

Lecture Outline for Relations

- Binary Relations
- Different kinds of relations (one-to-one, etc.)
- Properties of relations (transitive, etc.)
- Closures of relations
- Partial Orderings
- Equivalence Relations
- Section 4.1 of text

1

Binary Relation

- Cartesian product of a set S with itself, $S \times S$
- $S = \{1, 2, 3\}$
- $S \times S = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$
- A (binary) relation is a subset of $S \times S$
- Example: Equality relation
- Example: Less than relation

2

Definition of Binary Relation

- Binary Relation (ρ) on a set S is a subset of $S \times S$
- $x\rho y \leftrightarrow (x,y) \in \rho$
- A binary relation is often defined by giving a binary predicate that is satisfied by certain ordered pairs $\in S \times S$
- **Relations on multiple sets:** A binary relation from set S to T is a subset of $S \times T$

3

Example of Binary Relation

- For each of the following binary relations ρ on \mathbb{N} , decide which of the ordered pairs belong to ρ
1. $x\rho y \leftrightarrow x=y+1$; (2,2), (2,3), (3,3), (3,2)
 2. $x\rho y \leftrightarrow y$ is divisible by x ; (2,4), (2,5), (2,6)
 3. $x\rho y \leftrightarrow x$ is odd; (2,3), (3,4), (4,5), (5,6)
 4. $x\rho y \leftrightarrow x+y < 10$; (1,3), (2,3), (3,4), (4,5)
 5. $x\rho y \leftrightarrow x > y^2$; (1,2), (2,1), (5,2), (6,4), (4,3)

4

Types of Relations

A binary relation ρ consists of a set of ordered pairs (s_1, s_2)

One-to-one relation: Each s_1 is paired with only one s_2 and vice-versa

One-to-many relation: At least one s_1 is paired with more than one s_2

Many-to-one relation: At least one s_2 is paired with more than one s_1

Many-to-many relation: At least one s_1 is paired with more than one s_2 and at least one s_2 is paired with more than one s_1

Note that not all elements in the set need to appear in ρ

5

Examples of Relation Types

- $S=\{2,5,7,9\}$. Identify the type of each of the following relations
 1. $\{(5,2), (7,5), (9,2)\}$
 2. $\{(2,5), (5,7), (7,2)\}$
 3. $\{(7,9), (2,5), (9,9), (2,7)\}$

6

Operations on Relations

- Let there be two (binary) relations ρ and σ on a set S . Each is a subset of $S \times S$ and we can perform set operations on them to create new subsets (new binary relations).
- These are $\rho \cup \sigma$, $\rho \cap \sigma$, ρ'
- **Example:** Two relations ρ and σ on N such that $x\rho y \leftrightarrow x=y$ and $x\sigma y \leftrightarrow x < y$. Define the following:
 1. $\rho \cup \sigma$
 2. ρ'
 3. σ'
 4. $\rho \cap \sigma$

7

Properties of Relations

- Example relation $x\rho y \leftrightarrow x=y$ has the following properties
 1. For any $x \in S$, $(x,x) \in \rho$
 2. For any $x, y \in S$, $(x,y) \in \rho \rightarrow (y,x) \in \rho$
 3. For any $x, y, z \in S$, $(x,y) \in \rho$ and $(y,z) \in \rho \rightarrow (x,z) \in \rho$
- **Reflexive:** $(\forall x)(x \in S \rightarrow (x,x) \in \rho)$
- **Symmetric:** $(\forall x)(\forall y)(x \in S \wedge y \in S \wedge (x,y) \in \rho \rightarrow (y,x) \in \rho)$
- **Transitive:** $(\forall x)(\forall y)(\forall z)(x \in S \wedge y \in S \wedge z \in S \wedge (x,y) \in \rho \wedge (y,z) \in \rho \rightarrow (x,z) \in \rho)$

8

Example of Relation Property

- Relation \leq on \mathbb{N} :
 - Reflexive?
 - Symmetric?
 - Transitive?

9

Anti-symmetric Relation

- If a binary relation ρ on S is anti-symmetric, then

$$(\forall x)(\forall y)(x \in S \wedge y \in S \wedge (x,y) \in \rho \wedge (y,x) \in \rho \rightarrow x=y)$$

- Relation \leq on \mathbb{N} :
 - Anti-symmetric?

10

Example on Relation Property

- $S = \{1,2,3\}$
 1. If ρ is reflexive, what elements should it have?
 2. If ρ is symmetric, what elements should it have?
 3. If ρ is symmetric and $(a,b) \in \rho$, what elements should it have?
 4. If ρ is anti-symmetric and $(a,b) \in \rho$ and $(b,a) \in \rho$, what property must hold?
 5. Is the relation $\rho = \{(1,2)\}$ on S transitive?

11

Symmetry and anti-symmetry

- Give an example of a relation which is symmetric as well as anti-symmetric.
- Relations can be
 - symmetric and *not* anti-symmetric
 - anti-symmetric and not symmetric
 - both symmetric and anti-symmetric
 - neither symmetric nor anti-symmetric
- Example:
 - $\rho = \{(1,2), (2,1), (3,2), (2,3)\}$
 - $\rho = \{(1,2), (2,1), (3,2)\}$

12

Closures of Relations

- **Closure:** If a relation ρ on S fails to have a certain property (symmetry, etc.), what elements should be added to ρ to satisfy the property
- Closure is the original set **plus** the additional elements
- Formally, a binary relation ρ^* is the **closure of a relation** ρ with respect to property P if
 - 1) ρ^* has property P
 - 2) $\rho \subseteq \rho^*$
 - 3) ρ^* is the smallest such relation

13

Computing Transitive Closure

- Write the relation ρ as a matrix \mathbf{A} . If relation is on set S with n elements, then \mathbf{A} has dimension $n \times n$.
- $\mathbf{A}[i, j] = 1$ if $(i, j) \in \rho$
- Compute $\mathbf{A}^{(2)}$ where $\mathbf{A}^{(2)}[i, j] = \bigvee_{k=1}^n (a_{ik} \wedge a_{kj})$
- Now compute $\mathbf{A}^{(3)}, \dots, \mathbf{A}^{(n)}$ where
$$\mathbf{A}^{(p+1)}[i, j] = \bigvee_{k=1}^n (\mathbf{A}^{(p)}[i, k] \wedge a_{kj})$$
- Compute $\mathbf{R} = \mathbf{A} \vee \mathbf{A}^{(2)} \vee \mathbf{A}^{(3)} \vee \dots \vee \mathbf{A}^{(n)}$
- Convert \mathbf{R} back into a relationship ρ' . That is the transitive closure of ρ .

14

Example of Closure

- $S = \{1, 2, 3\}$
 - $\rho = \{(1, 1), (1, 2), (1, 3), (3, 1), (2, 3)\}$
1. What is the reflexive closure of ρ ?
 2. What is the symmetric closure of ρ ?
 3. What is the transitive closure of ρ ?
 4. What is the anti-symmetric closure of ρ ?

15

Partial Ordering on S

- A binary relation on S which is
 - Reflexive
 - Anti-symmetric
 - Transitive
- is called a **partial ordering** on S
- Examples
 - Set of all natural numbers, $x \leq y$
 - Set of all positive integers, x divides y

16

Partially ordered set

- (S, ρ) is called the **partially ordered set**, or **poset**
 - S is the set on which the relation is defined
 - ρ is the relation, such as “less than or equals”, “is a subset of”, etc.
- **Predecessor of y** : If $x\rho y$ and $x \neq y$, x is predecessor of y (denoted, $x < y$)
- **Immediate predecessor of y** : If $x < y$, and no z exists such that $x < z < y$, then x is immediate predecessor of y
- Example:
 - $S = \{1, 2, 3, 6, 12, 18\}$
 - $x\rho y \leftrightarrow x$ divides y

17

Total Ordering

- A partial ordering in which every element is related to every other element
- **Example**: The relation \leq on the set of natural numbers

18

Special Elements in a Poset

- **Least element:** If there is a $y \in S$ and $y < x$ for every other element $x \in S$, then y is the least element
 - A least element may not exist
 - A least element if it exists is unique
- **Minimal element:** An element $y \in S$ is minimal if there does not exist any element $x \in S$, such that $x < y$
 - There may be multiple minimal elements
 - A minimal element will always exist

19

Special Elements in a Poset

- **Greatest element:** If there is a $y \in S$ and $x < y$ for every other element $x \in S$, then y is the greatest element
 - A greatest element may not exist
 - A greatest element if it exists is unique
- **Maximal element:** An element $y \in S$ is maximal if there does not exist any element $x \in S$, such that $y < x$
 - There may be multiple maximal elements
 - A maximal element will always exist
- **Example:** From the relation “ x divides y ” from before

20

Equivalence Relation on S

- A binary relation on S which is
 - Reflexive
 - Symmetric
 - Transitive

is called an **equivalence relation** on S

- Examples
 - Set of all natural numbers, $x + y$ is even
 - Set of all students in this class, $x \rho y \leftrightarrow x$ sits in the same row as y

21

Partition of a Set

- **Partition of a set S** is the collection of subsets of S such that
 - The subsets are disjoint
 - No subset is empty
 - The union of all the subsets is the set S
- Example: Relation “sits in the same row”
- **Equivalence class of x**: For an equivalence relation, the set of all elements of S to which x is related (denoted $[x]$)

22

Theorem on Equivalence Relations and Partitions

- An **equivalence relation** ρ on a set S determines a **partition** on S
- A **partition** on S determines an **equivalence relation** on S
- Proof: Omitted
- **Example:**
 1. On set of natural numbers, $x\rho y \leftrightarrow x=y$
 2. On $\{1,2,3\}$,
 $\rho=\{(1,1),(2,2),(3,3),(1,2),(2,1)\}$

23

Congruence modulo n

- Congruence modulo 3, say
- Why is it an equivalence relation?
- What are the equivalence classes?
(Use all integers, positive and negative)

24

Examples

25

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 - $\rho = \{(1,2), (2,1), (3,2)\}$

26

Examples

- Identify the 4 kinds of elements

1. $S = \{a, b, c\}$

$$\rho = \{(a,a), (b,b), (c,c), (a,b), (b,c), (a,c)\}$$

2. $S = \{a, b, c, d\}$

$$\rho = \{(a,a), (b,b), (c,c), (d,d), (a,b), (a,c)\}$$