Lecture Outline for Functions

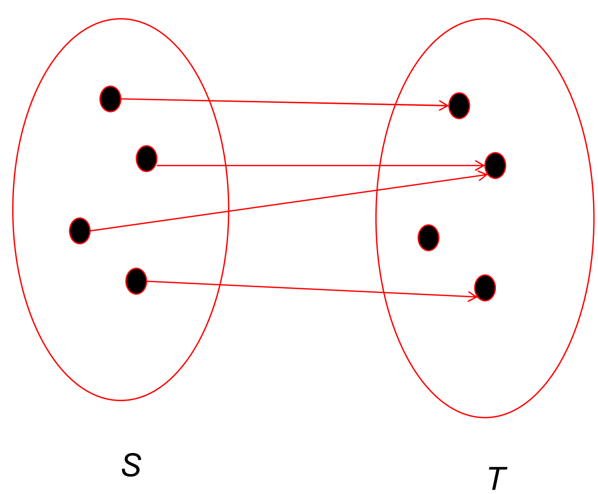
- Definition
- Properties of functions (onto, one-to-one, etc.)
- Composition of functions
- Inverse of functions
- Number of functions
- Magnitude of functions
- Section 4.4 of text

Definition of function

- f: S $\rightarrow$ T
- Every element of set S mapped to one and only one value in T
- Example:
  - $g(x) = x^3$, S: N, T: N
  - Described by $\{(x, g(x)) | g(x) = x^3\}$
- S is the domain, T is co-domain
- If (s, t) belongs to the function, t is the image of s, and s is the pre-image of t

Examples and non-examples of functions

1. f: $S \rightarrow T$ where S = T = {1, 2, 3}, f = {(1,1), (2,2), (3,3)}
2. f: $S \rightarrow T$ where S = T = {1, 2, 3, 4}, f = {(1,1), (2,2), (3,3), (4, 3)}
3. f: $S \rightarrow T$ where S = T = {1, 2, 3, 4}, f = {(1,1), (2,2), (3,3), (3, 4)}
4. f: $N \rightarrow N$ where
   - $f(x) = x, x \leq 3$
   - $f(x) = 2x, x \geq 3$



Images and Pre-images

- S: R, T: Z
- f(x) = $\lfloor x \rfloor$
- What is the image of 2.3, -3.4?
- What are the preimages of 2, -4?

[2, 3)
[-4, -3)

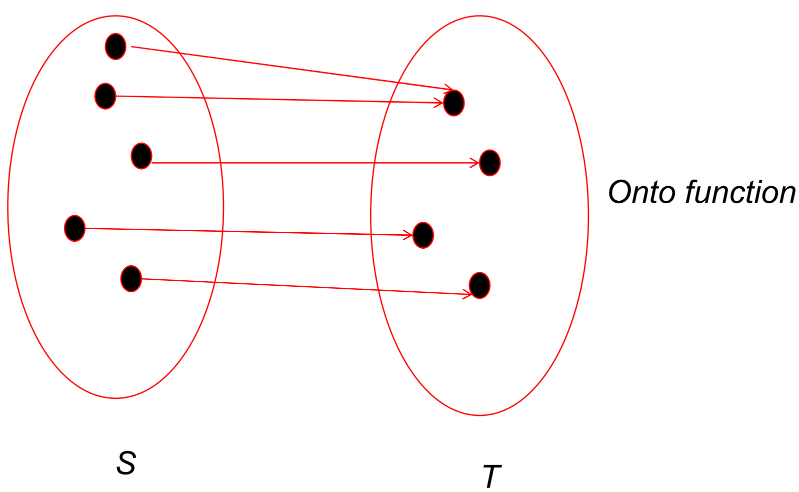
Functions on more than one variable

- f: $S_1 \times S_2 \times \ldots S_n \rightarrow T$
- Associates each ordered n-tuple of elements (s_1, s_2, …, s_n) with an element of T
- Example:
  - f: Z×N×{0,1} → Q where f(x,y,z) = x^{y+z}
  - f(2, -2, 0) = ?
  - f(2, 2, 0) = ? 4

x/y
7/2
8/2 = 4/1

Properties of functions: Onto

- Onto functions: Take function f: $S \rightarrow T$.
- The range of f is the set R of images of all members of S.
- If the range of f is identical to the co-domain, then it is an onto function.
- How to prove?
  - Always $R \subseteq T$.
  - If we can prove $T \subseteq R$, then we are done.
  - Take an arbitrary element in T and show it is the image of some member of S.



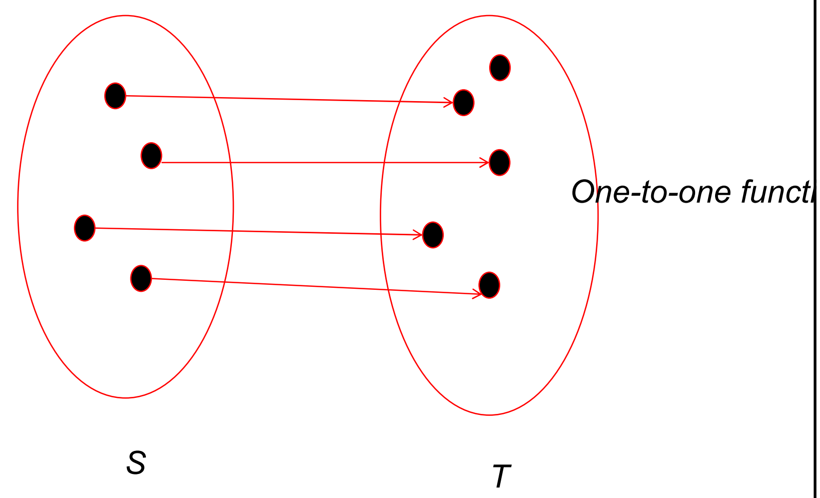
Properties of functions: onto

- $S = \{0, 2, 4, 6\}$
- $T = \{1, 3, 5, 7\}$
- Which of the following are onto functions?

1. {(0,2), (2,4), (4,6), (6,0)}
2. {(6,3), (2,1), (0,7), (4,5)}
3. {(6,1), (0,3), (4,1), (0,7), (2,5)}

Properties of functions: One-to-one

- A function $f: S \rightarrow T$ is one-to-one (or injective) if no member of T has more than one preimage in S
- How to prove?
  - Assume that $f(s_1) = f(s_2)$ and then show that $s_1=s_2$
- What is the difference from a one-to-one *relation* from S to T?



Examples of one-to-one functions

- S = {0, 2, 4, 6}
- T = {1, 3, 5, 7, 9}
- Which of the following are one-to-one functions?

1. {(0,1), (2,3), (4,5), (6,7)}
2. {(0,1), (2,1), (4,7), (6,9)}

Properties of functions: Bijective

- A function f: $S \rightarrow T$ which is both onto and one-to-one is called Bijective
- How to prove?
  - Step 1: Prove onto
  - Step 2: Prove one-to-one

Composition of functions

- f: $S \rightarrow T$ and g: $T \rightarrow U$.
- For any $s \in S$, $f(s) \in T$
- Therefore $g(f(s)) \in U$
- $g \circ f(x) \equiv g(f(x))$
- In other words, $g \circ f$ is a new function from S to U ($S \rightarrow U$)
- This is called the composition function.
- For composing, the domains and the ranges have to be compatible.

Composition of functions: Example

- f: R→R, g: R→R
- $f(x) = x^2$
- $g(x) = \lfloor x \rfloor$
- (g∘f) (2.3) = g(5.29)= 5
- (f∘g) (2.3) = f(2) = 4
- Order is important

Inverse Functions

- If f: S→T is bijective, we have another function g: T→S such that
  - an element t∈T maps to s∈S where f(s) = t
- g∘f maps each element of S to itself
  - Called Identity function on S (i_S)
- What is f∘g?
- Inverse function: f: S→T. There exists g:T→S, such that g∘f = i_S and f∘g = i_T, then g is the inverse function of f, denoted f^{-1}

Inverse Functions (Cont'd)

- Theorem:

1. If f is a bijection, then f^{-1} exists.
2. If f^{-1} exists, then f is a bijection.

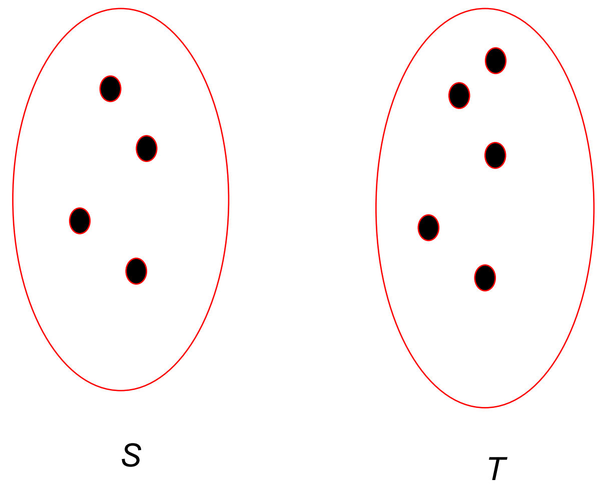
- Example: Each function is R→R. Find f^{-1}.

1. f(x) = 2x
2. f(x) = (x+4)/3

How Many Functions?

- We will use f: S→T
  - |S| = m, |T| = n
- How many such functions without any restriction? n^m
- How many such one-to-one functions?
  - $m \leq n$
  - # functions = n * (n-1) * (n-2) * … (n-m+1) = n!/(n-m)!

m = 4, n = 5   *No restriction: 5 * 5 * 5 * 5 = 5^4*



*One-to-one: 5 * 4 * 3 * 2*

How Many Functions? (Cont'd)

- We will use f: $S \rightarrow T$
  - |S| = m, |T| = n
- How many such onto functions?
  - $m \geq n$
  - Total number of functions – Total number of non-onto functions

  $= n^m - [C(n,1)(n-1)^m - C(n,2)(n-2)^m - \ldots - (-1)^{n-1}C(n,n-1)(1)^m]$

  $= n^m - C(n,1)(n-1)^m + C(n,2)(n-2)^m - \ldots + (-1)^{n-1}C(n,n-1)(1)^m$

Permutation of a Set

- For a given set A, consider bijective functions from A→A
- These are called permutations of A (Set of all permutations is S_A)
- Example:
  - A = {1, 2, 3, 4}
  - f = {(1,2), (2,3), (3,4), (4,1)}
  - f written in matrix form
  - Cyclical notation for a function
- How many such permutations are there?

More on Cycles

- Say, A = {1, 2, 3, 4, 5}
- f = (1, 4, 5)
- Write out f in matrix form.
- Example:
  - Composition of functions
  - Composition of cycles

More on Permutations

- Disjoint cycles: f and g are members of S_A and are disjoint cycles (i.e., have no element in common)
  - Then, $f \circ g = g \circ f$
- Identity permutation: Maps each element of A to itself
- Derangement: Permutation of A such that no element in A is mapped to itself

Number of Derangements

- $|A| = n$
- $A = \{a_1, a_2, \ldots, a_n\}$
- Remember in a derangement no element of A is mapped to itself
- # derangements = total number of permutation functions – total number of non-derangements

$= n! - n!/1! + n!/2! - n!/3! + \ldots + (-1)^n n!/n!$

$= n![1 - 1/1! + 1/2! - 1/3! + \ldots + (-1)^n 1/n!]$

Number of Derangements: Examples

- S = {a, b, c}
- Find the number of derangements
- Show the different derangements

Definition of Θ

- When do you say two functions are the same order of magnitude?
- Consider two functions f and g, whose domain and co-domain are non-negative real numbers.
- If you can find positive constants n_0, c_1, c_2 such that

$$c_1 g(x) \leq f(x) \leq c_2 g(x),\ \forall x \geq n_0$$

- Then $f=\Theta(g)$

Definition of O

- When do you say a function f is big-oh of another function g?
- Consider two functions f and g, whose domain and co-domain are non-negative real numbers.
- If you can find positive constants n_0, c such that

$$f(x) \leq c\, g(x),\ \forall x \geq n_0$$

- Then f=O(g)

Examples

- Example 1
  - $f(x) = 3x^3-7x$
  - $g(x) = x^3/2$
  - f(x) ? g(x)
- Example 2
  - $f(x) = 100x^2$
  - $g(x) = x^3$
  - f(x) ? g(x)

More Big-O Examples

a) $(n^2 + 8)(n + 1)$

b) $(n \log n + n^2)(n^3 + 2)$

c) $(n! + 2^n)(n^3 + \log(n^2 + 1))$

Give Big-O estimates for each of these functions.