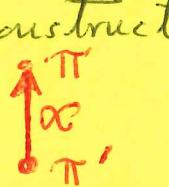


The Six Basic NP-Complete Problems

- Reviewing the previous lecture, we have SATISFIABILITY (SAT) as our first NP-Complete problem.
- It also follows from the previous lecture that if we want to prove that a decision problem Π is NP-Complete, we need to invoke the following two steps:
 - First show that $\Pi \in NP$
 - Next, select a known NP-Complete problem Π' and construct a polynomial transformation from Π' to Π .



- We will now present six problems that are commonly used as the "known NP-Complete problem" in proofs of NP-Completeness.

1 3-SATISFIABILITY (3SAT)

INSTANCE : Given a collection $C = \{c_1, c_2, \dots, c_m\}$ of clauses on a finite set U of Boolean variables such that $|c_i| = 3$ for all i .
(See the previous lecture for the definition of a clause.)

QUESTION : Is there a truth assignment for U that satisfies all clauses in C ?

- 3SAT is obviously a special case of SAT. But, as it turns out, 3SAT is at least as difficult as SAT.
- Recall that a clause is a disjunction of literals
- A literal is a Boolean variable or its negation

2 3-DIMENSIONAL MATCHING (3DM)

INSTANCE : A set $M \subseteq W \times X \times Y$ where W, X , and Y are disjoint sets having the same number of elements. That is, $|W| = |X| = |Y| = q$, for some $q \in \mathbb{Z}^+$.

QUESTION : Does M contain a matching, that is, a subset $M' \subseteq M$ such that ~~$M' \neq \emptyset$~~ $|M'| = q$, and no two elements of M' agree in any of the three coordinates?

Example : Consider a planet with a notion of marriage that is different from ours. Instead of a man and a woman pairing up, on this other planet a marriage consists of a man, a woman, and a family pet (typically a dog) teaming up as triples. Let W be the set of single men, X the set of single women, and Y the set of single dogs. Let's further assume that all the singles (meaning men, women, and dogs) have registered themselves with eHarmony.com website of the planet. By analyzing the personalities of all the singles involved, the eHarmony.com folks have put out a list M of triples. Obviously, $M \subseteq W \times X \times Y$. Now the question that 3DM tries to answer is whether all of the single men, the single women, and the single dogs can form blissful families. Consider $W = \{\text{Joe, Adam}\}$, $X = \{\text{Eve, Amber}\}$, and $Y = \{\text{Goldie, Huskie}\}$ and $M = \{(\text{Joe, Eve, Goldie}), (\text{Joe, Eve, Huskie}), (\text{Adam, Eve, Goldie}), (\text{Adam, Eve, Huskie})\}$. This M contains at least one matching $M' = \{(\text{Joe, Eve, Goldie}), (\text{Adam, Eve, Huskie})\}$.

3 VERTEX COVER (VC)

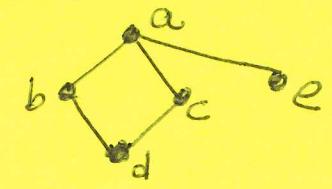
INSTANCE : A graph $G = (V, E)$ and a positive integer $K \leq |V|$

QUESTION : Is there a vertex cover of size K or less for G ?

That is, does there exist a subset $V' \subseteq V$ such that $|V'| \leq K$ and for each $\{u, v\} \in E$ at least one of u and v belongs to V' ?

Example : Consider the graph

If $K = 2$, the answer to the question is 'yes' since $V' = \{a, d\}$ is a vertex cover for this graph.



4 CLIQUE

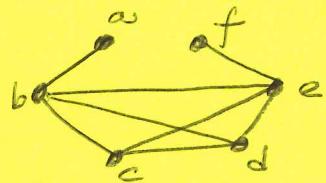
INSTANCE :

A graph $G = (V, E)$ and a positive integer $J \leq |V|$. Does G contain a clique of size J or more? That is, does there exist a subset $V' \subseteq V$ such that $|V'| \geq J$ and such that every two vertices in V' are joined by an edge in E ?

Example : The largest clique in



is, for example, $\{a, b\}$. It is certainly not $\{a, b, c\}$. On the other hand, the largest clique in



is $\{b, c, d, e\}$.

5 HAMILTONIAN CIRCUIT (HC)

INSTANCE : A graph $G = (V, E)$

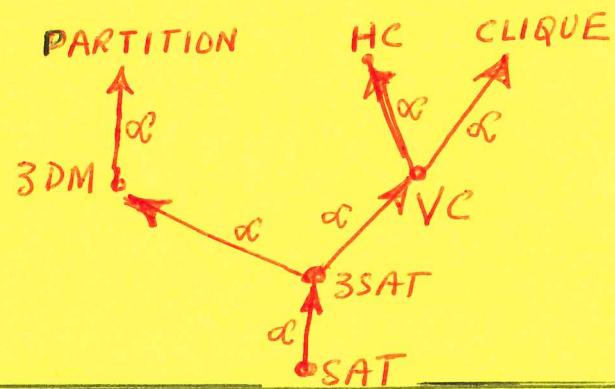
QUESTION : Does G contain a Hamiltonian circuit? That is, does there exist an ordering (v_1, v_2, \dots, v_n) of all the vertices in G such that $\{v_i, v_{i+1}\} \in E$ for all $i = 1, 2, \dots, n-1$ and also $\{v_n, v_1\} \in E$?

6 PARTITION

INSTANCE : A finite set A and a "size" function $s(a) \in \mathbb{Z}^+$ for each $a \in A$.

QUESTION : Does there exist a subset $A' \subseteq A$ such that $\sum_{a \in A'} s(a) = \sum_{a \in A - A'} s(a)$

- Given the recipe at the beginning of this lecture for how to prove that a problem is NP-Complete, we will start with the satisfiability (SAT) problem and work our way through the polynomial transformations indicated by the tree on the right to establish the NP-Completeness of the six basic problems listed above.



- THEOREM : 3SAT is NP-Complete.

PROOF : We want to show that every instance of SAT can be transformed (in poly time) into an instance of 3SAT in such a way that the SAT instance is satisfiable iff the instance of 3SAT (as yielded by the transformation) is satisfiable. (If you think of 3SAT as a baby version of SAT, there is a certain delicious irony in transforming SAT to 3SAT.) Recall that a SAT clause can have an arbitrary number of literals. On the other hand, a 3SAT clause has exactly 3 literals. By injecting additional Boolean variables we can show that any SAT clause can be expressed as a conjunction of 3-literal clauses in such a way that the truth values for the injected variables do NOT matter. Consider for example the case when a SAT clause has only one literal $\{z\}$. Injecting two additional variables y^1 and y^2 , we can express $\{z\}$ as $\{z, y^1, y^2\} \wedge \{\bar{z}, y^1, \bar{y}^2\} \wedge \{\bar{z}, \bar{y}^1, y^2\} \wedge \{\bar{z}, \bar{y}^1, \bar{y}^2\}$. If $\{z\}$ is true, the conjunction of the four 3-literal clauses is also true regardless of the truth values of y^1 and y^2 .