Combinatorics Problems Set 2 Solution

1. Chandra has a drawer full of 12 red and 14 green socks. In order to avoid waking up his roommate, he must grab a selection of clothes in the dark and get dressed. How many socks must he grab in order to be assured that he will have at least one matching pair?

Pigeon hole principle. 2 pigeon holes. Must select 3 to have the guarantee.

2. Hugo and Viviana work in an office with eight other coworkers. Out of these 10 workers, their boss needs to choose a group of four to work together on a project.
   a. How many different working groups of four can the boss choose?
   b. Suppose Hugo and Viviana absolutely refuse, under any circumstances, to work together. Under this restriction, how many different working groups of four can be formed?

(a) C(10, 4) = 210

(b) C(8, 3) + C(8, 3) + C(8, 4) = 182

3. Two teams, A and B, play a best-of-seven match. The match ends when one team wins four games. How many different win or loss scenarios are possible?

Solution: (Version #1.) The match could go four, five, six, or seven games, and these cases are all disjoint. There are only two ways the winners of a four-game match could go: AAAA or BBBB. In a five-game match, the winning team must lose one of the first four games, so there are 2 C(4, 1) = 8 ways this can happen; the C(4, 1) factor accounts for choosing which game to lose, and the factor of 2 accounts for either A or B winning the match. Similarly, there are 2.C(5,2)= 20 scenarios for a six-game match, and 2-C(6,3)= 40 scenarios for a seven-game match. By the addition principle, there are 2+8+20+40= 70 different possible win or loss scenarios.

This last solution is a nice illustration of using the addition principle in tandem with the selection principle. However, there is an alternate solution that is possibly simpler to understand.

Solution: (Version #2.) Regard every match as lasting seven games: once one team has won four games, that team forfeits the remaining games. This is the same as ending the match after four wins by one team, so the total number of win or loss scenarios should be the same. We must then count the number of seven-symbol strings using four A's and three B's (when A wins the match) and the number of seven-symbol strings using four B's and three A's (when B wins). This is just like Example 4.19; in each case there are C(7,4) = 35 such strings, for a total of 70 win or loss scenarios.

4. Form a seven-letter word by mixing up the letters in the word COMBINE.
   a. How many ways can you do this?
   b. How many ways can you do this if all the vowels have to be at the beginning?

Solution:

(a) 7! = 5040
(b) 3! * 4! = 144

5. Possible grades for a class are A, B, C, D, and F. (No +/-'s.)
   a. How many ways are there to assign grades to a class of seven students?
   b. How many ways are there to assign grades to a class of seven students, if nobody receives an F and exactly one person receives an A?

Solution:

(a) 5^7 = 78,125
(b) 7 . 3^6 = 5,103

6. A men's field lacrosse team consists of ten players: three attackmen, three midfielders, three defenders, and one goaltender. Given a set of 10 players, how many different ways are there to assign the roles of attackmen, midfielders, defenders, and goaltender?

Solution:

C(10, 3) . C(7, 3). C(4, 3) = 16,800

7. There are nine empty seats in a theater, and five customers need to find places to sit. How many different ways can these five seat themselves?

Solution:

P(9, 5) = 15,120

8. How many solutions, using only positive integers, are there to the following equation?

$$\sum_{i=1}^{5} x_i = 32$$

Solution:

C_{rep}(n=5, r=27) = C(31, 27) = 31,465

9. A certain brand of jellybean comes in four colors: red, green, purple, and yellow. These jellybeans are packaged in bags of 50, but there is no guarantee as to how the colors will be distributed; you might get a mixture of all four colors, or just some red and some green, or even (if you are very lucky) a whole bag of purple.

   Compute the total number of different possible color distributions.

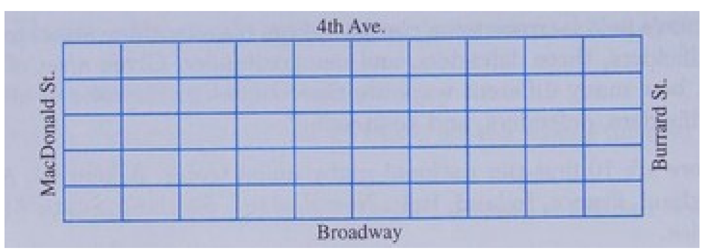
Solution:

C_{rep}(n=4, r=50) = C(53, 50) = 23,426

10. The streets of many cities (e.g., Vancouver, British Columbia) are based primarily on a rectangular grid. In such a city, if you start at a given street corner, how many different ways are there to walk directly to the street corner that is 5 blocks north and 10 blocks east?

    For example, how many ways can you walk from the corner of MacDonald and Broadway to the corner of 4th and Burrard? See Figure below.

    For the purpose of this problem, the order does not matter. So if I walk 1 block north, followed by 1 block east, that is identical to if I walk 1 block east, followed by 1 block north.



Streets of Vancouver, British Columbia

Solution:

C (15, 5) = 3,003

11. Compute the coefficient of x^8 in the expansion of $(3x – 2)^{13}$.

Solution:

-C(13, 5) * (3x)^8 * (12)^5 = -2,101,139,149,824

$$1594323x^{13} - 82904796x^{12} + 1989715104x^{11} - 29182488192x^{10} + 291824881920x^{9} - 2101139149824x^{8} + 1.12061E13x^{7} - 4.48243E13x^{6} + 1.34473E14x^{5} - 2.98829E14x^{4} + 4.78126E14x^{3} - 5.21592E14x^{2} + 12^{12} \cdot 39x - 12^{13}$$

Hide Steps

Solution steps

One step at a time

$(3x-12)^{13}$

Apply binomial theorem: $(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{(n-i)} b^i$

$a = 3x, \; b = -12$

$= \sum_{i=0}^{13} \binom{13}{i} (3x)^{(13-i)} (-12)^i$