EE369: Discrete Math
Propositional Logic

Outline

- Logic
- Propositional Logic
- Well formed formula
- Truth table
- Tautology & Contradiction
- Proof System for Propositional Logic
- Deduction method
- Formalizing English arguments
- **Text book chapters 1.1 and 1.2**

Logics

To define a *logic*, answer three questions:

1. What are the *models*?
2. What are the *formulas*?
3. Which formulas are true in which models?

A logic is a formal system relating
syntax (formulas) and
semantics (models of the world).

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Propositional Logic: Models

- A statement or a *proposition* is a sentence that is either true or false. Represented as upper cap letters of the alphabet.
  - "She is very talented"
  - "There are life forms on other planets in the universe"
  - Q: "Today is Tuesday"
- **Statement letters** can be combined into new statements using **logical connectives** of
  - **conjunction (AND, $\wedge$)**
  - **disjunction (OR, $\vee$)**
  - **implication ($\rightarrow$)**
  - **equivalence ($\leftrightarrow$)**
  - **negation ($'$ or $\neg$)**

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Propositional Logic: Formulas

- Truth tables define how each of the connectives operate on truth values.
- Truth table for implication ($\rightarrow$)
- Equivalence connective $A \leftrightarrow B$ is shorthand for $(A \rightarrow B) \wedge (B \rightarrow A)$
- Truth table for equivalence ($\leftrightarrow$)
- Binary & unary connective

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When is a formula true in a model?

- Each Boolean connective has a *truth table*

e.g.

P	Q	P ∨ Q
T	T	T
T	F	T
F	T	T
F	F	F

P	Q	P ∧ Q
T	T	T
T	F	F
F	T	F
F	F	F

P	¬P
T	F
F	T

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What about the other connectives?

P	Q	P → Q
T	T	T
T	F	F
F	T	T
F	F	T

Tricky cases

P	Q	P ↔ Q
T	T	T
T	F	F
F	T	F
F	F	T

"P is the same as Q"

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Abstract: What about new connectives?

☺

P	Q	P ☺ Q
T	T	T
T	F	T
F	T	F
F	F	T

Truth tables define the connective

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Connective Truth Table Summary

TABLE 1.4

A	B	$A \wedge B$	$A \vee B$	$A \rightarrow B$	$A \leftrightarrow B$	A'
T	T	T	T	T	T	F
T	F	F	T	F	F	
F	T	F	T	T	F	T
F	F	F	F	T	T	

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Connectives in English

TABLE 1.5

English Word	Logical Connective	Logical Expression
and; but; also; in addition; moreover	Conjunction	$A \wedge B$
or	Disjunction	$A \vee B$
If A, then B. A implies B. A, therefore B. B follows from A. A is a sufficient condition for B. B is a necessary condition for A.	Implication	$A \rightarrow B$
A if and only if B. A is necessary and sufficient for B.	Equivalence	$A \leftrightarrow B$
not A It is false that A ... It is not true that A ...	Negation	A'

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Example formulas and non-formulas

- "If the rain continues, then the river will flood." Express by implication.
- A: the rain continues
  B: the river floods
- If A, then B: ---- $A \rightarrow B$
- If the river floods, did the rain continue?

A	B	$A \rightarrow B$
T	T	T
F	T	T
F	F	T

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Example formulas and non-formulas

- "A good diet is a necessary condition for a healthy cat." Express by implication.
- C: the cat has a good diet
  D: the cat is healthy

  "B is a necessary condition for A": $A \rightarrow B$

  Thus $D \rightarrow C$

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Example formulas and non-formulas

- "Julie likes butter but hates cream." Negate this.
- A: Julie likes butter
  B: Julie hates cream
  "A but B" ---- $A \wedge B$

  Need $(A \wedge B)'$

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Example formulas and non-formulas

- "Julie likes butter but hates cream." Negate this.
- Negation:

A	B	$A \wedge B$	$(A \wedge B)'$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

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Example formulas and non-formulas

- "Julie likes butter but hates cream."
  Negate this.

A	B	A ∧ B	(A ∧ B)'	A'	B'	A'∨B'
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

- A': Julie does not like butter
  B': Julie does not hate (likes) cream
  Julie does not like butter or likes cream

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Additional Negation Examples

1. The processor is fast but the printer is slow
1. Either the processor is slow or the printer is fast

2. The processor is fast or else the printer is slow
2. The processor is slow and the printer is fast

3. If the processor is fast then the printer is slow
3. If the processor is fast but so is the printer

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Formulas & Precedence

- Well-formed formula: A valid string with statement letters, connectives, and parentheses
  - $(((A \wedge B))) \wedge D$
  - $(A' \rightarrow B) \wedge (C' \vee A)$
- Non-WFF
  - $(A' \rightarrow \rightarrow B))) \wedge \wedge (C' \vee A)$

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Formulas & Precedence

- Precedence
  - $A \vee B \rightarrow C \equiv (A \vee B) \rightarrow C$ OR $A \vee (B \rightarrow C)$?
- Rules:
  1. Innermost parenthesis () first
  2. Negation ‘
  3. $\wedge$,$\vee$
  4. $\rightarrow$
  5. $\leftrightarrow$

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Formulas & Precedence

- Main connective: Last connective to be applied in a WFF
  - $(A' \rightarrow B) \wedge (C' \vee A)$ $\wedge$
  - $(A \vee B) \rightarrow C$ $\rightarrow$
  - $(((A' \rightarrow B) \wedge (C')) \vee A)$ $\vee$

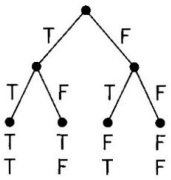
Example of Truth Table for a WFF

- Compound WFF $(A' \rightarrow B) \wedge (C' \vee A)$

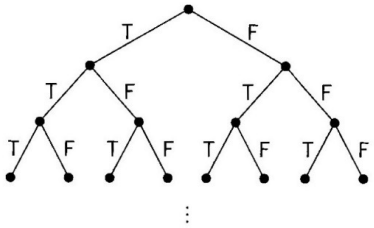
P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

- # rows in a truth table with *n* statement letters = ?

Truth Table Scale



Statement letters	Choices
1	$2 = 2^1$ branches
2	$4 = 2^2$ branches



Statement letters	Choices
1	$2 = 2^1$ branches
2	$4 = 2^2$ branches
3	$8 = 2^3$ branches
⋮	⋮
n	2^n branches

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Tautologies and Contradictions

- A *tautology* is a formula that is true in every model. (also called a *theorem*)
  - for example, $(A \vee \neg A)$ is a tautology
  - What about $(A \vee B) \vee (A' \vee B')$?
  - Look at tautological equivalences on pg. 8 of text
- A *contradiction* is a formula that is false in every model.
  - for example, $(A \wedge \neg A)$ is a contradiction

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De Morgan's Laws

- $(A \vee B)' = A' \wedge B'$
- Negate "The couch is either new or comfortable."
- $(A \wedge B)' = A' \vee B'$
- Negate "The book is thick and boring."

De Morgan's by Truth Table

- $(A \wedge B)' = A' \vee B'$

A	B	A ∧ B	(A ∧ B)'	A'	B'	A'∨B'
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

1.1 Objectives and Summary

- You can now:
  - Construct truth tables from WFF's
  - Define tautologies and contradictions
- Summary:
  - WFFs are symbolic representations of statements
    - "Julie likes milk but not cream." : $A \wedge B$
  - Truth values for compound WFF's depend on inputs and connectives
    - # of rows = 2^n

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A Proof System

- A *proof system* is a syntactic system for finding formulas implied by the hypotheses
  - "syntactic" means manipulating syntax
    - i.e. manipulating formulas rather than models.
  - $P_1 \wedge P_2 \wedge \ldots P_n \rightarrow Q$ is a tautology
- A *proof* is a sequence of formulas, where each formulas in the sequence is either
  - a hypothesis
  - a formula justified by previous formulas and a proof rule
- The sequence "proves" the last formula

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Example proof rule

$$\frac{P \qquad P \rightarrow Q}{Q}$$

modus ponens (*mp*)

Given the formulas above the line, we can add the formula below the line to the proof.

Example proof

- Hypotheses: C, B, $B \rightarrow (C \rightarrow A)$
- Conclusion: A

1.	B	*premise*
2.	$B \rightarrow (C \rightarrow A)$	*premise*
3.	$C \rightarrow A$	1, 2, *mp*
4.	C	*premise*
5.	A	3, 4, *mp*

Proof rules: Equivalence rules (Table 1.12)

Expression	Equivalent	Rule name	Abbr.
$P \vee Q$ $P \wedge Q$	$Q \vee P$ $Q \wedge P$	Commutative	comm
$(P \vee Q) \vee R$ $(P \wedge Q) \wedge R$	$P \vee (Q \vee R)$ $P \wedge (Q \wedge R)$	Associative	ass
$\neg(P \vee Q)$ $\neg(P \wedge Q)$	$\neg P \wedge \neg Q$ $\neg P \vee \neg Q$	De Morgan	dm
$P \rightarrow Q$	$\neg P \vee Q$	Implication	imp
$\neg(\neg P)$	P	Double Neg.	dn
$P \leftrightarrow Q$	$(P \rightarrow Q) \wedge$ $(Q \rightarrow P)$	Equivalence	equ

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Proof Rules: Inference Rules (Table 1.13)

From	Can derive	Rule name	Abbr.
$P,\ P \rightarrow Q$	Q	Modus ponens	mp
$P \rightarrow Q,\ \neg Q$	$\neg P$	Modus tollens	mt
$P,\ Q$	$P \wedge Q$	Conjunction	con
$P \wedge Q$	$P,\ Q$	Simplification	sim
P	$P \vee Q$	Addition	add

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An Example Proof

$[(A \vee B') \rightarrow C] \wedge (C \rightarrow D) \wedge A \rightarrow D$

($P_1 \wedge P_2 \wedge \ldots P_n \rightarrow Q$ is a tautology)

1. $(A \vee B') \rightarrow C$	Hyp/premise
2. $C \rightarrow D$	Hyp/premise
3. A	Hyp/premise
4. $A \vee B'$	3, add
5. C	1,4, mp
6. D	2,5,mp

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Deduction Method in Proofs

- When proving $P \rightarrow Q$…
  - add *P* to premises and prove *Q*.
- Repeat to prove $P \rightarrow (Q \rightarrow R)$
  - add *P* and *Q* to premises and prove *R*.
- To prove: $P_1 \wedge P_2 \wedge \ldots P_n \rightarrow (R \rightarrow S)$, prove: $P_1 \wedge P_2 \wedge \ldots P_n \wedge R \rightarrow S$

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Deduction Example

- To prove: $P_1 \wedge P_2 \wedge \ldots P_n \rightarrow (R \rightarrow S)$, prove: $P_1 \wedge P_2 \wedge \ldots P_n \wedge R \rightarrow S$
- $[A \rightarrow (B \rightarrow C)] \wedge (A \vee D') \wedge B \rightarrow (D \rightarrow C)$

1. $A \rightarrow (B \rightarrow C)$	Hyp/premise
2. $A \vee D'$	Hyp/premise
3. B	Hyp/premise
4. D	Hyp/premise *(Deduction)*
5. $D' \vee A$	2,comm
6. $D \rightarrow A$	5,imp
7. A	4,6,mp
8. $B \rightarrow C$	1,7,mp
9. C	3,8,mp

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Review from 1/24/2017: Logics

- Logics are models, formulas, and definitions that represent the world
  - Statements: A; B; C; "Today is Thursday"
  - Connectors: $\rightarrow$, $\vee$, $\wedge$, ', $\neg$, $\leftrightarrow$
  - Well-formed formulas (WFF)
- Truth tables define the logic
  - 2^n rows for n input statements
  - Can show equivalence (De Morgan's)
  - Implication can be tricky

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Connectives in English

TABLE 1.5

English Word	Logical Connective	Logical Expression
and; but; also; in addition; moreover	Conjunction	$A \wedge B$
or	Disjunction	$A \vee B$
If A, then B. A implies B. A, therefore B. A only if B. B follows from A. A is a sufficient condition for B. B is a necessary condition for A.	Implication	$A \rightarrow B$
A if and only if B. A is necessary and sufficient for B.	Equivalence	$A \leftrightarrow B$
not A It is false that A ... It is not true that A ...	Negation	A'

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Review from 1/24/2017: Implication

- "If the rain continues, then the river will flood." Express by implication.
- A: the rain continues
  B: the river floods
- If A, then B: ---- $A \rightarrow B$
- If the river floods, did the rain continue?

A	B	$A \rightarrow B$
T	T	**T**
T	F	F
F	T	**T**
F	F	T

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Review from 1/24/2017: Negation

- "Julie likes butter but hates cream." Negate this.
- A: Julie likes butter
  B: Julie hates cream
  "A but B" ---- $A \wedge B$

  Need $(A \wedge B)' \leftrightarrow (A' \vee B')$ – De Morgan's
- "Julie hates butter or likes cream"

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Review from 1/24/2017: Negation

- Negate "Cucumbers are green and seedy:"
  1. Cucumbers are not green and not seedy
  2. Cucumbers are not green or not seedy
  3. Cucumbers are green and not seedy
  – Which answer(s) are valid negations?
- "The carton is sealed or the milk is sour"
  1. The milk is not sour or the carton is not sealed
  2. The carton is not sealed and also the milk is not sour
  3. If the carton is not sealed, then the milk will be sour

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Review from 1/24/2017: Definitions

- A *tautology* is a formula that is true in every model. (also called a *theorem*)
  - for example, $(A \vee \neg A)$ is a tautology
- A *contradiction* is a formula that is false in every model.
  - for example, $(A \wedge \neg A)$ is a contradiction
- De Morgan's
  - $(A \vee B)' = A' \wedge B'$
  - $(A \wedge B)' = A' \vee B'$

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Review from 1/24/2017: Prop. Proofs

- A *proof system* is a syntactic system for finding formulas implied by the hypotheses
  - Show that $P_1 \wedge P_2 \wedge \ldots P_n \rightarrow Q$ is a tautology
- A *proof* is a sequence of formulas, where each formulas in the sequence is either
  - a hypothesis
  - a formula justified by previous formulas and a proof rule
- The sequence "proves" the last formula

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Review from 1/24/2017: The Proposition

- $P_1 \wedge P_2 \wedge \ldots P_n \rightarrow Q$
  - $P_{1\ldots}$ P_n are hypotheses or predicates
  - Q is the conclusion
  - If $P_{1\ldots}$ P_n are true, show that Q is true
    - Syntactic manipulation to form tautology
- Example proposition:
  - If Bob is **h**ungry, then he will **e**at. Bob is **h**ungry. Therefore, he will **e**at.
    - $(H \rightarrow E) \wedge H \rightarrow E$

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Review from 1/24/2017: Modus Ponens

$$\frac{P \qquad P \rightarrow Q}{Q}$$

modus ponens (*mp*)

- If Bob is **h**ungry, then he will **e**at. Bob is **h**ungry. Therefore, he will **e**at.
  - $(H \rightarrow E) \wedge H \rightarrow E$

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Review: Formal Proof

- If Bob is **h**ungry, then he will **e**at. Bob is **h**ungry. Therefore, he will **e**at.
  - $(H \rightarrow E) \wedge H \rightarrow E$

1.$(H \rightarrow E)$	hypothesis
2.H	hypothesis
3.E	1,2,modus ponens

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Proof rules: Equivalence rules (Table 1.12)

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$\neg(P \vee Q)$ $\neg(P \wedge Q)$	$\neg P \wedge \neg Q$ $\neg P \vee \neg Q$	De Morgan	dm
$P \rightarrow Q$	$\neg P \vee Q$	Implication	imp
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Proof Rules: Inference Rules (Table 1.13)

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$P \wedge Q$	$P,\ Q$	Simplification	sim
P	$P \vee Q$	Addition	add

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Prop. Logic Proof: Example 1

$$(A' \rightarrow B') \wedge B \wedge (A \rightarrow C) \rightarrow C$$

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Prop. Logic Proof: Example 2

$$(A \rightarrow B) \wedge [B \rightarrow (C \rightarrow D)] \wedge [A \rightarrow (B \rightarrow C)] \rightarrow (A \rightarrow D)$$

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Prop. Logic Proof: Example 3

$$[A \rightarrow (B \rightarrow C)] \rightarrow [B \rightarrow (A \rightarrow C)]$$

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Prop. Logic Proof: Example 4

$$(A \wedge B) \rightarrow (A \rightarrow B')'$$

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Prop. Logic Proof: Example 5

$$(A \rightarrow C) \wedge (C \rightarrow B') \wedge B \rightarrow A'$$

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Formalizing English Arguments

To formalize an English argument:

1. Find the minimal *statements* in the argument and symbolize them with propositional letters *A*, *B*, …
2. Convert English connectives to propositional ones.
3. Give a proof of the conclusion using the premises.

Examples

Jack went to fetch a pail of water. Jack fetches a pail of water only if Jack works hard. Jack works hard only when Jack is paid. Therefore Jack was paid.

Minimal True/False Statements

Jack went to fetch a pail of water. Jack fetches a pail of water only if Jack works hard. Jack works hard only if Jack is paid. Therefore Jack was paid.

A = Jack fetches a pail of water
B = Jack works hard
C = Jack is paid

A. *A* only if *B*. *B* only if *C*. Therefore *C*.

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Eliminating connectives

A. *A* only if *B*. *B* only if *C*. Therefore *C*.

becomes

Hypotheses: A, $A \rightarrow B$, $B \rightarrow C$.
Conclusion: C.

Easy to prove that these hypotheses give this conclusion using rule mp.

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Examples — a bad argument

Jack went to fetch a pail of water. Jack fetches a pail of water if Jack works hard. Jack works hard if Jack is paid. Therefore Jack was paid.

Hypotheses: A, $B \rightarrow A$, $C \rightarrow B$.
Conclusion: C.

Hypotheses do not entail conclusion…to show this, give a model that makes the conclusion false.

– A=T, B=T, C=F

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Another example

Fish can walk. Fish can walk only if elephants can fly. Elephants can fly only if eggplants can talk. Therefore, eggplants can talk.

Is this a valid argument?

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Another example

Fish can walk. Fish can walk only if elephants can fly. Elephants can fly only if eggplants can talk. Therefore, eggplants can talk.

A: Fish can walk
B: Elephants can fly
C: Eggplants can talk

Another example

A. A only if B. B only if C. Therefore, C.

A: Fish can walk
B: Elephants can fly
C: Eggplants can talk

Hypothesis: A, A $\rightarrow$ B, B $\rightarrow$ C
Conclusion: C

Formalizing: Example 1

- If chicken is on the menu, then don't order fish, but you should have either fish or salad. So if chicken is on the menu, have salad.

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Formalizing: Example 2

- The crop is good, but there is not enough water. If there is a lot of rain or not a lot of sun, then there is enough water. Therefore the crop is good and there is a lot of sun.

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Formalizing w/ Proof: Example 1

- If the program is efficient, then it executes quickly. Either the program is efficient, or it has a bug. However, the program does not execute quickly. Therefore it has a bug.

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Formalizing w/ Proof: Example 2

- If Jane is more popular, then she will be elected. If Jane is more popular, then Craig will resign. Therefore if Jane is more popular, she will be elected and Craig will resign.

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Formalizing w/ Proof: Example 3

- It is not the case that if electric rates go up, then usage will go down, nor is it true that either new power plants will be built or bills will not be late. Therefore usage will not go down and bills will be late.

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Examples where propositional logic fails

Every positive number is greater than zero. Five is a positive number. Therefore, five is greater than zero.

Minimal statements?

A = Every positive number is greater than zero.
B = Five is a positive number.
C = Five is greater than zero.

Hypotheses: *A*, *B*. Conclusion: *C*.
Conclusion not entailed (consider *A* = *B* = T, C = F)

Our logic does not model the internal structure of the propositions

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