representational space $\mathbb{R}^3$ to be of the form $\begin{pmatrix} u \\ v \\ w \end{pmatrix}$. Assuming, $w \in \mathbb{R} \setminus 0$ or $w \neq 0$ Now, $x = \frac{u}{w}$ and $y = \frac{v}{w}$

$\implies$ As $w \to 0$ then the physical points in $\mathbb{R}^2$ are x = infinity , y = infinity which means the x and y coordinates of the physical point moves moves away from the origin towards infinity in a particular direction. However, the direction in which the point approaches infinity in the physical space $\mathbb{R}^2$ is different for different points and is controlled by the u and v values.

For e.g, the point represented by $\begin{pmatrix} 1 \\ 0 \\ w \to 0 \end{pmatrix}$ approaches infinity in $\mathbb{R}^2$ along the x axis.

For e.g, the point represented by $\begin{pmatrix} 0 \\ 1 \\ w \to 0 \end{pmatrix}$ approaches infinity in $\mathbb{R}^2$ along the y axis.

3. Question 3 Solution

The Degenerate Conic C is represented by $C = lm^T + ml^T$,
where l and m are the 2 lines of the degenerate conic represented in the representational space $\mathbb{R}^3$ as $l = \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix}$ and $m = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}$.

Now $lm^T = \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix} \begin{pmatrix} m_1 & m_2 & m_3 \end{pmatrix} = \begin{pmatrix} l_1m_1 & l_1m_2 & l_1m_3 \\ l_2m_1 & l_2m_2 & l_2m_3 \\ l_3m_1 & l_3m_2 & l_3m_3 \end{pmatrix}$

Outer Product of l and m generates the matrix whose i^{th} column is $m_i \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix}$. So the columns are linearly dependent since they are vector $\begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix}$ multiplied by scalars m_i and hence $\begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix}$ forms the basis. Hence rank of lm^T is 1.

Similarly $ml^T = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_3 \end{pmatrix} = \begin{pmatrix} m_1l_1 & m_1l_2 & m_1l_3 \\ m_2l_1 & m_2l_2 & m_2l_3 \\ m_3l_1 & m_3l_2 & m_3l_3 \end{pmatrix}$

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Now we know that

$$rank(A+B) \leq rank(A) + rank(B)$$

The rank of a matrix can be defined as the dimension of the space that its column vectors span or the number of independent columns. Let $(a_1, \cdots, a_n)$ denote the column vectors of A and $(b_1, \cdots, b_n)$ denote the column vectors of B and $(r_1, \cdots, r_n)$ denote the column vectors of $A+B$. The column vectors r_i can be written as a linear combination of a_i and b_i

$$r_i = c_j a_i + c_k b_i \qquad i = 1, \cdots, n \qquad c_j, c_k = constants$$

$\implies$ the columns of the matrix $A+B$ is a linear combination of the rows of the matrices A and B. So the maximum number of independent columns in $A+B$ is bounded by the sum of the number of independent columns in A and B.
$\implies$ the dimension of the space spanned by $A+B$ cannot be greater than the sum of the dimensions of the spaces spanned by A and B.
$\implies$ the rank is the number of independent columns in the matrix and this thus proves the identity.

Applying this,

$$rank(C) = rank(lm^T + ml^T) \leq rank(lm^T) + rank(ml^T)$$

$$rank(lm^T) + rank(ml^T) = 1 + 1 = 2$$

$$\implies rank(C) \leq 2$$

4. Question 4 Solution

a. The points in the physical space $\begin{pmatrix} x \\ y \end{pmatrix}$ in $\mathbb{R}^2$ can be represented in Homogeneous Co-ordinates, as $\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$ in the representational space in $\mathbb{R}^3$.

Line passing through $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ is given by $l_1 = x_1 \times x_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix}$

Line passing through $\begin{pmatrix} -3 \\ 3 \end{pmatrix}$ & $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ is given by $l_2 = x_3 \times x_4 = \begin{pmatrix} -3 \\ 3 \\ 1 \end{pmatrix} \times \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$

The point of intersection of l_1 and l_2 is given by $x_{int} = l_1 \times l_2 = \begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} -6 \\ -9 \\ -8 \end{pmatrix}$

The point of intersection in the physical plane $\mathbb{R}^2$ is given by $x = \frac{-6}{-8} = \frac{3}{4}$ and $y = \frac{-9}{-8} = \frac{9}{8}$

b.

Method 1 (1 step)

By observation, this problem can be solved in one step.

By observation the points $\begin{pmatrix} 4 \\ 5 \end{pmatrix}$ & $\begin{pmatrix} -4 \\ -5 \end{pmatrix}$ are symmetric about the origin and hence the line connecting these 2 points must pass through the origin. According to the question, the line l_1 passes through the origin. $\implies$ The point of intersection is the origin $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Method 2 (2 steps)

The line passing through $\begin{pmatrix} 4 \\ 5 \end{pmatrix}$ & $\begin{pmatrix} -4 \\ -5 \end{pmatrix}$ is given by $l_3 = x_5 \times x_6 = \begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix} \times \begin{pmatrix} -4 \\ -5 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \\ -8 \\ 0 \end{pmatrix}$

Now the point x_1 is the origin $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ that lies on the line l_1 according to the question and $l_1^T x_1 = 0$.

Also we see, $l_3^T x_1 = \begin{pmatrix} 10 & -8 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0 \implies x_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is the point of intersection since 2 intersecting lines can interesect only once.

Method 3 (3 steps)

Also we could try in the conventional method, Line passing through $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ is given by $l_1 = x_1 \times x_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix}$

The line passing through $\begin{pmatrix} 4 \\ 5 \end{pmatrix}$ & $\begin{pmatrix} -4 \\ -5 \end{pmatrix}$ is given by $l_3 = x_5 \times x_6 = \begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix} \times \begin{pmatrix} -4 \\ -5 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \\ -8 \\ 0 \end{pmatrix}$

The point of intersection of l_1 and l_3 is given by $x_{int} = l_1 \times l_3 = \begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix} \times \begin{pmatrix} 10 \\ -8 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$

$\implies \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is the point of intersection by the concept of equivalence class.

The point of intersection in the physical plane $\mathbb{R}^2$ is given by $x = \frac{0}{1} = 0$ and $y = \frac{0}{1} = 0$

5. Question 5 Solution

Line passing through $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -2 \end{pmatrix}$ is given by $l_1 = x_1 \times x_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$

Line passing through $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$ & $\begin{pmatrix} 0 \\ -3 \end{pmatrix}$ is given by $l_2 = x_3 \times x_4 = \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 9 \end{pmatrix}$

The point of intersection of l_1 and l_2 is given by $x_{int} = l_1 \times l_2 = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} \times \begin{pmatrix} 3 \\ 3 \\ 9 \end{pmatrix} = \begin{pmatrix} 18 \\ -18 \\ 0 \end{pmatrix}$

The point of intersection in the physical plane $\mathbb{R}^2$ is given by:
$x = \frac{18}{0}$ = infinity and $y = \frac{-18}{0}$ = infinity $\implies$ the lines l_1 and l_2 never intersect.

Comments: For a line in the representational space, $l = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, the slope is given by the ratio of a and b. For lines l_1 and l_2, this ratio is 1 and hence the lines are parallel and will intersect at infinity.

From the mathematical solution presented above, we can see that the point of intersection $\begin{pmatrix} 18 \\ -18 \\ 0 \end{pmatrix}$ is an ideal point.

$\implies$ This means in the physical space, $\mathbb{R}^2$ the abscissa = infinity and ordinate = infinity.
$\implies$ The point of intersection is at infinity in $\mathbb{R}^2$, along the direction having a slope of -1 which is controlled by the values 18 and -18.

6. Question 6 Solution

The implicit representation of the circle in the physical plane $\mathbb{R}^2$ is:

$$
\begin{aligned}
(x-(-6))^2 + (y-(-6))^2 &= 1^2 \\
\implies (x+6)^2 + (y+6)^2 &= 1 \\
\implies x^2 + 12x + 36 + y^2 + 12y + 36 &= 1 \\
\implies x^2 + y^2 + 12x + 12y + 71 &= 0
\end{aligned}
$$

We substitute $x = x_1/x_3$ and $y = x_2/x_3$ in the equation. The implicit representation in Homogeneous Coordinates in the representational space in $\mathbb{R}^3$ is:

$$
\begin{aligned}
\implies \left(\frac{x_1}{x_3}\right)^2 + \left(\frac{x_2}{x_3}\right)^2 + 12\left(\frac{x_1}{x_3}\right) + 12\left(\frac{x_2}{x_3}\right) + 71 &= 0 \\
\implies x_1^2 + x_2^2 + 12x_1x_3 + 12x_2x_3 + 71x_3^2 &= 0
\end{aligned}
$$

Rewriting this as a Vector Matrix product, we get:

$$
\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 6 \\ 0 & 1 & 6 \\ 6 & 6 & 71 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0
$$

The HC representation of the circle is given by :

$$
C = \begin{pmatrix} 1 & 0 & 6 \\ 0 & 1 & 6 \\ 6 & 6 & 71 \end{pmatrix}
$$

According to the problem x is the origin in $\mathbb{R}^2$ physical space. $\implies x = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ in $\mathbb{R}^3$

The polar line l is given by:

$$
l = Cx = \begin{pmatrix} 1 & 0 & 6 \\ 0 & 1 & 6 \\ 6 & 6 & 71 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \\ 71 \end{pmatrix}
$$

In the representational space, the X axis is $l_x = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and Y axis is $l_y = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

The x-intercept is given by $l \times l_x = \begin{pmatrix} 6 \\ 6 \\ 71 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -71 \\ 0 \\ 6 \end{pmatrix} \implies x_{intercept} = -\frac{71}{6}$

The y-intercept is given by $l \times l_y = \begin{pmatrix} 6 \\ 6 \\ 71 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 71 \\ -6 \end{pmatrix} \implies y_{intercept} = -\frac{71}{6}$

$\implies$ The polar line l cuts the x-axis at $(-\frac{71}{6}, 0)$ and the y-axis at $(0, -\frac{71}{6})$