Lecture Outline for Proof Techniques

- Exhaustive Proof
- Refuting by counter-example
- Direct proof
- Indirect proof by contraposition
- Proof by contradiction
- Fallacies in proofs
- Rules of thumb
- Section 2.1 of text

Proofs — key concepts

Formal vs. informal proofs.

Inductive vs. Deductive reasoning.

- Claim: $n^2 - n + 41$ is prime

n	$n^2 - n + 41$
1	41
2	43
3	47
4	53
5	61

Inductive reasoning might conclude claim is true

But: is there a counter-example?

An Example Proof Technique

Exhaustive proof

- If an integer between 1 and 20 is divisible by 6, it is also divisible by 3.

Proof Techniques – key concepts

Choosing to prove or to refute.

- Counterexamples refute a claim
- e.g. Prove or refute
    - that every odd integer is prime.

More techniques

Direct proof using the deduction method

- To prove $P \to Q$, start with the hypothesis P and prove conclusion Q.
- The product of two even integers is even

Indirect proof: proving the *contrapositive*

- The contrapositive of $P \to Q$ is $Q' \to P'$
- $P \to Q \equiv Q' \to P'$
- *If n^2 is odd, then n is odd*
- So, instead prove:
  - ???

Proof by contradiction

- Prove that the square root of 2 is irrational.
  - Any rational number can be represented as m/n, where m,n are integers with no common factor and $n \neq 0$.
  - To prove by contradiction assume $\sqrt{2}$ is rational = m/n
  - Then $2 = (m/n)^2$
  - So, $2n^2 = m^2$
    - *m^2 is divisible by 2.*
    - *Therefore, m is divisible by 2. (Fundamental theorem in arithmetic)*
    - *Therefore, m^2 is divisible by 4.*

Proof by contradiction

- Prove that the square root of 2 is irrational.
  - So, $2n^2 = m^2$
    - *m^2 is divisible by 4.*
  - $2n^2 = 4x$
  - $n^2 = 2x$
  - n^2 is divisible by 2.
    - Therefore, n is divisible by 2.
    - Thus, m and n are both divisible by 2.
  - Contradiction of the assumption that m and n cannot have a common factor.

Contradiction vs. Contrapositive

- Proof by contradiction is more powerful than proving the contrapositive.
  - In trying to prove $P \rightarrow Q$,
    - Proving the contrapositive:
      - Assume Q', try to prove P'
    - Proof by contradiction:
      - Assume Q' *along with P*, try to prove a contradiction.
      - We have more premises (can only help)
      - The goal of proving a contradiction may seem harder than proving P', but it is not:
        - If we prove P', we'll have a contradiction…

Proof by Contradiction (Example)

- Prove that the product of two odd integers is an odd integer.

Proving if and only if statements

- To prove *P* iff *Q*, you must prove both directions separately.
  - Prove $P \to Q$ and $Q \to P$
  - Note: these are separate, independent statements that must be proven.
  - $Q \to P$ is called the *converse* of $P \to Q$
    - *e.g.* If it is raining out, the ground is wet.
      If the ground is wet, it is raining out.
- Proving a group of formulas equivalent:
  - Prove $P1 \to P2 \to P3 \to \ldots \to Pn \to P1$

Fallacies

- Affirming the conclusion
  - Q together with $P \to Q$ does <u>not</u> imply P !
  - e.g. If it is raining outside, the ground is wet.
    The ground is wet.
    Therefore, it is raining outside. (not true!)
- Denying the hypothesis
  - $\neg P$ together with $P \to Q$ does <u>not</u> imply $\neg Q$!
  - e.g. If it is raining outside, the ground is wet.
    It is not raining outside.
    Therefore, the ground is not wet. (not true!)

Another Fallacy

- Incorrect use of proof by contradiction
  - Assume $P \wedge Q'$ and without using Q' are able to prove Q.
  - Then we assert $Q \wedge Q'$ is a contradiction.
  - What we have effectively done is proven $P \to Q$.
  - Example:
    - "Prove by contradiction that if a number added to itself gives the number, then the number is 0."

Example

a) Draw conclusion(s) using rules of inference for the following statement.

   "If I play hockey, then I am sore the next day." "I use the whirlpool if I am sore." " I did not use the whirlpool."

b) Construct an argument using the rules of inference to show that the hypothesis "Randy works hard", "If Randy works hard, he is a dull boy", "If Randy is a dull boy, then he will not get the job" imply the conclusion "Randy will not get the job"

Examples (Cont)

Determine which of the following arguments are valid (give proper reasoning).

1. If n is a real number such that $n>1$,then $n^2 > 1$. Suppose that $n^2 > 1$ then $n > 1$.
2. The number $\log_2 3$ is irrational if it is not the ratio of two integers. Therefore, since $\log_2 3$ cannot be written in the form a/b where a and b are integers, it is irrational.
3. If n is a real number with $n>3$ then $n^2>9$. Suppose $n^2 \leq 9$ then $n \leq 3$.
4. If n is a real number with $n > 2$, then $n^2 > 4$. Suppose that $n \leq 2$. Then $n^2 \leq 4$.

Rules of Thumb

- When to use which proof technique

1. Exhaustive proof:
   - Can only be used for a finite number of cases.
   - Demonstrate P→Q for all cases of P
2. Direct proof:
   - The cleanest approach, use whenever possible
   - Assume P, prove Q

Rules of Thumb

- When to use which proof technique

3. Proof by contraposition:
   - If Q′ proves powerful to reason with.
   - Demonstrate P→Q by demonstrating Q′→P′
4. Proof by contradiction:
   - Use this when having P and Q′ in the premise set is helpful
   - Assume P and Q′ and show a contradiction