ECE661: Homework 1

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1. All the points $\begin{pmatrix} 0 \\ 0 \\ k \end{pmatrix}$ where $k! = 0$.

2. No, not all the points at infinity are the same, because they can approach infinity along different directions in $\mathbb{R}^2$.

   Let the homogeneous coordinate vector of a point at infinity be $\begin{pmatrix} u \\ v \\ 0 \end{pmatrix}$. The direction along which this point approaches infinity is determined by u and v. For example, the point $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ approaches infinity along the x-axis in $\mathbb{R}^2$, while the point $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ approaches infinity along the y-axis in $\mathbb{R}^2$.

3. - First step: get the 3D parameter vector of l_1 as the cross product of the two 3D homogeneous coordinate vectors of the two points on l_1:

   $$l_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

   Second step: get the 3D parameter vector of l_2 as the cross product of the two 3D homogeneous coordinate vectors of the two points on l_2:

   $$l_2 = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} \times \begin{pmatrix} -4 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ -7 \\ 7 \end{pmatrix} \equiv \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

   Third step: get the homogeneous coordinate vector of the intersection point x of l_1 and l_2 as the cross product of their 3D parameter vectors:

   $$x = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

   This means that the two lines intersect at infinity, i.e. they are parallel.

   - In the second case when l_2 passes through $(-3, -4)$ instead of $(-4, -3)$, we can find the intersection using two steps as follows:

   First step: get the 3D parameter vector of l_1 as the cross product of the two 3D homogeneous coordinate vectors of the two points on l_1:

$$l_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

Second step: get the 3D parameter vector of l_2 as the cross product of the two 3D homogeneous coordinate vectors of the two points on l_2:

$$l_2 = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} \times \begin{pmatrix} -3 \\ -4 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ -6 \\ 0 \end{pmatrix} \equiv \begin{pmatrix} 4 \\ -3 \\ 0 \end{pmatrix}$$

Since the third component of both the parameter vectors of l_1 and l_2 is 0, the two lines have zero y-intercept, i.e. they pass through the origin. As a result, the intersection point is the origin.

As a double check, get the homogeneous coordinate vector of the intersection point x of l_1 and l_2 as the cross product of their 3D parameter vectors:

$$x = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 4 \\ -3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

This confirms that the two lines intersect at the origin in $\mathbb{R}^2$.

4. The matrix C of a degenerate conic is written as $C = lm^T + ml^T$ where l and m are the 3D parameter vectors of two lines.

   Let $l = \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix}$, and $m = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}$. The columns of the outer product $lm^T = \begin{pmatrix} l_1m_1 & l_1m_2 & l_1m_3 \\ l_2m_1 & l_2m_2 & l_2m_3 \\ l_3m_1 & l_3m_2 & l_3m_3 \end{pmatrix}$ are scalar multiples of the first column. This means that there is only one independent column. Hence, $rank(lm^T) = 1$. Using the same argument, $rank(ml^T) = 1$.

   Adding two matrices can not result in a matrix whose rank is larger than the summation of the ranks of the two input matrixes. That's because the columns of the output matrix are linear combinations of the columns of the input matrixes, so the output matrix will not introduce new independent columns. As a result, $rank(C) \leq rank(lm^T) + rank(ml^T)$, i.e. $rank(C) \leq 2$. So, the rank of a degenerate conic matrix can not exceed 2.

5. The implicit form for the circle:

$$(x-5)^2 + (y-5)^2 = 1$$

$$x^2 + y^2 - 10x - 10y + 49 = 0$$

   The implicit form for a conic:

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

   Hence, $a = c = 1, b = 0, d = e = -10$, and $f = 49$. Given that the homogeneous coordinate matrix of a conic is:

$$C = \begin{pmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{pmatrix}$$

   As a result, the homogeneous coordinate matrix of the given circle is:

$$C = \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & -5 \\ -5 & -5 & 49 \end{pmatrix}$$

The polar line l is the result of multiplying the homogeneous coordinate representation of both the circle and the point x:

$$l = Cx = \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & -5 \\ -5 & -5 & 49 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -5 \\ -5 \\ 49 \end{pmatrix}$$

The homogeneous coordinate vector of the intersection point p of l and the y-axis ($x = 0$) is the cross product of their homogeneous coordinate vectors:

$$p = \begin{pmatrix} -5 \\ -5 \\ 49 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 49 \\ 5 \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 9.8 \\ 1 \end{pmatrix}$$

As a result, the polar line intersects the y-axis at the physical 2D point $(0, 9.8)$.