Lecture Outline for Relations

- Binary Relations
- Different kinds of relations (one-to-one, etc.)
- Properties of relations (transitive, etc.)
- Closures of relations
- Partial Orderings
- Equivalence Relations
- Section 4.1 of text

Binary Relation

- Cartesian product of a set S with itself, $S \times S$
- $S=\{1,2,3\}$
- $S \times S = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$
- A (binary) relation is a subset of $S \times S$
- Example: Equality relation
- Example: Less than relation

Definition of Binary Relation

- Binary Relation (ρ) on a set S is a subset of $S \times S$
- $x \rho y \quad \leftrightarrow (x,y) \in \rho$
- A binary relation is often defined by giving a binary predicate that is satisfied by certain ordered pairs $\in S \times S$
- Relations on multiple sets: A binary relation from set S to T is a subset of $S \times T$

Example of Binary Relation

- For each of the following binary relations ρ on N, decide which of the ordered pairs belong to ρ

1. $x \rho y \leftrightarrow x=y+1$; (2,2), (2,3), (3,3), (3,2)
2. $x \rho y \leftrightarrow$ y is divisible by x; (2,4), (2,5), (2,6)
3. $x \rho y \leftrightarrow$ x is odd; (2,3), (3,4), (4,5), (5,6)
4. $x \rho y \leftrightarrow x+y<10$; (1,3), (2,3), (3,4), (4,5)
5. $x \rho y \leftrightarrow x>y^2$; (1,2), (2,1), (5,2), (6,4), (4,3)

Types of Relations

A binary relation ρ consists of a set of ordered pairs (s_1, s_2)

One-to-one relation: Each s_1 is paired with only one s_2 and vice-versa

One-to-many relation: At least one s_1 is paired with more than one s_2

Many-to-one relation: At least one s_2 is paired with more than one s_1

Many-to-many relation: At least one s_1 is paired with more than one s_2 and at least one s_2 is paired with more than one s_1

Note that not all elements in the set need to appear in ρ

Examples of Relation Types

- S={2,5,7,9}. Identify the type of each of the following relations
  1. {(5,2), (7,5), (9,2)}
  2. {(2,5), (5,7), (7,2)}
  3. {(7,9), (2,5), (9,9), (2,7)}

Operations on Relations

- Let there be two (binary) relations ρ and σ on a set S. Each is a subset of $S \times S$ and we can perform set operations on them to create new subsets (new binary relations).

- These are $\rho \cup \sigma$, $\rho \cap \sigma$, ρ'

- Example: Two relations ρ and σ on N such that $x\rho y \leftrightarrow x=y$ and $x\sigma y \leftrightarrow x<y$. Define the following:
  1. $\rho \cup \sigma$
  2. ρ'
  3. σ'
  4. $\rho \cap \sigma$

Properties of Relations

- Example relation $x \rho y \leftrightarrow x=y$ has the following properties
  1. For any $x \in S$, $(x,x) \in \rho$
  2. For any $x, y \in S$, $(x,y) \in \rho \rightarrow (y,x) \in \rho$
  3. For any $x, y, z \in S$, $(x,y) \in \rho$ and $(y,z) \in \rho \rightarrow (x,z) \in \rho$
- Reflexive: $(\forall x)(x \in S \rightarrow (x,x) \in \rho)$
- Symmetric: $(\forall x)(\forall y)(x \in S \wedge y \in S \wedge (x,y) \in \rho \rightarrow (y,x) \in \rho)$
- Transitive: $(\forall x)(\forall y)(\forall z)(x \in S \wedge y \in S \wedge z \in S \wedge (x,y) \in \rho \wedge (y,z) \in \rho \rightarrow (x,z) \in \rho)$

Example of Relation Property

- Relation $\leq$ on N:
  - Reflexive?
  - Symmetric?
  - Transitive?

Anti-symmetric Relation

- If a binary relation ρ on S is anti-symmetric, then

$(\forall x)(\forall y)(x \in S \wedge y \in S \wedge (x,y) \in \rho \wedge (y,x) \in \rho \rightarrow x = y)$

- Relation $\leq$ on N:
  - Anti-symmetric?

Example on Relation Property

- $S = \{1,2,3\}$

1. If ρ is reflexive, what elements should it have?
2. If ρ is symmetric, what elements should it have?
3. If ρ is symmetric and $(a,b)\in\rho$, what elements should it have?
4. If ρ is anti-symmetric and $(a,b)\in\rho$ and $(b,a)\in\rho$, what property must hold?
5. Is the relation $\rho = \{(1,2)\}$ on S transitive?

Symmetry and anti-symmetry

- Give an example of a relation which is symmetric as well as anti-symmetric.

- Relations can be
  - symmetric and *not* anti-symmetric
  - anti-symmetric and not symmetric
  - both symmetric and anti-symmetric
  - neither symmetric nor anti-symmetric

- Example:

  $\forall$ $\rho=\{(1,2), (2,1), (3,2), (2,3)\}$

  $\forall$ $\rho=\{(1,2), (2,1), (3,2)\}$

Closures of Relations

- Closure: If a relation ρ on S fails to have a certain property (symmetry, etc.), what elements should be added to ρ to satisfy the property

- Closure is the original set **plus** the additional elements

- Formally, a binary relation ρ* is the closure of a relation ρ with respect to property P if

1) ρ* has property P
2) $\rho \subseteq \rho^*$
3) ρ* is the smallest such relation

Computing Transitive Closure

- Write the relation ρ as a matrix **A**. If relation is on set *S* with *n* elements, then **A** has dimension $n \times n$.
- **A**[*i*, *j*] = 1 if (*i*, *j*) $\in \rho$
- Compute $\mathbf{A}^{(2)}$ where $\mathbf{A}^{(2)}[i, j] =$
- Now compute $\mathbf{A}^{(3)}, \ldots, \mathbf{A}^{(n)}$ where $\bigvee_{k=1}^{n} (a_{ik} \wedge a_{kj})$

$\mathbf{A}^{(p+1)}[i, j] =$ $\bigvee_{k=1}^{n} (\mathbf{A}^{(p)}[i,k] \wedge a_{kj})$

- Compute $\mathbf{R} = \mathbf{A} \vee \mathbf{A}^{(2)} \vee \mathbf{A}^{(3)} \vee \ldots \vee \mathbf{A}^{(n)}$
- Convert **R** back into a relationship ρ'. That is the transitive closure of ρ.

Example of Closure

- S={1,2,3}

∀ ρ = {(1,1), (1,2), (1,3), (3,1), (2,3)}

1. What is the reflexive closure of ρ?
2. What is the symmetric closure of ρ?
3. What is the transitive closure of ρ?
4. What is the anti-symmetric closure of ρ?

Partial Ordering on S

- A binary relation on S which is
  - Reflexive
  - Anti-symmetric
  - Transitive

is called a partial ordering on S

- Examples
  - Set of all natural numbers, $x \leq y$
  - Set of all positive integers, x divides y

Partially ordered set

- (S, ρ) is called the partially ordered set, or poset
  - S is the set on which the relation is defined
  - ρ is the relation, such as "less than or equals", "is a subset of", etc.
- Predecessor of *y*: If xρy and x≠y, x is predecessor of y (denoted, x<y)
- Immediate predecessor of *y*: If x < y, and no z exists such that x < z < y, then x is immediate predecessor of y
- Example:
  - S = {1, 2, 3, 6, 12, 18}
  - xρy ↔ x divides y

Total Ordering

- A partial ordering in which every element is related to every other element
- Example: The relation $\leq$ on the set of natural numbers

Special Elements in a Poset

- Least element: If there is a $y \in S$ and $y < x$ for every other element $x \in S$, then y is the least element
  - A least element may not exist
  - A least element if it exists is unique
- Minimal element: An element $y \in S$ is minimal if there does not exist any element $x \in S$, such that $x < y$
  - There may be multiple minimal elements
  - A minimal element will always exist

Special Elements in a Poset

- Greatest element: If there is a $y \in S$ and $x<y$ for every other element $x \in S$, then y is the greatest element
  - A greatest element may not exist
  - A greatest element if it exists is unique
- Maximal element: An element $y \in S$ is maximal if there does not exist any element $x \in S$, such that $y<x$
  - There may be multiple maximal elements
  - A maximal element will always exist
- Example: From the relation "x divides y" from before

Equivalence Relation on S

- A binary relation on S which is
  - Reflexive
  - Symmetric
  - Transitive

is called an equivalence relation on S

- Examples
  - Set of all natural numbers, x + y is even
  - Set of all students in this class, $x \rho y \leftrightarrow$ x sits in the same row as y

Partition of a Set

- Partition of a set S is the collection of subsets of S such that
  - The subsets are disjoint
  - No subset is empty
  - The union of all the subsets is the set S
- Example: Relation “sits in the same row”
- Equivalence class of x: For an equivalence relation, the set of all elements of S to which x is related (denoted [x])

Theorem on Equivalence Relations and Partitions

- An equivalence relation ρ on a set S determines a partition on S
- A partition on S determines an equivalence relation on S
- Proof: Omitted
- Example:
  1. On set of natural numbers, $x \rho y \leftrightarrow x=y$
  2. On {1,2,3}, ρ={(1,1),(2,2),(3,3),(1,2),(2,1)}

Congruence modulo n

- Congruence modulo 3, say
- Why is it an equivalence relation?
- What are the equivalence classes?

(Use all integers, positive and negative)

Examples

Symmetry and anti-symmetry

- Give an example of a relation which is symmetric as well as anti-symmetric.
- Relations can be
  - symmetric and *not* anti-symmetric
  - anti-symmetric and not symmetric
  - both symmetric and anti-symmetric
  - neither symmetric nor anti-symmetric
- Example:

$\forall \rho=\{(1,2), (2,1), (3,2), (2,3)\}$

$\forall \rho=\{(1,2), (2,1), (3,2)\}$

Examples

- Identify the 4 kinds of elements

1. $S = \{a, b, c\}$

   $\rho = \{(a,a), (b,b), (c,c), (a,b), (b,c), (a,c)\}$

2. $S = \{a, b, c, d\}$

   $\rho = \{(a,a), (b,b), (c,c), (d,d), (a,b), (a,c)\}$