ECE 661
Purdue Univ
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Lecture 20

Camera Imaging of Various Geometrical Forms (including the Absolute Conic)

Reference: "Multiple View Geometry in Computer Vision" by Hartley and Zisserman

- In this lecture, we are interested in the images formed by a camera for various geometrical forms in the physical 3D space. We are specifically interested in the imaging of planes, lines, conics, and quadrics. We will also be interested in how a camera images the Absolute Conic because of the important role it plays in modern camera calibration algorithms.

How a Plane in 3D is Imaged by a Camera

- This section proves that the relationship between a planar scene and its camera image is always a homography — that is, a linear relationship when we use homogeneous 3-vectors for representing the coordinates — regardless of the pose of the camera with respect to the scene.

- We assume that the planar scene is in the Z=0 plane of the world coordinate frame, and that the camera is in an arbitrary pose with respect to this plane. Writing the 3x4 camera projection matrix P as $P=[\vec{p}_1\ \vec{p}_2\ \vec{p}_3\ \vec{p}_4]$, we have for the pixel coordinates

$$\underbrace{\vec{x}}_{\text{pixel}} = [\vec{p}_1\ \vec{p}_2\ \vec{p}_3\ \vec{p}_4]\underbrace{\vec{X}}_{\text{world point}} = [\vec{p}_1\ \vec{p}_2\ \vec{p}_3\ \vec{p}_4]\begin{pmatrix}x\\y\\0\\w\end{pmatrix} = [\vec{p}_1\ \vec{p}_2\ \vec{p}_4]\begin{pmatrix}x\\y\\w\end{pmatrix} = \underbrace{H}_{3\times3\text{ homography}}\vec{x}_w$$

where $\vec{x}_w$ is the homogeneous 3-vector representation of a world point in the Z=0 plane of the world frame. If we prefer to write P as $P=K[R|\vec{t}]$ (see page 18-3 of Lecture 18), it follows that the 3x3 homography H shown above would become $\boxed{H=K[\vec{r}_1\ \vec{r}_2\ \vec{t}]}$, where $\vec{r}_1$ and $\vec{r}_2$ are the first two columns of the 3x3 rotation matrix R, and $\vec{t}$ the translation vector from the world frame origin to the camera frame origin.

How a Line in 3D is Imaged by a Camera

- As you'd expect, it is trivial to show that the camera image of a 3D line is a line. Following Lecture 6, let's represent a 3D line as a vector span of two world points $\vec{A}$ and $\vec{B}$. Any world point on this line can be expressed as $\vec{X}=\lambda_1\vec{A}+\lambda_2\vec{B} \equiv \vec{A}+\lambda\vec{B}$ for arbitrary values of the coefficients involved. The image of such a world point is given by $\vec{x}(\lambda) = P\vec{A}+\lambda P\vec{B} = \vec{a}+\lambda\vec{b}$ where $\vec{a}$ and $\vec{b}$ are the coordinates of the pixels for the images of $\vec{A}$ and $\vec{B}$. Obviously, all the points $\vec{x}(\lambda)$ in the image form a straight line

- While we are on the subject of lines, let's talk about backprojecting image lines into the 3D space of the world coordinate frame. As you would expect, a line in the camera image backprojects into a plane in the world frame. If a 3-vector $\vec{l}$ is the homogeneous representation of a line in a camera image, the homogeneous representation of the world plane that $\vec{l}$ backprojects to is given by $P^T\vec{l}$. PROOF: The set of pixels $\vec{x}$ on $\vec{l}$ must obey $\vec{x}^T\vec{l} = 0$. Now let $\vec{x}$ be the image of some world point $\vec{X}$. We have $\vec{x} = P\vec{X}$. Substituting $\vec{x} = P\vec{X}$ in $\vec{x}^T\vec{l} = 0$, we get $\vec{X}^T P^T \vec{l} = 0$, which (from Lecture 6) implies that $\vec{X}$ is on a world plane $\vec{\pi}$ whose homogeneous representation is the 4-vector given by $\vec{\pi} = P^T \cdot \vec{l}$

Backprojecting Conics

- A reader might ask: Shouldn't we first talk about the camera imaging of conics in world-3D before addressing the problem of backprojecting conics? The reason we gloss over the problem of how a camera images a conic in world-3D is that there is nothing special about it. A conic in the world coordinate frame (the conic must reside in a plane in the world frame) is imaged by a camera using the same homography that you saw on page 20-1 when we talked about how a camera images a plane.

- So let's talk about backprojecting conics from the image plane into the world coordinate frame. Consider a conic whose homogeneous representation is a 3x3 matrix C.

- As you can imagine, the conic C will backproject into a cone-like object in the world frame. The apex of this cone will be at the camera center and its cross-sections in any plane parallel to the image plane a scaled version of C. Such a cone in the world frame is a degenerate quadric whose homogeneous representation is given by the 4x4 matrix $\boxed{Q_{co} = P^T C P}$. PROOF: All pixels $\vec{x}$ on the conic C obey $\vec{x}^T C \vec{x} = \vec{0}$. Now let $\vec{X}$ be a world point whose image is at the pixel $\vec{x}$. Obviously, $\vec{x} = P\vec{X}$. Substituting this in $\vec{x}^T C \vec{x} = \vec{0}$, we get $\boxed{\vec{X}^T P^T C P \vec{X} = \vec{0}}$, which completes the proof since all points $\vec{X}$ on a quadric Q must obey $\vec{X}^T Q \vec{X} = \vec{0}$.

- Note that $Q_{co} = P^T C P$ is NOT of full rank because P is only of rank 3. It is trivial to show that the null vector of Q_{co} is the same as that of P. You can show that by multiplying on the right both sides of $Q_{co} = P^T C P$ by the null vector of P. It is because Q_{co} is of reduced rank that it has a cone-like shape in 3D.

How a Quadric is Imaged by a Camera

- The camera image of a point quadric Q — the image is just a silhoutte of the quadric — is a point conic C. The Q and C are related by $\boxed{C^* = P Q^* P^T}$ where Q^* is the dual of the point quadric Q and C^* the dual of the point conic C. PROOF: The lines $\vec{l}$ that are tangent to the point conic C must obey $\vec{l}^T C^* \vec{l} = 0$. Additionally, as you saw in the first bullet on the previous page, each such image line backprojects into the plane $\vec{\pi} = P^T \vec{l}$ in the world frame. From the geometry of image formation, we know that such planes must be tangential to the point quadric Q. That is, we must have $\vec{\pi}^T Q^* \vec{\pi} = 0$. Substituting in this the relationship $\vec{\pi} = P^T \vec{l}$, we get $\boxed{\vec{l}^T P Q^* P^T \vec{l} = 0}$, which implies $C^* = PQ^*P^T$.

How Does a Camera Image The Absolute Conic

- In Lecture 7, we defined the Absolute Conic Ω_∞ as the intersection of $\vec{\pi}_\infty$ and any arbitrary sphere in world 3D. The points on $\vec{\pi}_\infty$ are all ideal and are defined by homogeneous vectors $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ 0 \end{pmatrix}$. On the plane $\vec{\pi}_\infty$, Ω_∞ is defined by $(x_1\ x_2\ x_3)\, I \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$ where I is the 3×3 identity matrix.

- To find the camera image of Ω_∞, let's first focus on the camera image of a single point $\vec{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ 0 \end{pmatrix}$ on the plane $\vec{\pi}_\infty$. The pixel coordinates of the image point for $\vec{X}$ are:

$$\text{pixel} \rightarrow \vec{x} = P\vec{X} = KR\left[I_{3\times3} \middle| -\tilde{C}\right]\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ 0 \end{pmatrix} = KR\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = KR\,\vec{x}_d = H\vec{x}_d$$

where $\vec{x}_d = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ means a "direction vector" to the point $\vec{X} = \begin{pmatrix} \vec{x}_d \\ 0 \end{pmatrix}$ on the plane $\vec{\pi}_\infty$. To see that we can associate the meaning of direction with the first three elements of $(x_1\ x_2\ x_3\ x_4)^T$, note that as we go from the point $\vec{X} = (x_1, x_2, x_3, x_4)$ to the point $(kx_1, kx_2, kx_3, x_4)^T$, we will continue to travel along the ray that joins $\vec{X}$ with the camera center.
Therefore, what remains invariant as we travel along the same ray all the way to $\vec{\pi}_\infty$ are the ratios of the three elements x_1, x_2, and x_3.

- Therefore, if the 3-vector $\vec{x}_d$ in the derivation shown above is interpreted as a direction vector, it is homogeneous in the same sense as the pixel coordinate vector $\vec{x}$ on the left. The derivation shown above says that the direction vectors to points on $\vec{\pi}_\infty$ are related to their corresponding pixels by the homography $H = KR$.

- The Absolute Conic Ω_∞ is defined by the direction vectors $\vec{x}_d$ that obey $\vec{x}_d^T I_{3\times3} \vec{x}_d = 0$. You know from Lecture 3 that under a homography H, a conic C transforms as $C' = H^{-T} C H^{-1}$. Since the image formation from the direction vectors $\vec{x}_d$ to the pixels $\vec{x}$ is the homography $H = KR$, the image of the Absolute Conic is given by

$$\text{Image of the Absolute Conic} \rightarrow \omega = H^{-T} I_{3\times3} H^{-1} = (KR)^{-T}(KR)^{-1} = ((KR)^T)^{-1}(KR)^{-1}$$
$$= (R^T K^T)^{-1}(KR)^{-1} = K^{-T}R^{-T}R^{-1}K^{-1} = K^{-T}(RR^T)^{-1}K^{-1} = K^{-T}K^{-1}$$

The actual pixels on the image conic ω would be $\vec{x}^T \omega \vec{x} = 0$. As you would expect, all these pixels are imaginary because $K^{-T}K^{-1}$ is positive definite.

A conic C must NOT be pos-def since $\vec{x}^T C \vec{x} = 0$

Inverse of upper triangular matrix is always upper triangular
" " lower " " " lower "

A matrix A is pos-def if $\vec{x}^T A \vec{x} > 0$ for all $\vec{x}$

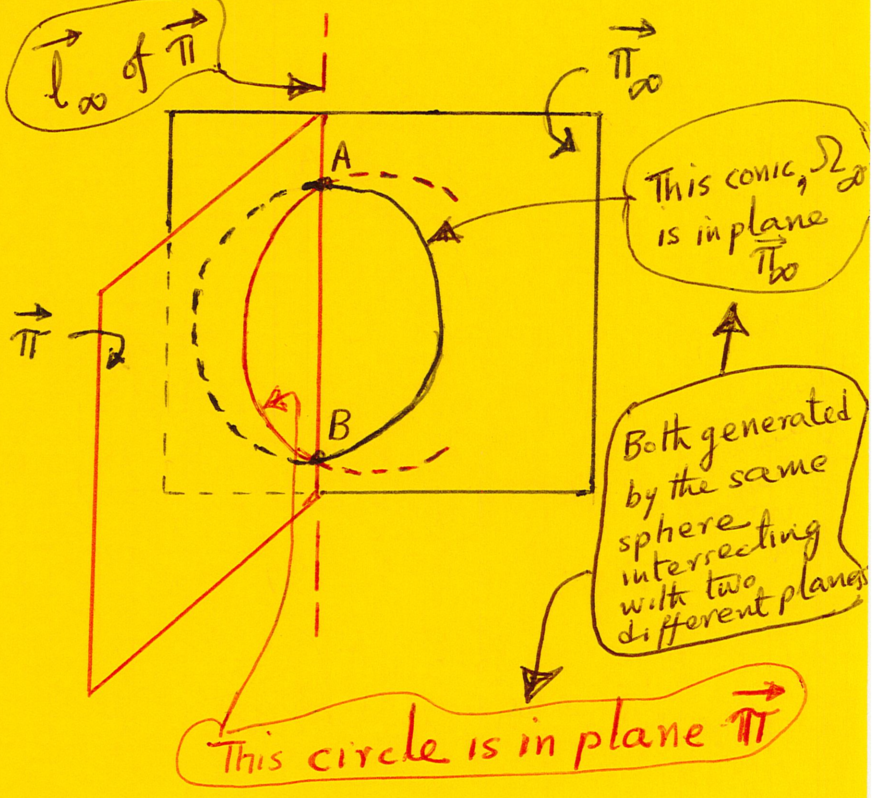
When you multiply a triangular by its transpose, you get a pos-def matrix

A Most Important Property of the Absolute Conic

- Any arbitrary plane $\vec{\pi}$ in world 3D samples the Absolute Conic at exactly two points — the two circular points $\vec{I}$ and $\vec{J}$ of $\vec{\pi}$. (See Lecture 4 for the definition of the Circular Points.)

- In order to establish this property, the first thing to note is that the Absolute Conic Ω_∞ resides in the plane $\vec{\pi}_\infty$ and any given plane $\vec{\pi}$ intersects the plane $\vec{\pi}_\infty$ in the former's line $\vec{l}_\infty$. Next, we need to show that $\vec{l}_\infty$ intersects the conic Ω_∞ at exactly two points — the circular points of $\vec{\pi}$. This is best done with the help of the figure shown below:

- The two points A and B shown are on the $\vec{l}_\infty$ line of plane $\vec{\pi}$ because that's where the plane $\vec{\pi}$ meets the plane $\vec{\pi}_\infty$. The blue conic Ω_∞ is formed by the intersection of an arbitrary sphere with $\vec{\pi}_\infty$. And the red circle is formed by the intersection of the same sphere with the plane $\vec{\pi}$. The two points A and B where the red circle on plane $\vec{\pi}$ meets the plane $\vec{\pi}_\infty$ must be on the line $\vec{l}_\infty$ of plane $\vec{\pi}$. That implies that A and B are the two Circular Points of the plane $\vec{\pi}$.



Calibrating a Camera's Intrinsic Parameters by Waving a 2D Pattern in Front of It

- The property of the Absolute Conic described leads to a novel algorithm for calculating the intrinsic parameters (as represented by the elements of the 3x3 matrix K) of a camera. Estimating K is a very important part part of camera calibration.

- The idea is to let the camera record at least three different images of a 2D pattern as you "wave" it in front of the camera. The orientation of the pattern must be different for each image that is recorded. The translation does not matter.

- The plane corresponding to each pose of the pattern will sample Ω_∞ at the two Circular Points, $\vec{I} = \begin{pmatrix}1\\i\\0\end{pmatrix}$ and $\vec{J} = \begin{pmatrix}1\\-i\\0\end{pmatrix}$, in that plane.

- Let's assume the pattern is visually rich enough to allow us to compute the homography $H = [\vec{h}_1\ \vec{h}_2\ \vec{h}_3]$ from the plane of the pattern to the camera image plane. Applying H to $\vec{I}$ and $\vec{J}$, we get $H\cdot\vec{I}$ and $H\cdot\vec{J}$ as two points on the image conic ω defined at the bottom of the previous page. Since $H\cdot\vec{I} = \vec{h}_1 + i\vec{h}_2$ and $H\cdot\vec{J} = \vec{h}_1 - i\vec{h}_2$, both these points on ω must satisfy the $\vec{x}^T\omega\vec{x} = 0$ condition. So we get the two equations:

$$\left.\begin{aligned}(\vec{h}_1 + i\vec{h}_2)^T\omega(\vec{h}_1 + i\vec{h}_2) &= 0\\ (\vec{h}_1 - i\vec{h}_2)^T\omega(\vec{h}_1 - i\vec{h}_2) &= 0\end{aligned}\right\} \Rightarrow \left.\begin{aligned}\vec{h}_1^T\omega\vec{h}_1 - \vec{h}_2^T\omega\vec{h}_2 + i2\vec{h}_1^T\omega\vec{h}_2 &= 0\\ \vec{h}_1^T\omega\vec{h}_1 - \vec{h}_2^T\omega\vec{h}_2 - i2\vec{h}_1^T\omega\vec{h}_2 &= 0\end{aligned}\right\} \Rightarrow \boxed{\begin{aligned}\vec{h}_1^T\omega\vec{h}_1 - \vec{h}_2^T\omega\vec{h}_2 &= 0\\ \vec{h}_1^T\omega\vec{h}_2 &= 0\end{aligned}}$$

where we get the middle pair of equations by ω being symmetric, and the final pair by setting to zero separately the real and the imaginary parts.

- Each image gives us 2 equations for the 5 unknowns of ω. We apply Cholesky decomposition (only for pos-def) to this ω to recover K^{-1}. We invert that to get K. Cholesky breaks ω into LL^T where L is lower triangular