ECE 661
Purdue U.
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Lecture 4

Characterization of the Distortions Caused by Projective Imaging and The Principle of Point/Line Duality

Reference: "Multiview Geometry in Computer Vision" by Hartley and Zisserman

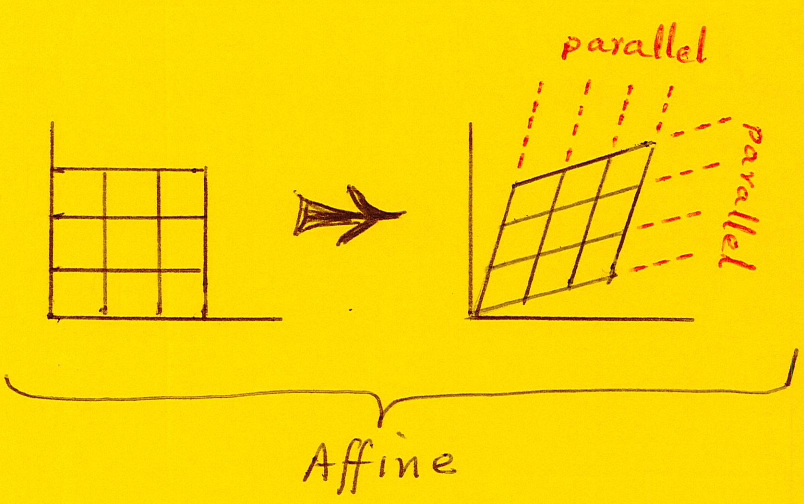
- When you use your camera to take a photo of a large planar scene — such as a building facade, or a wall decorated with picture hangings, or a great expanse of the flat outdoors — the sort of distortions you will see in your photo are the subject of this lecture. As to why a camera may produce images with these distortions will become clear later in this class.

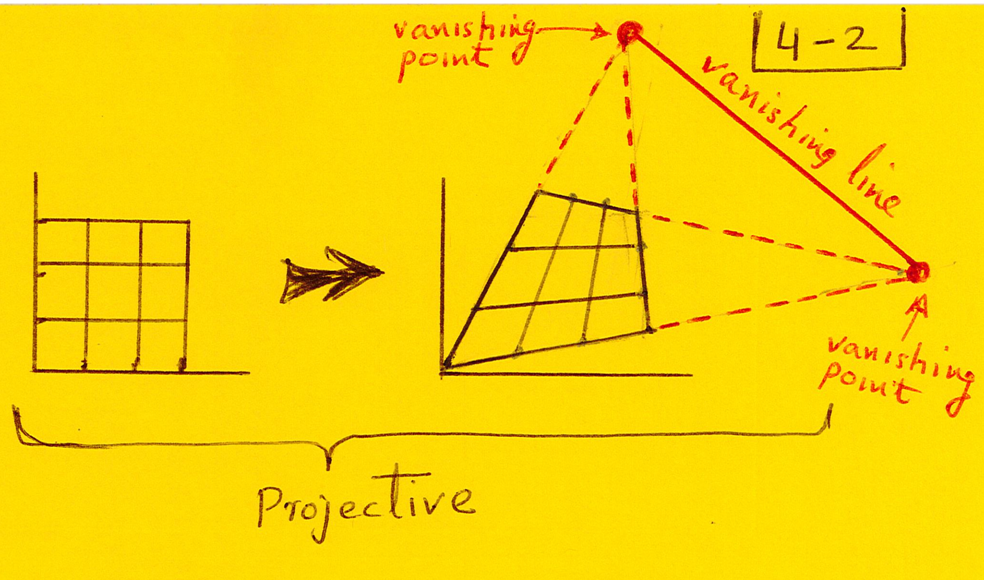
- However, before we get into the specifics of the distortions, note the following: A homography H is affine if and only if l_∞ is mapped to l_∞. Recall from Lecture 2 that all the points at ∞ form a single straight line given by $\boxed{l_\infty = \begin{pmatrix}0\\0\\1\end{pmatrix}}$.

- For proving the above assertion in the forward direction, we first recall from Lecture 3 that if points transform according to H, then lines transform according to H^{-T}. So let $\boxed{H = \begin{bmatrix} A & \vec{t} \\ \vec{0}^T & 1 \end{bmatrix}}$ be a general affine transform (note that A is non-singular). We can easily verify that $\boxed{H^{-1} = \begin{bmatrix} A^{-1} & -A^{-1}\vec{t} \\ \vec{0}^T & 1 \end{bmatrix}}$ Therefore, $\boxed{H^{-T} = \begin{bmatrix} A^{-T} & \vec{0} \\ (-A^{-1}\vec{t})^T & 1 \end{bmatrix}}$. Obviously, it is always true that $\boxed{H^{-T}\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}}$.

- For proving the converse of the underlined assertion in the second bullet, assume that a general projective transform $\boxed{H = \begin{bmatrix} A & \vec{t} \\ \vec{v}^T & v \end{bmatrix}}$ maps l_∞ to l_∞. Now we must show that $\vec{v} = \begin{pmatrix}0\\0\end{pmatrix}$ and $v \neq 0$. In a proof by contradiction, assume $\vec{v} = \begin{pmatrix}v_1\\v_2\end{pmatrix}$ with one or both of v_1 and v_2 as non-zero. Since this H maps l_∞ to l_∞, it must take a point at ∞ to a point at ∞. Now consider the point at ∞ along the x-axis: $\begin{pmatrix}1\\0\\0\end{pmatrix}$. This point will be mapped to $\boxed{\begin{bmatrix} A & \vec{t} \\ v_1\ v_2 & v \end{bmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}a_{11}\\a_{21}\\v_1\end{pmatrix}}$ which is NOT a point at infinity. Thus we have a contradiction, implying that v_1 must be zero. Reasoning similarly about the image of the point at infinity along the physical y-axis, this is the point represented by $\begin{pmatrix}0\\1\\0\end{pmatrix}$, we can establish that v_2 must also be zero. With $v_1 = v_2 = 0$, we can<u>not</u> allow v to become zero since otherwise H will become singular (which is not allowed for a homography).

- Whereas every affine transform maps l_∞ to l_∞, a general projective transform maps l_∞ to a physical line that we call the <u>vanishing line</u>. This is illustrated in the figure at the top of the next page.

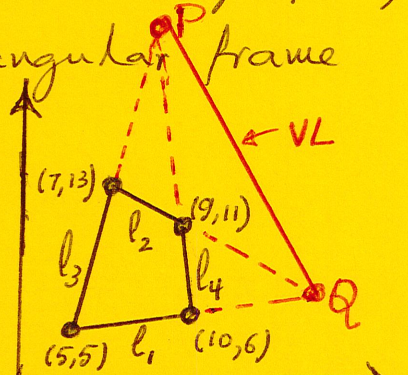




- The distortion in an image that results in the formation of one or more vanishing points and vanishing lines in the plane of the image is specifically projective, meaning that it is over and above the distortion introduced by affine part of the overall transformation.
- If we apply a homography to an image that sends the vanishing line(s) back to l_∞, the remaining distortion in the image will be purely affine.
- What's interesting is that the homography for removing the projective part of the distortion can be estimated from the parameters of the vanishing line. That is, if the vanishing line is $\boxed{l = \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix}}$, the homography that takes the vanishing line back to l_∞ is given by $\boxed{H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_1 & l_2 & l_3 \end{bmatrix}}$. To prove this, recall that when points are transformed by H, lines are transformed by H^{-T}. So we write first the matrix for $\boxed{H^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -l_1/l_3 & -l_2/l_3 & 1/l_3 \end{bmatrix}}$ implying $\boxed{H^{-T} = \begin{bmatrix} 1 & 0 & -l_1/l_3 \\ 0 & 1 & -l_2/l_3 \\ 0 & 0 & 1/l_3 \end{bmatrix}}$

The last step of the proof consists of showing that $\boxed{H^{-T} \cdot \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = l_\infty}$

Exercise: In a photograph of a wall on which pictures are mounted very high on the wall and off to the right of the photographer, the pixel coordinates of the corners of a rectangular frame are at (5,5), (10,6), (7,13), and (9,11). The photo obviously suffers from projective distortion. What 3X3 homography will get rid of this distortion?



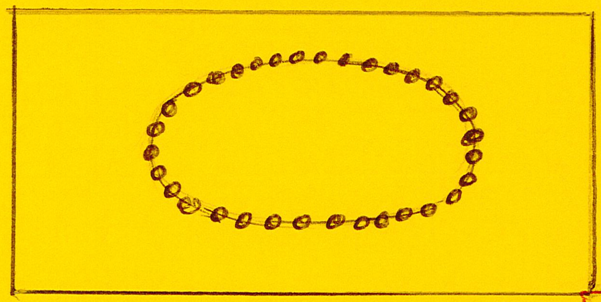
Solution: $\boxed{l_1 = \begin{pmatrix} 5 \\ 5 \\ 1 \end{pmatrix} X \begin{pmatrix} 10 \\ 6 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \\ -20 \end{pmatrix}}$ $\boxed{l_2 = \begin{pmatrix} 7 \\ 13 \\ 1 \end{pmatrix} X \begin{pmatrix} 9 \\ 11 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ 38 \end{pmatrix}}$ $\boxed{l_3 = \begin{pmatrix} 5 \\ 5 \\ 1 \end{pmatrix} X \begin{pmatrix} 7 \\ 13 \\ 1 \end{pmatrix} = \begin{pmatrix} -8 \\ 2 \\ 30 \end{pmatrix}}$ $\boxed{l_4 = \begin{pmatrix} 10 \\ 6 \\ 1 \end{pmatrix} X \begin{pmatrix} 9 \\ 11 \\ 1 \end{pmatrix} = \begin{pmatrix} -5 \\ -1 \\ 56 \end{pmatrix}}$

$\boxed{P = l_3 X l_4 = \begin{pmatrix} 142 \\ 298 \\ 18 \end{pmatrix}}$ $\boxed{Q = l_1 X l_2 = \begin{pmatrix} 110 \\ -2 \\ -6 \end{pmatrix}}$ Vanishing Line $= P X Q = \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix}$ $H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_1 & l_2 & l_3 \end{bmatrix}$

- So we have at least one approach for getting rid of distortion that is specifically projective. After an image is rectified with respect to this distortion, the image will still contain affine distortion — the primary manifestation of which is unequal scaling along two orthogonal directions in the image. To see how one might remove that kind of distortion, we need additional theoretical tools — tools that require you to understand the principle of duality with respect to points and lines in projective geometry

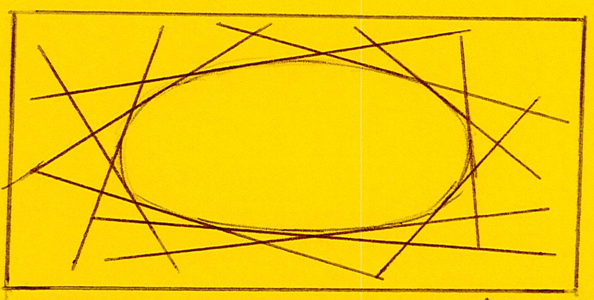
The Principle of Point/Line Duality in Homogeneous Coordinate Formulation of Projective Geometry

- Two of the key results we have derived so far — that the line joining two points x_1 and x_2 is given by $l = x_1 \times x_2$, and that the point of intersection of two lines l_1 and l_2 is $x = l_1 \times l_2$ — should be your first clue as to what we mean by the point/line duality. We can think of the second relation as a dual to the first.
- We can take any relation in the homogeneous coordinate formulation of projective geometry and obtain its dual by switching points and lines. The dual relation thus obtained is geometrically valid — provided you take care in the interpretation of the relation.
- Another reason for our interest in point/line duality is that a geometrical form in a 2D plane can be defined either as a set of points or through a set of lines. For example, a conic can be defined as a set of points, as shown at left below, or through a set of tangent lines, as shown at right:



conic: C

A conic defined with points: $\boxed{x^T C x = 0}$



dual conic: C^*

A conic defined with lines: $\boxed{l^T C^* l = 0}$

- The description of the conic on the right is a dual of its description at left. But what is C^* in the line-based definition of the conic?
- We can derive the dual conic C^* from the point conic C as follows: The tangent line to conic C at its perimeter point x is given by $l = Cx$ as shown in Lecture 2. Therefore, $x = C^{-1} l$, implying $x^T = l^T C^{-T} = l^T C^{-1}$ since C is symmetric. Substituting these forms for x and x^T in the conic definition $x^T C x = 0$, we get $\boxed{l^T C^{-1} l = 0}$. This implies $\boxed{C^* = C^{-1}}$
- Our next goal is to introduce a very special dual conic — this one will be dual to the degenerate conic that was introduced on page 2-4 of Lecture 2. We will call this new dual conic as the Dual Degenerat Conic and denote it C^*_∞. However, before defining C^*_∞, we must introduce the notion of Circular Points.

Circular Points

- These are going to seem like the oddest things you'd have seen ever. But, as you'll see later, they will serve a useful purpose.
- The circular points are the points of intersection of $\boxed{\text{any}}$ circle in a plane with l_∞, the line at infinity. Since there can be at most two intersection points between a line and a circle, we have exactly <u>two</u> circular points.
- To obtain the circular points, we start with the general equation for a conic in homogeneous coordinates: $\boxed{ax_1^2 + bx_1x_2 + cx_2^2 + dx_1x_3 + ex_2x_3 + fx_3^2 = 0}$. For a circle, $b=0$ and $a=c$. So we are left with $\boxed{x_1^2 + x_2^2 + dx_1x_3 + ex_2x_3 + fx_3^2 = 0}$ where we set $a=1$. Now, the line at infinity is the set of ideal points, with each such point being expressable as $\begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}$. Therefore, at the intersection of l_∞ with any circle it must be the case that $x_3 = 0$. The equation of a circle for the solution for the circular points therefore reduces to $\boxed{x_1^2 + x_2^2 = 0}$, implying that $\boxed{x_2^2 = -x_1^2}$ at the intersection points.
- The two solutions to the constraint $x_2^2 = -x_1^2$ at the Circular Points are both imaginary — as you'd expect. Keeping in mind that only ratios matter, we arbitrarily set $x_1 = 1$. With that, x_2 is either $+i$ or $-i$. Using $\vec{I}$ and $\vec{J}$ to denote the two circular points, we can write

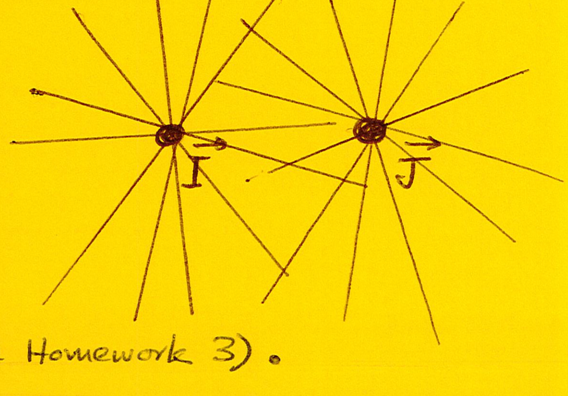
$$\vec{I} = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \qquad \vec{J} = \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}$$

The Dual Degenerate Conic C_∞^*

- As introduced on page 2-4 of Lecture 2, a point degenerate conic is defined by two straight lines l and m: $\boxed{C = lm^T + ml^T}$. To construct its dual, we must replace each line by a point. You get the dual degenerate conic if you choose the circular points $\vec{I}$ and $\vec{J}$ for this purpose. Denoting the resulting structure by C_∞^*, with the superscript telling us that we are dealing with a dual of a point conic and the subscript telling us that the anchor points $\vec{I}$ and $\vec{J}$ are at infinity, we get

$$\boxed{C_\infty^* = \vec{I}\vec{J}^T + \vec{J}\vec{I}^T}$$

- Since the dual of $\boxed{x^T C x = 0}$ is $\boxed{l^T C^* l = 0}$, the conic defined by C_∞^* consists of all lines l that satisfy $\boxed{l^T C_\infty^* l = 0}$. As it turns out, these are all the lines that pass through either the point $\vec{I}$ or the point $\vec{J}$, as shown here



- For a matrix representation of C_∞^*:

$$C_\infty^* = \vec{I}\vec{J}^T + \vec{J}\vec{I}^T = \boxed{\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}(1\ -i\ 0) + \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}(1\ i\ 0)}$$

$$= \begin{bmatrix} 1 & -i & 0 \\ i & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & i & 0 \\ -i & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- C_∞^* is invariant to a Similarity Homography.
- You can estimate the image of C_∞^* under a projective homography (see Homework 3).