14. Q: $j = a_n x^{n-i} + a_{n-1}x^{n-i-1} + \ldots + a_i x^0$

Q(0): $j_0 = a_n x^{n-i_0} + a_{n-1}x^{n-i_0-1} + \ldots + a_{i_0} x^0$ true since $j = a_n$ and $i = n$ before loop is entered, so equation becomes $a_n = a_n x^{n-n}$, or $a_n = a_n$

Assume Q(k): $j_k = a_n x^{n-i_k} + a_{n-1}x^{n-i_k-1} + \ldots + a_{i_k} x^0$

Show Q(k + 1): $j_{k+1} = a_n x^{n-i_{k+1}} + a_{n-1}x^{n-i_{k+1}-1} + \ldots + a_{i_{k+1}} x^0$

$$j_{k+1} = j_k * x + a_{i_k-1} = (a_n x^{n-i_k} + a_{n-1}x^{n-i_k-1} + \ldots + a_{i_k} x^0)x + a_{i_k-1}$$
$$= a_n x^{n-i_k+1} + a_{n-1}x^{n-i_k} + \ldots + a_{i_k} x^1 + a_{i_k-1}$$
$$= a_n x^{n-i_{k+1}} + a_{n-1}x^{n-i_{k+1}-1} + \ldots + a_{i_k} x^1 + a_{i_{k+1}}$$

At loop termination,

$j = a_n x^{n-i} + a_{n-1}x^{n-i-1} + \ldots + a_i x^0$ and $i = 0$ so

$j = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0 x^0$

15. Q: j = max of (a[1], … , a[i])

Q(0): j_0 = max(a[1], …, a[i_0]) true since $i_0 = 1$, so the right side becomes max(a[1]) and j_0 = a[1]

Assume Q(k): j_k = max(a[1], …, a[i_k])

Show Q(k + 1): j_{k+1} = max (a[1], …, a[i_{k+1}])

j_{k+1} = max(j_k, a[i_{k+1}]) = max(max(a[1], …, a[i_k]), a[i_k + 1])
= max(a[1], …, a[i_k + 1]) = max(a[1], …, a[i_{k+1}])

At loop termination, j = max(a[1], …, a[i]) and i = n, so j = max(a[1], …, a[n])

16.(a) makes one too many passes through the loop and adds a[n + 1] to the sum.

(b) is correct:

Q: j = a[1] + … + a[i – 1]

Q(0): j_0 = a[1] + … + a[i_0 – 1] true since j = 0, i = 1 before loop is entered, so equation becomes 0 = a[1] + … + a[0]; the right side has no terms, so has the value 0

Assume Q(k): j_k = a[1] + … + a[i_k – 1]

Show Q(k + 1): j_{k+1} = a[1] + … + a[i_{k+1} – 1]

Note that j_{k+1} = j_k + a[i_k] and i_{k+1} = i_k + 1

j_{k+1} = j_k + a[i_k] = a[1] + … + a[i_k – 1] + a[i_k]
= a[1] + … + a[i_k – 1] + a[i_{k+1}-1]

At loop termination, j = a[1] + … + a[i – 1] and i = n + 1, so j = a[1] + … + a[n]

(c) begins the sum with a[0]

(d) adds a[n + 1] to the sum

EXERCISES 2.4

*1. 10, 20, 30, 40, 50

2. 2, 1/2, 2, 1/2, 2

3. 1, 5, 14, 30, 55

*4. $1, 1+\frac{1}{2}, 1+\frac{1}{2}+\frac{1}{3}, 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}, 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$

5. 1, 2, 6, 24, 120

6. 1, 5, 47, 755, 18879

*7. 2, 2, 6, 14, 34

8. 3, 5, 13, 49, 235

9. 2, 3, 6, 18, 108

10. 1, 2, 3, 10, 22

*11. $F(n+1) + F(n-2) = F(n-1) + F(n) + F(n-2)$
$= [F(n-2) + F(n-1)] + F(n)$
$= F(n) + F(n) = 2F(n)$

12. $F(n) = F(n-2) + F(n-1)$
$= F(n-3) + F(n-4) + F(n-2) + F(n-3)$
$= 2F(n-3) + F(n-4) + F(n-2)$
$= 2[F(n-4) + F(n-5)] + F(n-4) + [F(n-3) + F(n-4)]$
$= 4F(n-4) + 2F(n-5) + F(n-3)$
$= 4F(n-4) + 2F(n-5) + [F(n-4) + F(n-5)]$
$= 5F(n-4) + 3F(n-5)$

13. $[F(n+1)]^2 = [F(n-1) + F(n)]^2$
$= [F(n-1)]^2 + 2F(n-1)F(n) + [F(n)]^2$
$= F(n-1)[F(n-1) + F(n) + F(n)] + [F(n)]^2$
$= F(n-1)[F(n+1) + F(n)] + [F(n)]^2$
$= F(n-1)F(n+2) + [F(n)]^2$

14. $F(n+3) = F(n+2) + F(n+1)$
$= F(n+1) + F(n) + F(n+1)$
$= 2F(n+1) + F(n)$

15. $F(n + 6) = F(n + 5) + F(n + 4)$
$= F(n + 4) + F(n + 3) + F(n + 4)$
$= [F(n + 3) + F(n + 2)] + F(n + 3) + [F(n + 3) + F(n + 2)]$
$= 3F(n + 3) + F(n + 2) + F(n + 2)$
$= 3F(n + 3) + [F(n + 3) - F(n + 1)] + F(n + 2)$
$= 4F(n + 3) - F(n + 1) + [F(n + 1) + F(n)]$
$= 4F(n + 3) + F(n)$

*16. $n = 1$: $F(1) = F(3) - 1$ or $1 = 2 - 1$ true
Assume true for $n = k$: $F(1) + \dots + F(k) = F(k + 2) - 1$
Show true for $n = k + 1$: $F(1) + \dots + F(k + 1) = F(k + 3) - 1$

$F(1) + \dots + F(k + 1)$
$= F(1) + \dots + F(k) + F(k + 1)$
$= F(k + 2) - 1 + F(k + 1)$ inductive hypothesis
$= F(k + 3) - 1$ recurrence relation

17. $n = 1$: $F(2) = F(3) - 1$ or $1 = 2 - 1$ true
Assume true for $n = k$: $F(2) + \dots + F(2k) = F(2k + 1) - 1$
Show true for $n = k + 1$: $F(2) + \dots + F(2(k + 1)) = F(2(k + 1) + 1) - 1$
$F(2) + \dots + F(2(k + 1))$
$= F(2) + \dots + F(2k) + F(2(k + 1))$
$= F(2k + 1) - 1 + F(2(k + 1))$ inductive hypothesis
$= F(2k + 1) + F(2k + 2) - 1$
$= F(2k + 3) - 1$ recurrence relation
$= F(2(k + 1) + 1) - 1$

18. $n = 1$: $F(1) = F(2)$ or $1 = 1$ true
Assume true for $n = k$: $F(1) + F(3) + \dots + F(2k - 1) = F(2k)$
Show true for $n = k + 1$: $F(1) + F(3) + \dots + F(2(k+ 1) - 1) = F(2(k + 1))$

$F(1) + F(3) + \dots + F(2(k+ 1) - 1)$
$= F(1) + F(3) + \dots + F(2k - 1) + F(2(k+ 1) - 1)$
$= F(2k) + F(2(k + 1) - 1)$ inductive hypothesis
$= F(2k) + F(2k + 1)$
$= F(2k + 2)$ recurrence relation
$= F(2(k + 1))$

19. $n = 1$: $[F(1)]^2 = F(1)F(2)$ or $1^2 = (1)(1)$ true
Assume true for $n = k$: $[F(1)]^2 + [F(2)]^2 + \dots + [F(k)]^2 = F(k)F(k + 1)$
Show true for $n = k + 1$: $[F(1)]^2 + [F(2)]^2 + \dots + [F(k + 1)]^2 = F(k + 1)F(k + 2)$

$[F(1)]^2 + [F(2)]^2 + \dots + [F(k + 1)]^2$
$= [F(1)]^2 + [F(2)]^2 + \dots + [F(k)]^2 + [F(k + 1)]^2$
$= F(k)F(k + 1) + [F(k + 1)]^2$ inductive hypothesis
$= F(k + 1)[F(k) + F(k + 1)]$
$= F(k + 1)F(k + 2)$ recurrence relation

*36. $S(1) = a$
$S(n) = rS(n - 1)$ for $n \geq 2$

37. $S(1) = a$
$S(n) = S(n - 1) + d$ for $n \geq 2$

*38. a. $A(l) = 50{,}000$
$A(n) = 3A(n - 1)$ for $n \geq 2$
b. 4

39. a. $P(l) = 500$
$P(n) = (1.1)P(n - 1)$ for $n \geq 2$
b. 4

40. b and c

41. a, b, c, f, and g

*42 a, b, and e

43. a, b, and d

*44. 1. Any unary predicate in x is a wff.
2. If P and Q are unary predicate wffs in x, so are $(P \wedge Q)$, $(P \vee Q)$, $(P \rightarrow Q)$, (P'), $(P \leftrightarrow Q)$, $(\forall x)P$, and $(\exists x)P$.

Note that this allows expressions such as $(\forall x)(\exists x)P(x)$ to be wffs, meaning that such expressions are syntactically correct. Within any interpretation, the truth value is unaffected by the outermost quantifiers. For example, if the domain is the integers and $P(x)$ is the predicate $x = 0$, then $(\forall x)(\exists x)P(x)$ is true because $(\exists x)P(x)$ is true.

45. 1. Any integer is a well-formed formula.
2. If P and Q are well-formed formulas, so are $(P + Q)$, $(P - Q)$, (P^*Q), and (P/Q).

46. 1. λ is well-balanced.
2. If A and B are strings of well-balanced parentheses, so are (A) and AB.

47. 1. The string 0 belongs to the set.
2. If x belongs to the set, so do 1x, x1, and 0x0.

You can add additional rules such as 00x and x00 - they still allow you to grow only legitimate strings, and may give you more flexibility in the choice of starting point, but they are redundant. To see that the three rules of part (2) are sufficient, consider working backwards. Take any binary string: if it has a 1 on either end, knock it off (because you got there using 1x or x1). If it has a 0 on both ends, knock them off (because you got there using 0x0). These are the only possibilities. Keep going until you get down to simple cases - if you get to a single 1, the string was not legitimate, if you get down to 2 0's, the string was not legitimate (note that if you continue to apply the rules, both of these cases reduce to the empty string, which is not legitimate). If you get down to a single 0, you are OK and that was the starting point.

*14. The recurrence relation is $A(n) = (1.01)A(n - 1) - 80$ with a base case of $A(1) = 5000$. This is a linear, first-order recurrence relation with constant coefficients, so Equation (8) applies and gives the solution $A(n) = (1.01)^{n-1}(5000) - 80[1.01^{n-2} + 1.01^{n-3} + \dots + 1.01 + 1] = (1.01)^{n-1}(5000) - 80[1 - (1.01)^{n-1}]/[1 - 1.01]$ from the formula for the sum of a geometric sequence. At the end of 18 months (the beginning of the 19th month), the loan balance remaining is $A(19) = (1.01)^{18}(5000) - 80[1 - (1.01)^{18}]/(1 - 1.01)$ which equals \$4411.56.

15. The recurrence relation is $S(n) = 0.98S(n - 1) - 10{,}000$ with a base case of $S(1) = 1{,}000{,}000$. This is a linear, first-order recurrence relation with constant coefficients, so Equation (8) applies and gives the solution $S(n) = (0.98)^{n-1}(1{,}000{,}000) - 10{,}000[(0.98)^{n-2} + (0.98)^{n-3} + \dots + (0.98) + 1] = (0.98)^{n-1}(1{,}000{,}000) - 10{,}000[1 - (0.98)^{n-1}]/(1 - 0.98)$ from the formula for the sum of a geometric sequence. At the end of 9 years (the beginning of the 10th year), the population is $A(10) = (0.98)^9(1{,}000{,}000) - 10000[1-(0.98)^9]/(1-0.98)$ which equals 750622.

16. The recurrence relation for the total number of infected machimes each day is $T(n) = T(n - 1) + 5T(n - 1) - 6^{n-2} = 6T(n - 1) - 6^{n-2}$ with a base case of $T(1) = 3$. This is a linear, first-order recurrence relation with constant coefficients, so Equation (8) applies; $c = 6$ and $g(n) = -6^{n-2}$. By Equation (8), the solution is

$$
\begin{aligned}
T(n) &= 6^{n-1}(3) + \sum_{i=2}^{n} 6^{n-i}\left(-6^{i-2}\right) \\
&= 6^{n-1}(3) - \sum_{i=2}^{n} 6^{n-2} \\
&= 6^{n-1}(3) - (n-1)6^{n-2} \\
&= 6^{n-2}[6 * 3 - (n-1)]
\end{aligned}
$$

Now setting $6*3 - (n - 1)$ equal to 0 and solving for n, the result is $n = 19$. The virus disappears after 19 days.

17. a. $M(1) = 1$ (one move for one disk)
$M(n) = 2M(n - 1) + 1$ (restack the top $n - 1$ disks on the third peg, requiring $M(n - 1)$ disk moves, move the bottom disk to the target peg, restack the $n - 1$ disks on the target peg, which also requires $M(n - 1)$ disk moves)

b. This is a linear first order recurrence relation with constant coefficients, so Equation (8) applies and gives the solution

$$
\begin{aligned}
M(n) &= 2^{n-1}M(1) + \sum_{i=2}^{n} 2^{n-i}(1) \\
&= 2^{n-1} + 2^{n-2} + 2^{n-3} + \dots + 2^{0}
\end{aligned}
$$

28. The characteristic equation is $t^2 - 10t + 25 = 0$ with repeated root $r = -5$. The solution is

$$F(n) = p(-5)^{n-1} + q(n-1)(-5)^{n-1}$$

where

$$p = -10$$
$$p(-5) + q(-5) = 40$$

so $p = -10$, $q = 2$, and the solution is

$$F(n) = -10(-5)^{n-1} + 2(n-1)(-5)^{n-1}$$

*29. The characteristic equation is $t^2 - 2t + 2 = 0$ with roots $r_1 = 1 + i$, $r_2 = 1 - i$. The solution is

$$A(n) = p(1+i)^{n-1} + q(1-i)^{n-1}$$

where

$$p + q = 8$$
$$p(1+i) + q(1-i) = 8$$

so $p = 4$, $q = 4$, and the solution is

$$A(n) = 4(1+i)^{n-1} + 4(1-i)^{n-1}$$

30. The characteristic equation is $t^2 + 4t + 5 = 0$ with roots $r_1 = -2 + i$, $r_2 = -2 - i$. The solution is

$$S(n) = p(-2+i)^{n-1} + q(-21 - i)^{n-1}$$

where

$$p + q = 4$$
$$p(-2+i) + q(-2-i) = -8$$

so $p = 2$, $q = 2$, and the solution is

$$S(n) = 2(-2+i)^{n-1} + 2(-2-i)^{n-1}$$

*31. The characteristic equation is $t^2 - t - 1 = 0$ with roots $r_1 = \dfrac{1+\sqrt{5}}{2}$, $r_2 = \dfrac{1-\sqrt{5}}{2}$. The solution is

$$F(n) = p\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} + q\left(\frac{1-\sqrt{5}}{2}\right)^{n-1}$$

where

$$p + q = 1$$
$$p\left(\frac{1+\sqrt{5}}{2}\right) + q\left(\frac{1-\sqrt{5}}{2}\right) = 1$$

Solve for $p = 1 - q$, then substitute that into the second equation and solve for q, giving $q = \dfrac{1-\sqrt{5}}{-2\sqrt{5}}$ and therefore $p = 1 - q = \dfrac{1+\sqrt{5}}{2\sqrt{5}}$. The solution is