ECE 661
Purdue U.
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Lecture 6



# World 3D: Representing Points, Planes, and Lines

Reference: "Multiple View Geometry in Computer Vision" by Hartley and Zisserman

- We represent a point in 3D physical space $\mathbb{R}^3$ by $(X, Y, Z)^T$. Its homogeneous coordinates in $\mathbb{R}^4 - (0,0,0,0)^T$ will be denoted by the vector $(x_1, x_2, x_3, x_4)^T$ with $\boxed{X = x_1/x_4}$, $\boxed{Y = x_2/x_4}$, $\boxed{Z = x_3/x_4}$.
- All points $k \cdot (x_1, x_2, x_3, x_4)^T$ for all non-zero $k$ form an equivalence class in $\mathbb{R}^4$, in the sense that they all represent the same physical point in $\mathbb{R}^3$.
- Using the same arguments as in Lecture 2, a point at infinity in physical $\mathbb{R}^3$ is represented by homogeneous coordinates of type $(x_1, x_2, x_3, 0)^T$
- Using homogeneous coordinates, a transformation of the physical space $\mathbb{R}^3$ can be represented by a 4x4 homography: $\boxed{X' = HX}$. Since only the ratios matter, in general, the homography $H$ has 15 degrees of freedom.

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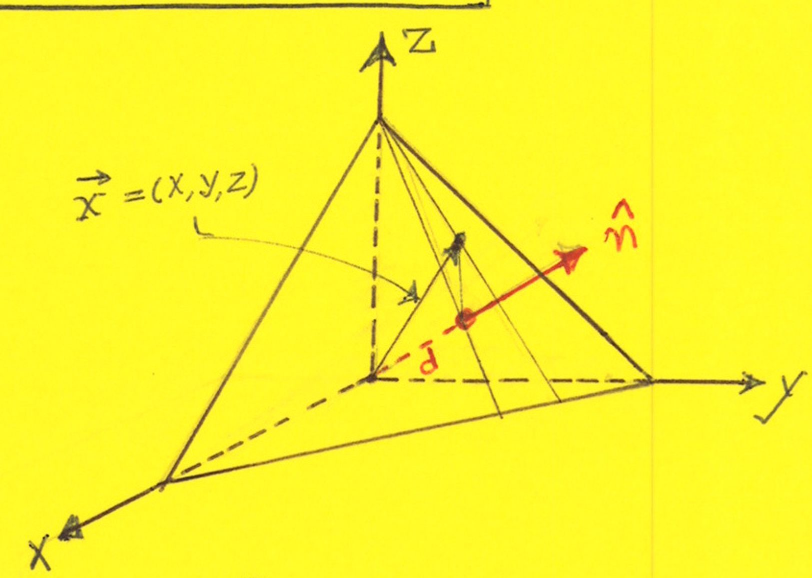
## Representing a Plane in 3D




- Perhaps the most common way to represent a plane is through the geometrical construction at right:
- The plane is characterized by a unit normal vector $\hat{n}$ passing through the origin and the shortest distance $d$ to the plane from the origin (this is along $\hat{n}$). Now given any point $\vec{X} = (X, Y, Z)^T$ on the plane, it follows straightforwardly that $\boxed{\vec{X} \cdot \hat{n} = d}$. This equation can be written as $\boxed{n_x X + n_y Y + n_z Z - d = 0}$ where $n_x, n_y, n_z$ are the direction cosines associated with the unit normal $\hat{n}$.
- Therefore, we can write $\boxed{\pi_1 X + \pi_2 Y + \pi_3 Z + \pi_4 = 0}$ as a general representation of a plane through a 4-vector parameter $(\pi_1, \pi_2, \pi_3, \pi_4)^T$. Using homogeneous coordinates, this translates into $\boxed{\pi_1 x_1 + \pi_2 x_2 + \pi_3 x_3 + \pi_4 x_4 = 0}$
- We thus have the following compact representation of a plane in homogeneous coordinates $\boxed{\pi^T x = 0}$ wher $\pi = (\pi_1, \pi_2, \pi_3, \pi_4)^T$ and $x = (x_1, x_2, x_3, x_4)^T$. We also recognize that the unit normal to the plane is encoded in $(\pi_1, \pi_2, \pi_3)$ and the perpendicular distance to the plane in $\pi_4$. More precisely, $(\pi_1, \pi_2, \pi_3)$ is the normal to the plane in physical $\mathbb{R}^3$ and $\boxed{\pi^4 / \|(\pi_1, \pi_2, \pi_3)^T\|}$ the perpendicular distance

- We therefore say that for a point $x$ to fall on a plane $\pi$, the point must satisfy the condition $\boxed{\pi^T x = 0}$, or equivalently, $\boxed{x^T \pi = 0}$.
- A plane is obviously defined by <u>any</u> three non-collinear points. So, given three such points, $x_1$, $x_2$, and $x_3$, what is the equation of the plane passing through these three points? The following follows trivially from the previous bullet:

$$\begin{bmatrix} x_1^T \\ x_2^T \\ x_3^T \end{bmatrix} \pi = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

A modern SVD based approach can be used to find the null-space solution vector $\pi$. You decompose the $3\times4$ matrix on the left by writing it as $U\Sigma V^T$. The last eigenvector in $V$, the one corresponding to the near-zero singular value, is your solution for $\pi$

$U$ is $3\times4$
$\Sigma$ is $4\times4$
$V$ is $4\times4$

This says that the plane must define the one-dimensional null space of the $3\times4$ matrix formed by the homogeneous coordinates of the 3 points

Note that we are talking about SVD of an under-determined system. $(m < n)$

That's why $V$ contains the basis of the null space

- Given three points, the approach described above defines a plane implicitly. Here is a more explicit approach: We recognize that if we construct a $4\times4$ matrix $\boxed{M = [x, x_1, x_2, x_3]}$ where $x$ is a general point (as a vector variable) on the plane, and $x_1$, $x_2$, and $x_3$, the three given specific points, the determinant of such a matrix $M$ must be zero since each point can be expressed as a linear combination of the other three. That is, $\boxed{\det(M) = 0}$. We have

$$M = \begin{bmatrix} x & x_1 & x_2 & x_3 \\ y & y_1 & y_2 & y_3 \\ z & z_1 & z_2 & z_3 \\ w & w_1 & w_2 & w_3 \end{bmatrix} \rightarrow \tilde{M}$$

variables

known values

- Let $\tilde{M}$ denote the $4\times3$ submatrix formed by the <u>known</u> last three column vectors of $M$. And let $D_{ijk}$ stand for the determinant calculated from the $i^{th}$, the $j^{th}$, and the $k^{th}$ rows of $\tilde{M}$. Now we can express the determinant of $M$ as:

$$\boxed{\det(M) = x \cdot D_{234} - y \cdot D_{134} + z \cdot D_{124} - w \cdot D_{123} = 0}$$

- Comparing the above equation with the definition $\boxed{x^T \pi = 0 = x\pi_1 + y\pi_2 + z\pi_3 + w\pi_4}$ we get the following explicit solution for the plane $\pi$:

$$\boxed{\pi = \begin{pmatrix} D_{234} \\ -D_{134} \\ D_{124} \\ -D_{123} \end{pmatrix}}$$

- It is interesting to note that when $x_1$, $x_2$, and $x_3$ are all at infinity (that is, $w_1$, $w_2$ and $w_3$ are all zero), $D_{234}$, $D_{134}$, and $D_{124}$ will all be zero. In this special case, the solution plane will be given by $\boxed{\pi_\infty = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}$. This then is the definition of the plane at infinity.

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# Finding the Point at the Intersection of Three Planes

- Recall if a point $x$ is on a plane $\pi$, then $\boxed{\pi^T x = 0}$. Given planes $\pi_1$, $\pi_2$, and $\pi_3$, the common point of intersection must obey

$$\boxed{\begin{bmatrix} \pi_1^T \\ \pi_2^T \\ \pi_3^T \end{bmatrix} x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}}$$

- As you notice, this is <u>dual</u> to the problem of finding the plane that pass through three given points. The solution strategies are therefore the same.
- This is a good time to mention that points and planes are duals in 3D — in much the same manner that points and lines are duals in 2D. Lines are self-duals in 3D.

# Applying a Projective Transformation to a Plane

- We want to know what happens to the homogeneous representation of a plane ~~the~~ when the space in which the plane resides undergoes a 4x4 homography, H.
- By definition $\boxed{X' = HX}$. Therefore, $\boxed{X = H^{-1}X'}$
- We know that for $X$ to belong to plane $\pi$, $\boxed{\pi^T X = 0}$. Substituting for $X$, we get $\boxed{\pi^T H^{-1} X' = 0}$, which can be expressed as $\boxed{(H^{-T}\pi)^T X' = 0}$ implying that the "image" point $X'$ resides on a plane $\boxed{\pi' = H^{-T}\pi}$.

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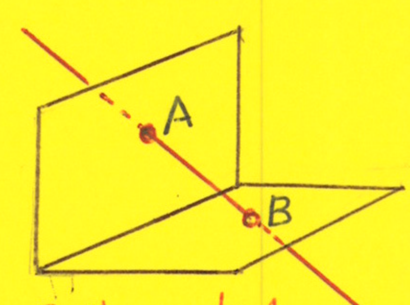
# Representing Lines in 3D

- To say the least, the homogeneous representation for 3D lines is NOT as easy, or as straightforward, as it is for 3D points and planes in 3D.
- The fact that a 3D line has four degrees of freedom requires ideally that we use a five-dimensional homogeneous reprentations for lines in 3D. But such a representation would make them incompatible with the homogeneous representations for 3D points and planes. (In the figure at right, we do not factor in what it takes to specify the locations and the orientations of the two planes. Those are completely arbitrary, in the sense that we can choose any two planes.)




Both points A and B have two degrees of freedom. Therefore line AB has 4 DoF.

- The desire to keep the homogeneous representation of 3D lines compatible with those used for 3D points and for the planes in 3D has resulted in a number of different workarounds for the case of 3D lines. We will review some of these in what follows.

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## Representation of a 3D Line With a Vector Span

- A line may be represented either as the join of two points or as the intersection of two planes. The latter is the dual of the former and one can easily be derived from the other. A working algorithm frequently needs both forms as each is "optimum" for answering certain kinds of questions.

| A 3D Line Formed by Joining Two Points A and B | A 3D Line Formed by the Intersection of Two Planes P and Q |
|---|---|
| – The line is represented by a 2x4 matrix: $$W = \begin{bmatrix} A^T \\ B^T \end{bmatrix} \updownarrow 2 \quad \xleftrightarrow{4}$$ | – The line is represented by a 2x4 matrix $$W^* = \begin{bmatrix} P^T \\ Q^T \end{bmatrix} \updownarrow 2 \quad \xleftrightarrow{4}$$ |
| – Being underdetermined, the 2x4 W possesses a 2D null space. If P and Q are the basis vectors of the null space, then | – Being underdetermined, the 2x4 $W^*$ possesses a 2D null space.<br>– If A and B are the basis vectors of the null space, then |

$WP = \begin{pmatrix}0\\0\end{pmatrix}$ and $WQ = \begin{pmatrix}0\\0\end{pmatrix}$

– The above implies

$\begin{bmatrix}A^TP\\B^TP\end{bmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$ and $\begin{bmatrix}A^TQ\\B^TQ\end{bmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$

– Our definition of what it means to be a point on a plane implies that the null vector P and Q must be planes, and that the points A and B must reside on both these planes.

$W^*A = \begin{pmatrix}0\\0\end{pmatrix}$ and $W^*B = \begin{pmatrix}0\\0\end{pmatrix}$

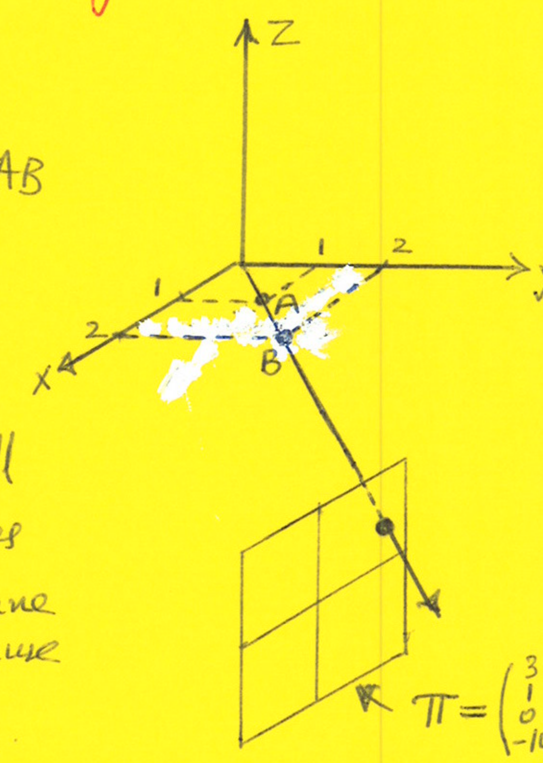
– the above implies

$\begin{bmatrix}P^TA\\Q^TA\end{bmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$ and $\begin{bmatrix}P^TB\\Q^TB\end{bmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$

– Our definition of what it means to be a point on a plane implies that the null vectors A and B are points that reside on both the planes P and Q. The intersection of the planes P and Q contains the line joining A and B.

$$\boxed{W^*W^T = \begin{bmatrix}P^T\\Q^T\end{bmatrix}\begin{bmatrix}A & B\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}}$$

- **EXERCISE:** You are given two points $A = \begin{pmatrix}1\\1\\0\\1\end{pmatrix}$ and $B = \begin{pmatrix}2\\2\\0\\1\end{pmatrix}$ and a plane $\pi = \begin{pmatrix}3\\1\\0\\-10\end{pmatrix}$. At what point does the line joining A and B pierce the plane $\pi$?




**SOLUTION:** A vector span representation of line AB is $W = \begin{bmatrix}A^T\\B^T\end{bmatrix} = \begin{bmatrix}1 & 1 & 0 & 1\\2 & 2 & 0 & 1\end{bmatrix}$

However, this problem is best solved with the dual representation $W^* = \begin{bmatrix}P^T\\Q^T\end{bmatrix}$ for the line where P and Q are two linearly independent basis vectors in the null space of W. Since we know that P and Q represent planes that contain the points A and B, for P we choose the plane XY. So $P = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$ because its normal is along Z and because it passes through the origin. For Q, we choose the plane perpendicular to XY plane and that contains line AB. The normal to this plane is along $(-1, 1, 0)$ and since it also passes through origin, $Q = \begin{pmatrix}-1\\1\\0\\0\end{pmatrix}$. [In "real-life" you'd compute P and Q by an SVD analysis of W.] So we have $W^* = \begin{bmatrix}0 & 0 & 1 & 0\\-1 & 1 & 0 & 0\end{bmatrix}$. Now the point at which line AB pierces $\pi$ must be common to all three planes P, Q, and $\pi$. Therefore, the solution point is given by

$$\begin{bmatrix}P^T\\Q^T\\\pi^T\end{bmatrix}X = \begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow \begin{bmatrix}0 & 0 & 1 & 0\\-1 & 1 & 0 & 0\\3 & 1 & 0 & -10\end{bmatrix}\begin{pmatrix}X\\Y\\Z\\W\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \xrightarrow[W=1]{} \begin{matrix}X = 2.5\\Y = 2.5\\Z = 0\end{matrix} \Rightarrow X = \begin{pmatrix}2.5\\2.5\\0\\1\end{pmatrix}$$

## Representing a 3D Line With a Plucker Matrix

Rank of an outer product is always 1

DoF: How many vars for specifying

Rank: How many linearly ind. vecs. in col. or row space

- It is possible to represent the line formed by joining the two points A and B by the 4x4 skew-symmetric matrix $\boxed{L = AB^T - BA^T}$. (Its dual consists of representing a line formed by the intersection of two planes P and Q by the 4x4 skew-symmetric matrix $\boxed{L^* = PQ^T - QP^T}$) L and $L^*$ are examples of Plucker matrices. Suppose $A = \begin{pmatrix}a_1\\a_2\\a_3\\1\end{pmatrix}$ and $B = \begin{pmatrix}b_1\\b_2\\b_3\\1\end{pmatrix}$, then

$$L = \begin{bmatrix}0 & a_1b_2 - b_1a_2 & a_1b_3 - b_1a_3 & a_1 - b_1\\a_2b_1 - b_2a_1 & 0 & a_2b_3 - b_2a_3 & a_2 - b_2\\a_3b_1 - b_3a_1 & a_3b_2 - b_3a_2 & 0 & a_3 - b_3\\b_1 - a_1 & b_2 - a_2 & b_3 - a_3 & 0\end{bmatrix}$$

L is of the form $\begin{bmatrix}0 & z_1 & z_2 & z_3\\-z_1 & 0 & z_4 & z_5\\-z_2 & -z_4 & 0 & z_6\\-z_3 & -z_5 & -z_6 & 0\end{bmatrix}$. It has only 6 independent elements. L has only 4 degrees of freedom and a rank of only 2. DoF being 4 follows from the fact that only ratios matter and the property $\boxed{\det(L) = 0}$. Note that L and $L^*$ are related because P and Q are basis vectors of 2D null space of L.

One adv. of Plucker: Eqn of plane passing thru line L and point X: $\pi = L^*X$