Lecture Outline for Functions

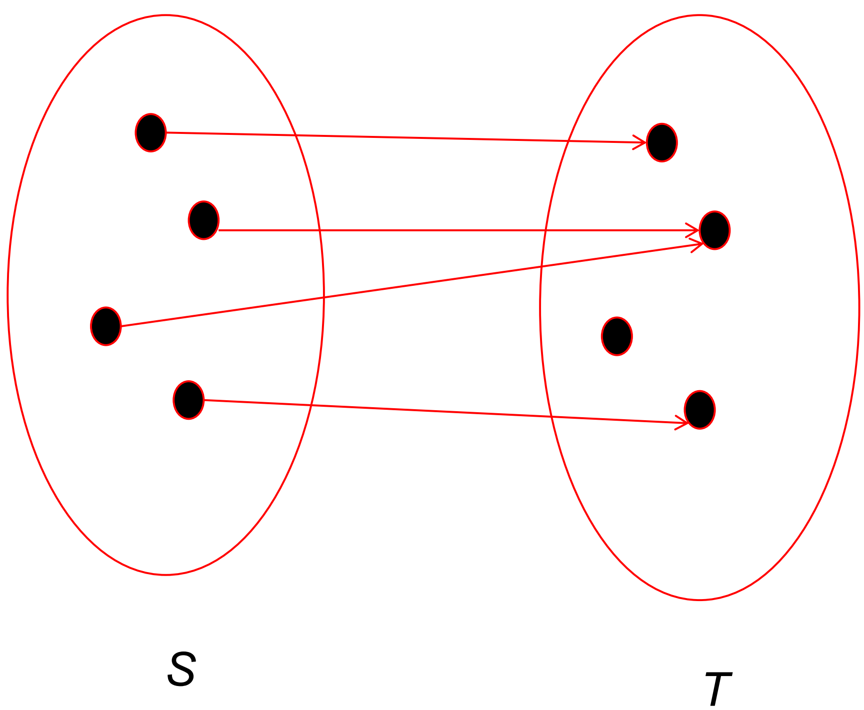
- Definition
- Properties of functions (onto, one-to-one, etc.)
- Composition of functions
- Inverse of functions
- Number of functions
- Magnitude of functions
- Section 4.4 of text

Definition of function

- f: S $\rightarrow$ T
- Every element of set S mapped to one and only one value in T
- Example:
  - $g(x) = x^3$, S: N, T: N
  - Described by $\{(x, g(x)) | g(x) = x^3\}$
- S is the domain, T is co-domain
- If (s, t) belongs to the function, t is the image of s, and s is the pre-image of t

Examples and non-examples of functions

1. $f: S \rightarrow T$ where $S = T = \{1, 2, 3\}$, $f = \{(1,1), (2,2), (3,3)\}$
2. $f: S \rightarrow T$ where $S = T = \{1, 2, 3, 4\}$, $f = \{(1,1), (2,2), (3,3), (4, 3)\}$
3. $f: S \rightarrow T$ where $S = T = \{1, 2, 3, 4\}$, $f = \{(1,1), (2,2), (3,3), (3, 4)\}$
4. $f: N \rightarrow N$ where
   - $f(x) = x, x \leq 3$
   - $f(x) = 2x, x \geq 3$



Images and Pre-images

- S: R, T: Z
- $f(x) = \lfloor x \rfloor$
- What is the image of 2.3, -3.4?
- What are the preimages of 2, -4? *[2, 3)*

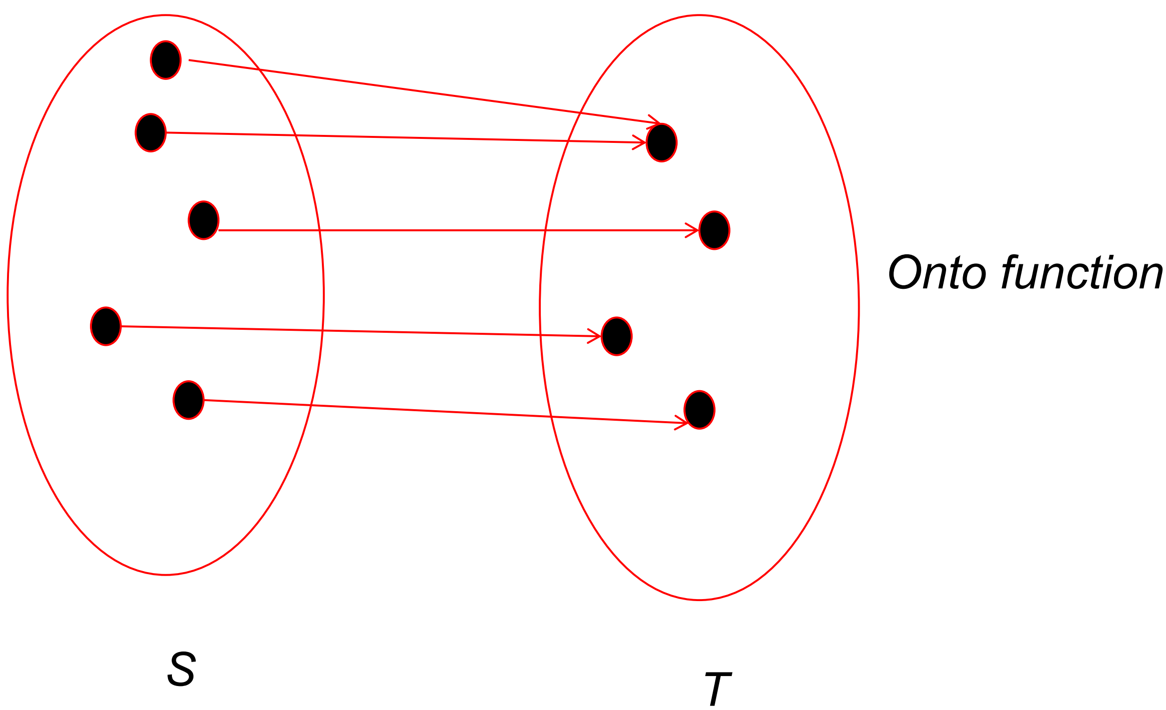
[-4, -3)

Functions on more than one variable

- f: $S_1 \times S_2 \times \ldots S_n \to T$
- Associates each ordered n-tuple of elements $(s_1, s_2, \ldots, s_n)$ with an element of T
- Example:
  - f: $Z \times N \times \{0,1\} \to Q$ where $f(x,y,z) = x^{y+z}$ *x/y*
  - f(2, -2, 0) = ? *7/2*
  - *8/2 = 4/1*
  - f(2, 2, 0) = ? 4

Properties of functions: Onto

- Onto functions: Take function f: S→T.
- The range of f is the set R of images of all members of S.
- If the range of f is identical to the co-domain, then it is an onto function.
- How to prove?
  - Always R ⊆ T.
  - If we can prove T ⊆ R, then we are done.
  - Take an arbitrary element in T and show it is the image of some member of S.



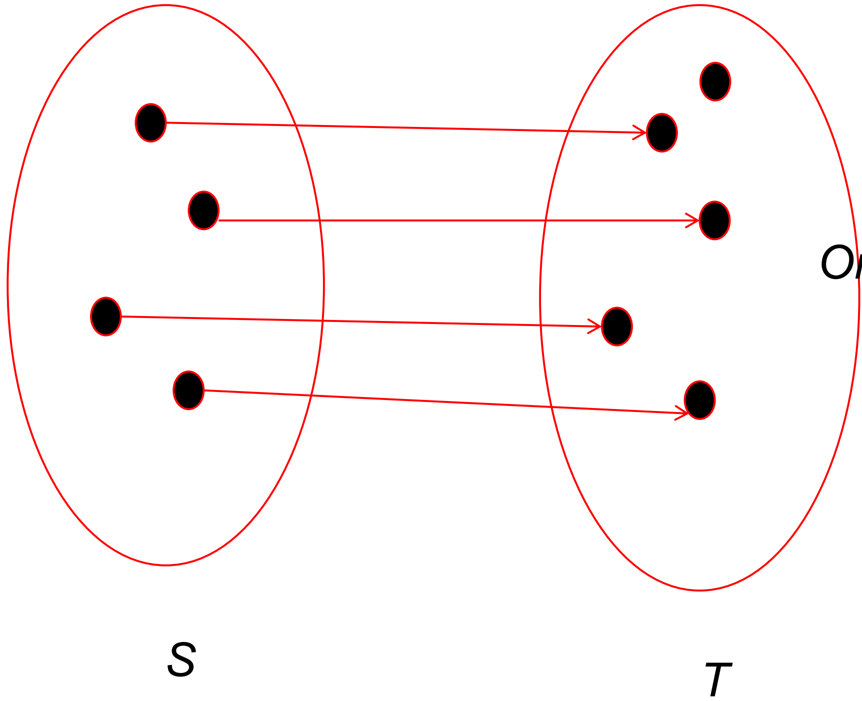
Properties of functions: onto

- $S = \{0, 2, 4, 6\}$
- $T = \{1, 3, 5, 7\}$
- Which of the following are onto functions?

1. {(0,2), (2,4), (4,6), (6,0)}
2. {(6,3), (2,1), (0,7), (4,5)}
3. {(6,1), (0,3), (4,1), (0,7), (2,5)}

Properties of functions: One-to-one

- A function $f: S \rightarrow T$ is one-to-one (or injective) if no member of T has more than one preimage in S
- How to prove?
  - Assume that $f(s_1) = f(s_2)$ and then show that $s_1 = s_2$
- What is the difference from a one-to-one *relation* from S to T?



Examples of one-to-one functions

- S = {0, 2, 4, 6}
- T = {1, 3, 5, 7, 9}
- Which of the following are one-to-one functions?

1. {(0,1), (2,3), (4,5), (6,7)}
2. {(0,1), (2,1), (4,7), (6,9)}

Properties of functions: Bijective

- A function f: $S \rightarrow T$ which is both onto and one-to-one is called Bijective
- How to prove?
  - Step 1: Prove onto
  - Step 2: Prove one-to-one

Composition of functions

- f: $S \rightarrow T$ and g: $T \rightarrow U$.
- For any $s \in S$, $f(s) \in T$
- Therefore $g(f(s)) \in U$
- $g \circ f(x) \equiv g(f(x))$
- In other words, $g \circ f$ is a new function from S to U ($S \rightarrow U$)
- This is called the composition function.
- For composing, the domains and the ranges have to be compatible.

Composition of functions: Example

- $f: R \rightarrow R$
- $f(x) = x^2$
- $g(x) = \lfloor x \rfloor$
- $(g \circ f)\ (2.3) = g(5.29) = 5$
- $(f \circ g)\ (2.3) = f(2) = 4$
- Order is important

Inverse Functions

- If $f: S \rightarrow T$ is bijective, we have another function $g: T \rightarrow S$ such that
  - an element $t \in T$ maps to $s \in S$ where $f(s) = t$
- $g \circ f$ maps each element of S to itself
  - Called Identity function on S (i_S)
- What is $f \circ g$?
- Inverse function: $f: S \rightarrow T$. There exists $g: T \rightarrow S$, such that $g \circ f = i_S$ and $f \circ g = i_T$, then g is the inverse function of f, denoted f^{-1}

Inverse Functions (Cont'd)

- Theorem:

1. If f is a bijection, then f^{-1} exists.
2. If f^{-1} exists, then f is a bijection.

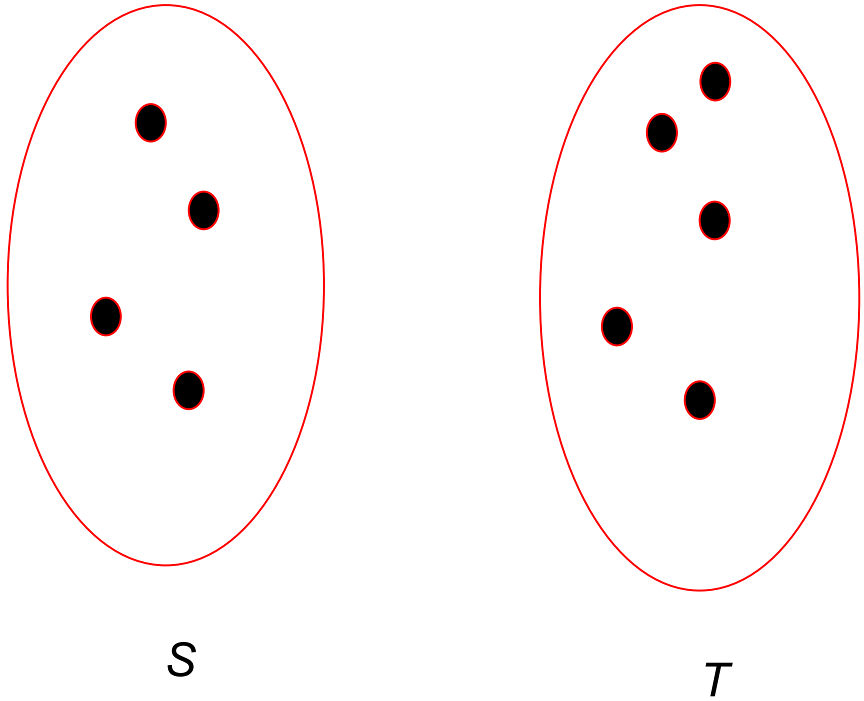
- Example: Each function is R→R. Find f^{-1}.

1. $f(x) = 2x$
2. $f(x) = (x+4)/3$

How Many Functions?

- We will use f: S→T
  - $|S| = m$, $|T| = n$
- How many such functions without any restriction? n^m
- How many such one-to-one functions?
  - $m \leq n$
  - # functions = $n * (n-1) * (n-2) * \ldots (n-m+1) = n!/(n-m)!$

$m = 4, n = 5$ No restriction: $5 * 5 * 5 * 5 = 5^4$



One-to-one: $5 * 4 * 3 * 2$

How Many Functions? (Cont’d)

- We will use $f: S \rightarrow T$
  - $|S| = m$, $|T| = n$
- How many such onto functions?
  - $m \geq n$
  - Total number of functions – Total number of non-onto functions

$= n^m – [C(n,1)(n-1)^m - C(n,2)(n-2)^m - \ldots - (-1)^{n-1}C(n,n-1)(1)^m]$

$= n^m – C(n,1)(n-1)^m + C(n,2)(n-2)^m - \ldots + (-1)^{n-1}C(n,n-1)(1)^m$

Permutation of a Set

- For a given set A, consider bijective functions from $A \rightarrow A$
- These are called permutations of A (Set of all permutations is S_A)
- Example:
  - $A = \{1, 2, 3, 4\}$
  - $f = \{(1,2), (2,3), (3,4), (4,1)\}$
  - f written in matrix form
  - Cyclical notation for a function
- How many such permutations are there?

More on Cycles

- Say, $A = \{1, 2, 3, 4, 5\}$
- $f = (1, 4, 5)$
- Write out f in matrix form.
- Example:
  - Composition of functions
  - Composition of cycles

More on Permutations

- Disjoint cycles: f and g are members of S_A and are disjoint cycles (i.e., have no element in common)
  - Then, $f \circ g = g \circ f$
- Identity permutation: Maps each element of A to itself
- Derangement: Permutation of A such that no element in A is mapped to itself

Number of Derangements

- $|A| = n$
- $A = \{a_1, a_2, \ldots, a_n\}$
- Remember in a derangement no element of A is mapped to itself
- # derangements = total number of permutation functions – total number of non-derangements

$= n! - n!/1! + n!/2! - n!/3! + \ldots + (-1)^n n!/n!$

$= n![1 - 1/1! + 1/2! - 1/3! + \ldots + (-1)^n 1/n!]$

Number of Derangements: Examples

- S = {a, b, c}
- Find the number of derangements
- Show the different derangements

Definition of Θ

- When do you say two functions are the same order of magnitude?
- Consider two functions f and g, whose domain and co-domain are non-negative real numbers.
- If you can find positive constants n_0, c_1, c_2 such that

$$c_1 g(x) \leq f(x) \leq c_2 g(x),\ \forall x \geq n_0$$

- Then $f=\Theta(g)$

Definition of O

- When do you say a function f is big-oh of another function g?
- Consider two functions f and g, whose domain and co-domain are non-negative real numbers.
- If you can find positive constants n_0, c such that

$$f(x) \leq c\, g(x),\ \forall x \geq n_0$$

- Then f=O(g)

Examples

- Example 1
  - $f(x) = 3x^3 - 7x$
  - $g(x) = x^3/2$
  - f(x) ? g(x)
- Example 2
  - $f(x) = 100x^2$
  - $g(x) = x^3$
  - f(x) ? g(x)