Homework 1

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1 Problem 1

Homogeneous coordinates of the origin in the physical space is given by: $\begin{bmatrix} 0 \\ 0 \\ w \end{bmatrix}$

where $w \in \mathbb{R}$ and $w \neq 0$

2 Problem 2

No, all points at infinity in the physical plane are not the same. Difference arises because of the difference in direction. For example, consider the point $[a, b, c]$ in R^3 representational space which is the homogeneous coordinates of the point $[a/c, b/c]$ in the R^2 physical space. Now as c approaches 0, the x and y coordinates of the physical point move further away from the origin and approach infinity. Now the point approaches infinity in a specific direction governed by a and b. And so points at infinity differ in the direction in which they approach infinity.

3 Problem 3

To find the line passing through two points we take the cross product of the homogeneous coordinate representations of the two points.

Step 1: Find $l1$

$$l1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

Step 2: Find $l2$

$l2 = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} \times \begin{bmatrix} -4 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ -7 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

Step 3: Find the intersection of $l1$ and $l2$

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$$

Now if the second line passes through (3,4) and (-3,-4) instead we can find the point of intersection in either a single step or in two steps.

First, let us look at the method requiring two steps. In this method we calculate the line $l2$ as done before: $l2 = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} \times \begin{bmatrix} -3 \\ -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ -6 \\ 0 \end{bmatrix}$

Now observe that the third coordinate in the homogeneous coordinate representation is zero for $l1$ and $l2$. This implies that both lines have the form $ax + by = 0$, which implies that both of them pass through the origin. Hence in two steps we are able to infer the point of intersection.

Second, if we observe the two points through which the second line passes, i.e. (3,4) and (-3,-4), and think about the position vectors of these points, we see that both the position vectors have the same slope (the angle it makes to x-axis is the same, its also true for the angle made with y-axis). Which means the line they lie on passes through the origin. We already know that the other line $l1$ passes through the origin and hence the point of intersection is the origin.

4 Problem 4

The homogeneous representation of a degenerate conic is given by $C = lm^T + ml^T$, where l and m are the homogeneous coordinates of the two lines obtained when we slice the double cones with a plane that passes through the axis of the double cones.

If we consider $l = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix}$ and $m = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix}$ the outer product is given by:

$$lm^T = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} [m_1 \ m_2 \ m_3] = \begin{bmatrix} l_1m_1 & l_1m_2 & l_1m_3 \\ l_2m_1 & l_2m_2 & l_2m_3 \\ l_3m_1 & l_3m_2 & l_3m_3 \end{bmatrix}$$

As we can see the second column is m_2/m_1 times the first column and the third column is m_3/m_1 times the first column. Hence the dimension of the column space is one, which means that the rank of the outer product matrix is one. Same is the case with the outer product matrix ml^T. The homogeneous representation of the degenerate conic is the sum of two outer product matrices. If A and B are two matrices then $Rank(A + B) \leq Rank(A) + Rank(B)$. To prove this consider $(a_1, a_2, ...a_n)$ to represent the columns of A and similarly $(b_1, b_2, ...b_n)$

to represent the columns of B. Each column is a $1 \times m$ vector, which implies A and B have dimension $m \times n$. Therefore $A + B$ will have columns $(a_1 + b_1, a_2 + b_2, ...a_n + b_n)$. The rank of a matrix is equal to the dimension of the column space. Any vector in the column space of $A + B$ can be written as the $v = k_1(a_1 + b_1) + k_2(a_2 + b_2) + ...k_n(a_n + b_n) = k_1 a_1 + k_2 a_2 + ...k_n a_n + k_1 b_1 + k_2 b_2 + ...k_n b_n$. Hence the $Rank(A + B)$ is at most equal to the dimension of the linear subspace generated by the $2n$ vectors $(a_1, a_2, ...a_n, b_1, b_2, ...b_n)$. Now we can choose at most $Rank(A)$ linearly independent vectors from columns of A and $Rank(B)$ linearly independent vectors from columns of B. Which means at most $Rank(A) + Rank(B)$ linearly independent vectors are required to span the space spanned by the $2n$ vectors, i.e. the dimension of the space spanned by $(a_1, a_2, ...a_n, b_1, b_2, ...b_n)$ has an upper bound of $Rank(A) + Rank(B)$. And since the space spanned by columns of $A + B$ is contained within the space spanned by $(a_1, a_2, ...a_n, b_1, b_2, ...b_n)$ the dimensions of the column space of $A + B$ is upper bounded by $Rank(A) + Rank(B)$. Which proves that $Rank(A + B) \leq Rank(A) + Rank(B)$.
In the case of the degenerate conic therefore we can say that the rank cannot exceed 2 as it is the sum of two matrices each with rank = 1.

5 Problem 5

The equation of the circle centered at (5,5) and having a radius 1 is given by:

$$(x - 5)^2 + (y - 5)^2 = 1$$

$$x^2 + y^2 - 10x - 10y + 49 = 0$$

The homogeneous coordinates of the conic is then given by:

$$\begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & -5 \\ -5 & -5 & 49 \end{bmatrix}$$

The polar line is given by:

$$CX = \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & -5 \\ -5 & -5 & 49 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ -5 \\ 49 \end{bmatrix}$$

The Y-axis can be represented in homogeneous coordinates by: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

The point of intersection is then just the cross product given by:

$$\begin{bmatrix} -5 \\ -5 \\ 49 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 49 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 9.8 \\ 1 \end{bmatrix}$$