Lecture Outline for Recurrences

- Already Covered: Recursive definition of sequences
- Recursive definition of sets
- Recursive definition of operations
- Recursive definition of algorithms
- Now: Solving recurrences
  - Expand, guess, verify
  - Solution formula
- Section 2.4 of text

Solving recurrences

- To solve the following recurrence relation
  1. S(1) = 2
  2. S(n) = 2 S(n-1), $n \geq 2$

  we developed an iterative and a recursive solution

- But, a one-line solution is possible!
- You tell me what

Solving recurrences

- Objective: Solve S(n) without needing to compute for lower values.
- Method 1: Expand – Guess – Verify
- Example:
  - T(1) = 1
  - T(n) = T(n-1) + 3, n≥2

Solving recurrences – EGV method

- Expand
  - T(1) = 1
  - T(2) = 1+3 = 4
  - T(3) = 4+3 = 1+2*3 = 7
  - T(4) = 7+3 = 1+3*3 = 10
- Guess
  - T(n) = 3n-2
- Verify
  - Use principle of induction

Solving recurrences – EGV method

$F(1) = 2$

$F(n) = 2F(n-1) + 2^n, n \geq 2$

Solving recurrences - formula

- Applicable to

  $S(n) = c\, S(n-1) + g(n)$

  Base case is S(1) which is known

- This is a *first order*, *linear* equation with *constant coefficients*

- If g(n)=0, then equation is *homogeneous*

Solving recurrences - formula

$S(n) = c\,S(n-1) + g(n)$

$= c[c\,S(n-2) + g(n-1)] + g(n)$

$= c^2\,S(n-2) + c\,g(n-1) + g(n)$

$= \ldots$

$= c^{n-1}\,S(1) + c^{n-2}\,g(2) + \ldots + c\,g(n-1) + g(n)$

[See proof details in book page 133-134]

$= c^{n-1}\,S(1) + c^{n-2}\,g(2) + \ldots + c^1\,g(n-1) + c^0\,g(n)$

$= c^{n-1}\,S(1) + \sum_{i=2}^{n} c^{n-i}\,g(i)$

Solving recurrences: formula

- Example:
  1. $T(1) = 1$
  2. $T(n) = T(n-1) + 3,\ n \geq 2$

  [We already solved it using the EGV method.]

Solving recurrences: formula

- Example:
  1. $A(1) = 1$
  2. $A(n) = A(n-1) + n^2, n \geq 2$

Solving recurrences

1. $F(1) = 1$
2. $F(n) = n\,F(n-1), n \geq 2$

Solving recurrences - formula

- Applicable to

  $S(n) = c\,S(n/2) + g(n)$

  Base case is $S(1)$ which is known

- See proof in text pages 162-163
- $S(n) = c^{\log n} S(1) + \sum_{i=1}^{\log n} c^{(\log n)-i} g(2^i)$

Homogeneous Recurrence Relations: First Order

$S(n) = c\,S(n-1)$

- Homogeneous first-order recurrence relation
- Base case: $S(1)$
- Solution: $S(n) = c^{n-1} S(1)$

Homogeneous Recurrence Relations: Second order

$S(n) = c_1 S(n-1) + c_2 S(n-2)$

- Homogeneous second-order recurrence relation
- Base cases: $S(1)$ and $S(2)$
- To find solution, create the characteristic equation: $t^2 - c_1 t - c_2 = 0$
- Case 1: There are distinct roots to the characteristic equation, say r_1 and r_2
- Then $S(n) = pr_1^{n-1} + qr_2^{n-1}$

Homogeneous Recurrence Relations: Second order

- Case 1: Two distinct roots to the characteristic equation, say r_1 and r_2 (Cont'd)
- How to solve for p and q?
- Use base cases S(1) and S(2) whose values are given to you
- $S(1) = p + q$ -------- Eqn (1)
- $S(2) = pr_1 + qr_2$ ----- Eqn (2)
- Solve Eqns (1) and (2) for p and q

Homogeneous Recurrence Relations: Second order

- Case 2: There is a single repeated root for the characteristic equation, say r
- Then solution is $S(n) = pr^{n-1} + q(n-1)r^{n-1}$
- How to solve for p and q?
- Use base cases S(1) and S(2) whose values are given to you
- $S(1) = p$ ---------- Eqn (1)
- $S(2) = pr + qr$ ----- Eqn (2)
- Solve Eqns (1) and (2) for p and q