Homework 2 Sample Solution

Problem I

After calculating the Geometric Mean (GM) and the Harmonic Mean (HM) for a few different pairs of positive integers, we hypothesize that:

$$HM(a,b) < GM(a,b) \text{ if } a \neq b.$$

(If $a = b$, then clearly $HM(a,b) = GM(a,b)$)

So we want to verify if

$$\frac{2ab}{a+b} \overset{?}{<} \sqrt{ab}.$$

Multiply both sides by $\dfrac{a+b}{2\sqrt{ab}}$

$$\therefore \quad \frac{2ab}{2\sqrt{ab}} \overset{?}{<} \frac{a+b}{2}$$

$$\text{i.e.} \quad \sqrt{ab} \overset{?}{<} \frac{a+b}{2}$$

Squaring both sides:

$$ab \overset{?}{<} \frac{(a+b)^2}{4}$$

$$4ab \overset{?}{<} (a+b)^2$$

$$a^2+b^2+2ab-4ab \overset{?}{>} 0$$

$$a^2-2ab+b^2 \overset{?}{>} 0$$

$$(a-b)^2 \overset{?}{>} 0$$

This last statement is true because we are squaring a non-zero integer.

$\therefore$ our original hypothesis $HM(a,b) < GM(a,b)$, $a \neq b$ is proved.

Problem II

A set that has n elements will have

$$\frac{n(n-1)}{2}$$

subsets that have exactly two elements.

Base case

$n = 2$. There is only one subset meeting this condition, the set itself.

Inductive hypothesis

Let a set of K elements have $\frac{K(K-1)}{2}$ subsets. — (1)
Call this set S.

Inductive step

To prove that a set of $(K+1)$ elements has $\frac{(K+1)(K+1-1)}{2}$
$= \frac{K(K+1)}{2}$ subsets. — (A)

Consider that I form the set of $(K+1)$ elements by adding an element $\{a\}$ to the original set of K elements. Call this set of $(K+1)$ elements T.

$$\therefore \quad T = S \cup \{a\}$$

To form the subsets of T that meet the above condition, I can do this in one of two ways:

1) Not include the new element $\{a\}$ $\Rightarrow$ There are $\frac{K(K-1)}{2}$ such subsets. [From (1)]
2) Include the new element $\Rightarrow$ There are K such subsets
$\rightarrow$ Include $\{a\}$ and any one of the K elements of the set S.

$\therefore$ The number of subsets of T =

$$\frac{K(K-1)}{2} + K = \frac{K^2 - K + 2K}{2} = \frac{K^2 + K}{2} = \frac{K(K+1)}{2}$$

$\therefore$ we have proven (A).

$\therefore$ By the 1st principle of mathematical induction, we have proven that # subsets with exactly two elements is $\frac{n(n-1)}{2}$.