ECE 661
Purdue U.
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Lecture 2

World 2D : Using Homogeneous Coordinates to Represent and Manipulate Points, Lines and Conics

Reference: "Multiple View Geometry in Computer Vision" by Hartley and Zisserman

- Homogeneous coordinates based representations for geometrical entities lead to compact (and easy to program) formulas for many useful operations on the entities.
- Let $\mathbb{R}^2$ be a 2D space of reals.
- A line in $\mathbb{R}^2$ is a set of points $(x,y) \in \mathbb{R}^2$ that obey $ax+by+c=0$ for some values for the parameters a, b, and c.
- Let's call $l = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ as our 3D parameter vector.
- Let's represent the 2D physical point (x,y) by a 3D vector $X = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$.
- Obviously, the algebraic equation of a line, $ax+by+c=0$, can be expressed as: $l^T X = 0$ or $X^T l = 0$.
- The 3D vector $X = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \in \mathbb{R}^3$ is the homogeneous coordinate representation of a physical 2D point $(x,y) \in \mathbb{R}^2$.
- Exploring the 3D representational space $\mathbb{R}^3$ we note that for any multiple $k \in \mathbb{R}$, $k \neq 0$, $k \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$ is the same physical point in $\mathbb{R}^2$ as that corresponding to $\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$. [That is certainly the case with regard to the use of points in the algebraic form $ax+by+c=0$.]
- So in the $\mathbb{R}^3$ representational space, all the points $k \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$, with $k \neq 0$, form an <u>equivalence class</u>, in the sense that they represent the same physical point $(x,y) \in \mathbb{R}^2$. That is, each radial line passing through the origin, but not including the origin, in $\mathbb{R}^3$ represents a unique physical point in $\mathbb{R}^2$. [Q: How is the origin of the physical $\mathbb{R}^2$ space represented in the $\mathbb{R}^3$ HC space?]

HC: From now on, let's use HC as an abbreviation for "homogeneous coordinates"

- Let's now talk about HC representation for lines in physical $\mathbb{R}^2$.
- Referring again to the algebraic form $ax+by+c=0$, it turns out that the $\mathbb{R}^3$ space in which the parameter vectors $l = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ are situated can serve as the HC representational space for the physical lines in $\mathbb{R}^2$.
- Paralleling what we saw for the HC representation of physical points in $\mathbb{R}^2$, all vectors $k \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, $k \neq 0$, in $\mathbb{R}^3$ represent the <u>same</u> physical line in $\mathbb{R}^2$.
- Again paralleling the HC representation of physical points, every radial line in $\mathbb{R}^3$ (that is used for the parameter vectors $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$), but without the point at its origin, represents a unique line in physical $\mathbb{R}^2$.

- Here are a couple of immediate payoffs of our HC representation for points and lines in physical $\mathbb{R}^2$:

  (1) Given any two lines $l_1 = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}$ and $l_2 = \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}$, the point of intersection of the two lines is given by $x = l_1 \times l_2$ where '$\times$' is the vector cross product.

  What's l for x-axis? $l = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$
  origin; $(1,0)$ pt.

  (2) Given any two points $x_1 = \begin{pmatrix} x_1 \\ y_1 \\ 1 \end{pmatrix}$ and $x_2 = \begin{pmatrix} x_2 \\ y_2 \\ 1 \end{pmatrix}$, ~~the~~ the line that passes through the two points is given by $l = x_1 \times x_2$

Both of the above assertions follow immediately from the scalar triple product identity $\boxed{\vec{A} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{A} \times \vec{B}) = 0}$. The first assertion follows from the fact that the intersection point X must be on both l_1 and l_2, implying that $l_1^T X = 0$ and $l_2^T X = 0$. Substituting for $X = l_1 \times l_2$, gives us the first assertion. The second assertion follows from ~~the~~ the fact that both X_1 and X_2 must be on l, implying $X_1^T l = 0$ and $X_2^T l = 0$.

- Let's now talk about parallel lines: Since the slope of a physical line in $\mathbb{R}^2$ is given by the ratio a/b (or, more precisely, by the pair (a,b)), the two distinct lines $l_1 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $l_2 = \begin{pmatrix} a \\ b \\ c' \end{pmatrix}$ must be parallel. You can easily show that the point of intersection of these two lines is $X = \begin{pmatrix} b \\ -a \\ 0 \end{pmatrix}$. But what does it mean when the third coordinate in the homogeneous representation of a point is zero?

- To answer the question in red above, let's consider an arbitrary point $\begin{pmatrix} u \\ v \\ w \end{pmatrix}$ in the $\mathbb{R}^3$ representational space for the homogeneous coordinates of the $\mathbb{R}^2$ physical points. Assuming $w \neq 0$, the coordinates of the physical point are given by $(u/w, v/w)$. It is obvious that as $w \to 0$, the x and y coordinates of the physical point move farther and farther away from the origin in $\mathbb{R}^2$ and approach infinity. The important thing to note here is that this approach to infinity is along a specific direction in $\mathbb{R}^2$, the direction being controlled by the values of u and v. When, say, $u = 1$ and $v = 0$, the approach to infinity will be along the x-axis in $\mathbb{R}^2$.

- Therefore, the intersection of the two parallel lines $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\begin{pmatrix} a \\ b \\ c' \end{pmatrix}$ is at $\begin{pmatrix} b \\ -a \\ 0 \end{pmatrix}$, which is a point at infinity in $\mathbb{R}^2$, along a specific direction controlled by the pair (a, b).

- The physical points in $\mathbb{R}^2$ whose HC representations are of the form $\begin{pmatrix} u \\ v \\ 0 \end{pmatrix}$ are known as $\boxed{\text{Ideal Points}}$

- Here is something that could keep you awake at night: All ideal points form a straight line in $\mathbb{R}^2$.

- To prove the above, consider any two ~~point~~ ideal points in $\mathbb{R}^2$: $\begin{pmatrix} u_1 \\ v_1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} u_2 \\ v_2 \\ 0 \end{pmatrix}$. The line that passes through these two points is $l = \begin{pmatrix} u_1 \\ v_1 \\ 0 \end{pmatrix} \times \begin{pmatrix} u_2 \\ v_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ which we get by keeping in mind the notion of equivalence classes in the representation space $\mathbb{R}^3$. Being independent of (u_1, v_1) and (u_2, v_2) parameters, this line remains the same for all pairs of ideal points.

- Why might the fact that all Ideal Points in a plane form a unique straight line keep you awake at night? As you try to go to sleep tonight, imagine yourself standing in the middle of an infinitely large flat plane as you are gazing at the horizon. (Note that what you experience visually will not be the same as what you see when you gaze at the horizon from a beach. There the experience of the horizon is affected strongly by the curvature of the earth.) Now imagine yourself turning as you continue to gaze at the horizon in different directions. Where is that unique straight line that all the far-away points are supposed to fall on? [Yes, you do have to keep in mind the limitations on the range of the human visual system.]
- The unique line $\begin{pmatrix}0\\0\\1\end{pmatrix}$ is referred to as the Line at Infinity and denoted l_∞.
- The l_∞ line will play an important role in our camera models. The camera image of the l_∞ line is called the **Vanishing Line**.

- There is one more 2D geometric entity that will prove important to us in camera modeling and camera calibration work: Conics.
- A conic is a curve that is described by <u>a second-degree</u> implicit algebraic form in a plane.
- Euclidean geometry talks about the following three main types of conics: ellipse, hyperbola, and parabola (circle is a special case of ellipse)

> All of these conics are obtained when you slice a double cone with a plane. A double cone consists of two cones that are vertically aligned and that make a one-point contact at their vertices. When the slicing plane cuts through only one of the cones, you get either a circle, or an ellipse, or a parabola. When the plane slices through both cones, you get a hyperbola.

- In the physical plane $\mathbb{R}^2$, the implicit form for a conic is

$$\boxed{ax^2 + bxy + cy^2 + dx + ey + f = 0}$$

- To express this implicit form in HC, we will first express the point coordinates (x, y) as a 3-vector $\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$ with $x = x_1/x_3$ and $y = x_2/x_3$. Substituting these expressions for x and y in the implicit form, we get

$$\boxed{ax_1^2 + bx_1x_2 + cx_2^2 + dx_1x_3 + ex_2x_3 + fx_3^2 = 0}$$

Rewriting this as a vector-matrix product, we get

$$\boxed{(x_1 \quad x_2 \quad x_3)\begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = 0}$$

or, as, $\quad x^T C x = 0 \quad$ where $x = \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$ and $\boxed{C = \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix}}$

- C is the HC representation of a conic.

- Note that, as with the HC representations for points and lines, only the ratios of the matrix elements are important in the definition of C. In other words, for all $k \neq 0$, $k \cdot C$ represents the same conic as C.

- The HC representation of a conic gives us compact formulas for the tangent lines to a conic.
- The line l that is the tangent to a conic C at point x (all entities expressed in HC) is given by $\boxed{l = Cx}$
- To prove the above formula, we first note that if x is on C, then x is also on l. That is because $\boxed{l^T x = (Cx)^T x = x^T C x = 0}$ (C is symmetric)
We must next show that l makes only a single point contact with C. To prove that by contradiction, let there exit another point y for which $y^T C y = 0$ and $l^T y = 0$. These two equations can be combined with $x^T C x = 0$ and $l^T x = 0$ to write $\boxed{(x+\alpha y)^T C (x + \alpha y) = 0 \text{ for all } \alpha > 0}$, implying that every point on the line joining $\vec{x}$ and y must be on C. That is obviously impossible. Note: $\underbrace{x^T C x}_{=0} + \underbrace{\alpha^2 y^T C y}_{=0} + \underbrace{\alpha y^T C x}_{y^T l = 0} + \underbrace{\alpha x^T C y}_{x^T l = 0}$

- We will now consider Degenerate Conics. When you ~~slide~~ slice the double cones with a plane that passes through the axis of the double cones, instead of seeing a hyperbola, you will now see two intersecting straight lines. We say that these two lines constitute a degenerate conic. Let l and m be the HC representations of the two lines. That is, every point on ~~the~~ the degenerate conic C is either on ~~l~~ line l or on line m. We can show that such a C is given by

$$\boxed{C = l m^T + m l^T}$$

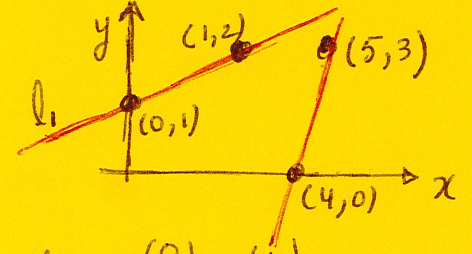
degenerate conic

It is trivial to show that such a C satisfies the conic equation $x^T C x = 0$. for all x that are on l, that is for which $l^T \vec{x} = 0$, and for all x that are on m, that is for which ~~l~~ $m^T x = 0$. Note: Each of the two terms in the above addition is a vector outer product. Also note that since each outer product has a rank of only 1, the rank of C is 2.

because every column is constant times every other column

REVIEW:

① Derive in just 3 steps the intersection of the two lines shown below:



$l_1 = \begin{pmatrix}0\\1\\1\end{pmatrix} \times \begin{pmatrix}1\\2\\1\end{pmatrix} =$

$l_2 = (\ ) \times (\ ) =$

intersection $x = l_1 \times l_2 =$

② Why is the rank of an outer product matrix always equal to 1?

$l = \begin{pmatrix}l_1\\l_2\\l_3\end{pmatrix} \quad m = \begin{pmatrix}m_1\\m_2\\m_3\end{pmatrix}$

$$l m^T = \begin{pmatrix}l_1\\l_2\\l_3\end{pmatrix}(m_1\ m_2\ m_3) = \begin{bmatrix} l_1 m_1 & l_1 m_2 & l_1 m_3 \\ l_2 m_1 & l_2 m_2 & l_2 m_3 \\ l_3 m_1 & l_3 m_2 & l_3 m_3 \end{bmatrix}$$

Every column is a constant times the first column. E.g., the second col. is m_2/m_1 times the first col. So all cols. are linearly dependent, implying a rank of 1.

③ Did you notice that HC not only give us compact and easy to program formulas, but also that we can represent entities at infinity precisely, that is, without approximation and without error?

④ Given a conic $x^2 + y^2 - 25 = 0$, where does the tangent to this conic at the perimeter point (3,4) intersect with x-axis?

$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -25 \end{bmatrix}$ and x-axis $= \begin{pmatrix}0\\0\\1\end{pmatrix} \times \begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix}$