# Week 4

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## I. (Continued)

- Unconstrained convex problem:

$$\min_{x\in\mathbb{R}^n} f(x)$$

necessary and sufficient condition for convex problems becomes

$$\nabla f(\bar{x}) = 0$$

(replace with subgradients if function is not differentiable)

- $\nabla f(\bar{x}) \in N_C(\bar{x})$. But since feasible set $\mathcal{F} = \mathbb{R}^n$

$\Rightarrow N_C(\bar{x}) = \{0\}$ and $\nabla f(\bar{x}) = 0$

- Equality constrained convex problems: (assume $A$ is $m\times n$, $m\le n$, full-rank)

$$\min_{x\in\mathbb{R}^n} f(x) \text{ s.t. } Ax = b$$

- null-space of $A$, $N(A)$, is defined as the set of vectors $v$ such that $Av = 0$

- Assume that $A\bar{x} = b$, then we can define any other feasible $x$ as $x = \bar{x} + v,\ \exists v \in N(A)$

To characterize $N(A)$ and $N(A)^\perp$, it is convenient to use SVD of $A = \underbrace{U}_{m\times m \text{ unitary}}\underbrace{[D, 0]}_{\text{Diagonal with } D_{ii}>0}\cdot \underbrace{V^T}_{n\times n \text{ unitary}} = \sum_{i=1}^m d_i \underbrace{u_i v_i^T}_{i\text{-th columns}}$

$\Rightarrow N(A) = \left\{\sum_{i=m+1}^n \alpha_i v_i,\ \exists \alpha_i \in \mathbb{R}\right\}$

and $N(A)^\perp = \left\{\sum_{i=1}^m \alpha_i v_i,\ \exists \alpha_i \in \mathbb{R}\right\}$

$$V\cdot\begin{bmatrix}\alpha_1\\ \vdots\\ \alpha_m\\ 0\\ \vdots\\ 0\end{bmatrix} = V\begin{bmatrix}D\\ 0\end{bmatrix}\begin{bmatrix}\frac{\alpha_1}{d_{11}}\\ \vdots\\ \frac{\alpha_m}{d_m}\end{bmatrix} = \underbrace{V\begin{bmatrix}D\\ 0\end{bmatrix}U^T}_{A^T}\cdot\underbrace{U\cdot\begin{bmatrix}\vdots\\ \frac{\alpha_i}{d_{ii}}\\ \vdots\end{bmatrix}}_{-v}$$

$$\boxed{\Rightarrow y \in N(A)^\perp \Leftrightarrow \exists v \in \mathbb{R}^m : y = -A^T\cdot v}$$

- In order for $\bar{x}$ to be optimal, $-\nabla f(\bar{x})$ must belong to the normal cone $N_C(\bar{x})$:

$$-\nabla f(\bar{x}) \in N_C(\bar{x})$$

- The normal cone in this case is defined as

$$N_C(\bar{x}) = \{d : d^T(x - \bar{x}) \leq 0, \forall x \text{ s.t. } Ax = b\}$$

$\Rightarrow -\nabla f(\bar{x})(x - \bar{x}) \leq 0 \ \forall x \text{ s.t. } Ax = b$

$\Updownarrow$

$\nabla f(\bar{x}) v \geq 0 \quad \forall v \text{ s.t. } Av = b \Leftrightarrow \nabla f(\bar{x}) v \geq 0 \quad \forall v \in N(A)$

However, since $v \in N(A) \Leftrightarrow -v \in N(A)$, this condition becomes
$\nabla f(\bar{x}) v = 0 \ \forall v \in N(A)$ or equiv. $\nabla f(\bar{x}) \perp N(A)$

- We conclude that, in order for $\bar{x}$ to be optimal, $\nabla f(\bar{x}) = -A^T \nu$, for some $\nu \in \mathbb{R}^m$, and the optimality conditions become

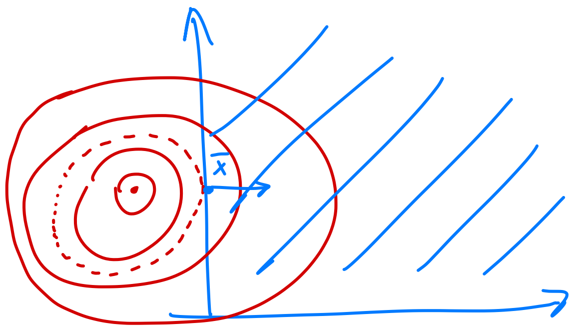
$A\bar{x} = b$ (feasibility)

$\nabla f(\bar{x}) + A^T v = 0, \ \exists v \in \mathbb{R}^m$

- This is the Lagrange multiplier optimality condition, more on this later

- Minimization over the non-negative quadrant

$$\min f(x), \text{ s.t. } x \geq 0$$




global if problem convex

- $\bar{x}$ is a local minimizer if $-\nabla f(\bar{x}) \in N_C(\bar{x})$, defined as

$$N_C(\bar{x}) = \{d : d^T(x - \bar{x}) \leq 0, \forall x \geq 0\}$$

$\Rightarrow \nabla f(\bar{x})(x-\bar{x}) \geq 0 \quad \forall x \geq 0 \iff \sum_i \nabla_i f(\bar{x})(x_i - \bar{x}_i) \geq 0 \quad \forall x \geq 0$

Let $x = [\bar{x}_1 \dots \bar{x}_{i-1}, x_i, \bar{x}_{i+1}, \dots \bar{x}_n]$ ($x$ differs from $\bar{x}$ only in its i-th component)

$\Rightarrow \nabla_i f(\bar{x})(x_i - \bar{x}_i) \geq 0 \quad \forall x_i \geq 0$. In particular, must hold for $x_i = 0$ and $x_i = +\infty$

$x_i = 0$: $\nabla_i f(\bar{x}) x_i \leq 0$ $\Rightarrow$ Two cases: $\begin{cases} x_i = 0 \text{ and } \nabla_i f(\bar{x}) \geq 0 \\ x_i > 0 \text{ and } \nabla_i f(\bar{x}) = 0 \end{cases}$ $\Rightarrow$ If this holds $\forall i$, it follows that $\nabla f(\bar{x})(x - \bar{x}) \geq 0 \quad \forall x \geq 0$

$x_i = \infty$: $\nabla_i f(\bar{x}) \geq 0$

- This leads to the optimality conditions

$$\nabla_i f(\bar{x})\bar{x}_i = 0, \ \forall i$$

$$\nabla f(\bar{x}) \geq 0, \ \bar{x} \geq 0$$

called *complementary slackness conditions*, more on this later

- There are many known forms of convex problems:
  - Linear programs (objective and constraints are linear)
  - Linear fractional programs
  - Quadratic program
  - Geometric program
  - Second order cone program
  - Etc.
- Geometric programming (Boyd Ch. 4.3-4.4): useful when there are products/fractions
  - Monomial function: $f(x) = c\prod_i x_i^{a_i}$, $c > 0$, $a_i \in \mathbb{R}$
  - Posynomial: $f(x) = \sum_k c_k \prod_i x_i^{a_{i,k}}$, $c_k > 0$
  - Geometric program is of the form

$$\min_{x>0} f_0(x)$$
$$\text{s.t. } f_i(x) \leq 1$$
$$h_j(x) = 1$$

  where $f_i$ are posynomials and $h_j$ are monomials
  - Example:

$$\min_{x,y>0} x/y$$
$$\text{s.t. } 2 < x < 3$$
$$x^2 + 3y/z \leq \sqrt{y}$$
$$x/y = z^2$$

- Convert to convex problem by letting $x_i = e^{y_i}$ and take the log

$x = e^{\alpha}$ $y = e^{\beta}$ $z = e^{\gamma}$

$$\begin{aligned} \min \quad & e^{\alpha-\beta} \\ \text{s.t.} \quad & e^{\alpha} < 3 \\ & e^{\alpha} > 2 \\ & e^{2\alpha} + 3e^{\beta-\gamma} \leq e^{\beta/2} \\ & e^{\alpha-\beta} = e^{2\gamma} \end{aligned} \quad \xRightarrow{\log} \quad \begin{aligned} \min \quad & \alpha-\beta \\ \text{s.t.} \quad & \alpha < \ln 3 \\ & \alpha > \ln 2 \\ & \ln\left(e^{2\alpha-\beta/2} + 3e^{\beta/2-\gamma}\right) \leq 0 \quad \text{(convex)} \\ & \alpha - \beta - 2\gamma = 0 \end{aligned}$$

- Linear-fractional programming (Boyd p.151)

$$\min \frac{c^T x + d}{e^T x + f}$$

$$\text{s.t. } e^T x + f > 0$$

$$Gx \leq h$$

$$Ax = b$$

- Can be converted to convex by letting $z = \frac{1}{e^T x + f}$ and $y = zx$, hence $x = y/z$:

$$\min c^T y + dz$$

$$\text{s.t. } z > 0 \qquad \text{(linear program)}$$

$$Gy - hz \leq 0$$

$$Ay - bz = 0$$

- Projection: let $C$ be closed and convex; $x$ is the projection of $x_0$ on $C$ if

$$\|x - x_0\| = \in \{\|y - x_0\|, y \in C\}$$

- Later, when we discuss iterative algorithms for solving convex problems, we will need the projection to the feasible set:

$$\min \|x - x_0\|_2^2$$
$$\text{s.t. } f(x) \leq 0$$
$$Ax = b$$

This is a convex problem.

- Example: projection to a quadrant

$$\min \|x - x_0\|_2^2 \quad \Rightarrow \quad \nabla f(\bar{x}) = 2(x - x_0)$$
$$\text{s.t. } x \geq 0$$

Solution is $x_i = (x_{0,i})^+$

We have seen before:

$\nabla f(\bar{x}) \geq 0 \iff \bar{x} - x_0 \geq 0$

$\bar{x} \geq 0$

and $\nabla f_i(\bar{x}) \cdot \bar{x}_i = 0 \iff (\bar{x}_i - x_{0i})\bar{x}_i = 0$

$\Rightarrow$

If $x_{0i} < 0, \Rightarrow \bar{x}_i = 0$

If $x_{0i} = 0 \Rightarrow \bar{x}_i^2 = 0 \iff \bar{x}_i = 0$

If $x_{0i} > 0 \Rightarrow \bar{x}_i \geq x_{0i} > 0 \Rightarrow \bar{x}_i = x_{0i}$

- Example: projection to polyhedra (Boyd p.390)

$$\min \|x - x_0\|_2^2$$
$$\text{s.t. } Ax \leq b$$

This is a quadratic program.

Special cases: projection on a hyperplane $a^T x = b$, solution is

$$\bar{x} = x_0 + (b - a^T x_0)a/\|a\|_2^2$$

- $\nabla f(\bar{x})^T(x-\bar{x}) \geq 0 \quad \forall x:\ a^T x = b \Rightarrow (\bar{x}-x_0)^T(x-x_0) \quad \forall x:\ a^T x = b$
- Check that above solution satisfies conditions

1) Feasibility: $a^T\bar{x} = a^T x_0 + (b-a^T x_0)a^T a/\|a\|^2 = b$ (ok!)

2) Optimality: $(\bar{x}-x_0)^T(x-x_0) = (b-a^T x_0)\frac{1}{\|a\|^2}(\underbrace{a^T x}_{= b} - a^T x_0) = \frac{(b-a^T x_0)^2}{\|a\|^2} \geq 0$ (ok!)

Other special case: projection on a half-space $a^T x \leq b$, solution is

$$x = x_0 \quad \text{(of course, since } x_0 \in F\text{)}$$

if $a^T x_0 \leq b$ and

$$\bar{x} = x_0 + (b - a^T x_0)a/\|a\|_2^2$$

otherwise

Let's look at the case $x_0 \notin F$ i.e. $a^T x_0 > b$, and check optimal solution

1) Feasibility: $a^T\bar{x} = a^T x_0 + (b-a^T x_0)a^T a/\|a\|^2 = b$ (ok!)

2) Optimality: $\nabla f(\bar{x})^T(x-\bar{x}) \geq 0 \quad \forall x:\ a^T x \leq b$

$\iff (\bar{x}-x_0)^T(x-\bar{x}) \geq 0 \quad \forall x:\ a^T x \leq b$

$\iff \overbrace{(b-\underbrace{a^T x_0}_{> b})}^{<0} \frac{1}{\|a\|^2} \overbrace{(\underbrace{a^T x}_{\leq b} - \underbrace{a^T x_0}_{> b})}^{<0} \geq 0$ (ok!)

- Example: projection to a ball

$$\min \|x - x_0\|_2^2$$
$$\text{s.t. } \|x\|_2 \le r$$

Solution is: if $\underline{\|x_0\| \le r}$, then $x = x_0$; otherwise, $x = rx_0/\|x_0\|$

of course since $x_0 \in F$

Consider the case $x_0 \notin F$ and $\bar{x} = r\dfrac{x_0}{\|x_0\|}$

1) Fesibility. $\|\bar{x}\| = r\dfrac{\|x_0\|}{\|x_0\|} = r$ (OK!)

2) Optimality: $\nabla f(\bar{x})^T(x - \bar{x}) \geq 0 \quad \forall x : \|x\| \le r$

$\iff (\bar{x} - x_0)^T(x - \bar{x}) \geq 0 \quad \forall x : \|x\| \le r$

$\iff \underbrace{\left(\dfrac{r}{\|x_0\|} - 1\right)}_{<0} x_0^T\left(x - \dfrac{r}{\|x_0\|}x_0\right) = -\left(1 - \dfrac{r}{\|x_0\|}\right)\overbrace{\left(x_0^T x - r\|x_0\|\right)}^{\le 0} \geq 0$ (OK!)

$|x_0^T x| \le \sqrt{x_0^T x_0} \cdot \sqrt{x^T x} \le r\|x_0\|$

## II. Formulations of convex optimization problems in applications

- Norm approximation problem

$$\min_x \|y - Ax\|$$

$y$: observations

$x$: parameter to be estimated

Solution for $L_2$ norm: assume $A$ is $m \times n$, $m \geq n$ full rank

$\min_x \|y - Ax\|_2^2 \longrightarrow$ differentiable ($\|y - Ax\|$ is not!)

$\Rightarrow \nabla = 0 \Leftrightarrow A^T(y - Ax) = 0 \Leftrightarrow A^T y = A^T A x \Leftrightarrow x = (A^T A)^{-1} A^T y$

- Power control in D2D networks aided by single base stateion

  - Wireless transmissions require a certain min SINR to be successful

  - many users pairs are communicating with their respective receiver over the same frequency band; far users need a higher transmission power; interference can reduce the SINR

  - Q: how to choose power that achieves the target SINR for all users?

  - Let $p_i$ be the transmission power for user $i$ and $g_{i,j}$ the channel gain between transmitter $i$ and receiver $j$.

  - The goal is to minimize sum-power under minimum SINR constraint for each user.

SINR of user i: $\dfrac{p_i g_{ii}}{\sum_{j \neq i} p_j g_{ji} + N_i}$ must satisfy $\geq \gamma_i$

$$\Rightarrow \quad \min_{p} \sum_i p_i$$

$$\text{s.t.} \quad \frac{p_i g_{ii}}{\sum_{j \neq i} p_j g_{ji} + N_i} \geq \gamma_i \; \forall i \iff -p_i g_{ii} + \gamma_i \left[ \sum_{j \neq i} p_j g_{ji} + N_i \right] \leq 0$$

$\downarrow$

LINEAR PROGRAM

- Reading: G. J. Foschini and Z. Miljanic, "A simple distributed autonomous power control algorithm and its convergence," in IEEE Transactions on Vehicular Technology, vol. 42, no. 4, pp. 641-646, Nov. 1993.

(Boyd p.245)

- Water-filling for fading channels. Channel gain varies over time i.i.d. with distribution $\mathbb{P}(g)$; the transmitter with CSI information may adapt the power $p(g)$ based on the current channel condition

  - What is the optimal power control strategy to maximize the average rate, under power constraint?

Avg power: $\sum_g \mathbb{P}(g) p(g)$

Avg rate: $\sum_g \mathbb{P}(g) \log_2\left(1+\frac{g p(g)}{N}\right)$

$\Longrightarrow$

$$\max_{p} \sum_g \mathbb{P}(g) \log_2\left(1+\frac{g p(g)}{N}\right)$$

$$\text{s.t. } p(g) \geq 0 \quad \forall g$$

$$\sum_g \mathbb{P}(g) p(g) \leq \bar{P}_{avg}$$

  - Solution (more on this later): there exists a number $\lambda > 0$ such that

$$p(g) = \left(\frac{1}{\lambda} - \frac{N}{g}\right)^+$$

$\lambda$ is chosen such that

$$\mathbb{E}[p(g)] = \bar{P}$$

(you can always find such $\lambda$)

1) Feasibility: $\sum_g \mathbb{P}(g)p(g) = \bar{P}$ (by the choice of $\lambda$) and $\left(\frac{1}{\lambda} - \frac{N}{g}\right)^+ \geq 0 \Rightarrow$ feasible

2) Optimality: $\nabla f(\bar{p})^T(p - \bar{p}) \leq 0$ (max problem) $\forall p \in \mathcal{F}$

$$\nabla_g f(\bar{p}) = \frac{d f(\bar{p})}{d p_g} = \mathbb{P}(g) \frac{1}{1+g\frac{P(g)}{N}} \frac{g}{N} = \frac{g}{N+gP(g)} \mathbb{P}(g)$$

$$\Longrightarrow \sum_g \mathbb{P}(g) \frac{g}{N+g\bar{p}(g)} \left(p(g) - \bar{p}(g)\right) \leq 0.$$

$$\Longleftrightarrow \sum_{g: g>\lambda N} \mathbb{P}(g)\lambda\left(p(g) - \bar{p}(g)\right) \leq \lambda \underbrace{\sum_g \mathbb{P}(g)p(g)}_{\leq P_{avg}} - \lambda \underbrace{\sum_g \mathbb{P}(g)\bar{p}(g)}_{= P_{avg}} \leq 0$$

$$+ \sum_{g: g\leq\lambda N} \mathbb{P}(g) \cdot \frac{\overbrace{g}^{\leq \lambda N}}{N} p(g)$$

- Opportunistic scheduling with fading and multi-user (consider 2 users)

  - the channel can be in one of $M$ states; when user $i$ is in state $s$, it receives the payoff $U_{i,s}$ if served; the payoff is zero for the unserved user.

  - Q: how to choose which user to transmit?

  - Choose uniformly at random:

$$\sum_s \mathbb{P}(s)\left[\frac{1}{2} U_{1s} + \frac{1}{2} U_{2s}\right]$$

Drawback: performance is poor (utility)

  - Choose user with best payoff:

$$\sum_s \mathbb{P}(s) \max\left\{U_{1,s}, U_{2,s}\right\}$$

Drawback: fairness

  - Need to consider trade-off between performance and fairness

  - Let $p_{\cancel{i},s} = \mathbb{P}(\cancel{\text{user } i}$ system is in state $s)$; $Q_{i,s}$ be the scheduling policy : $Q_{1,s} + Q_{2s} \leq 1$

$Q_{is}$: fraction of time "$i$" is scheduled in state $s$

Expected payoff of user $i$

$$\sum_s p(s) Q_{i,s} U_{is}$$

System payoff

$$\sum_i \sum_s p(s) Q_{is} U_{is}$$

Different types of fairness:

- Temporal fairness (fraction of time user $i$ is scheduled)

$$\max_{U} \sum_i \sum_s p(s) Q_{is} U_{is}$$

$$\text{s.t.} \quad \sum_s Q_{is} p_s \geq r_i \quad \forall i$$

$$\sum_i Q_{is} \leq 1 \quad \forall s \quad \Rightarrow LP$$

$$Q_{is} \in [0,1] \quad \forall i,s$$

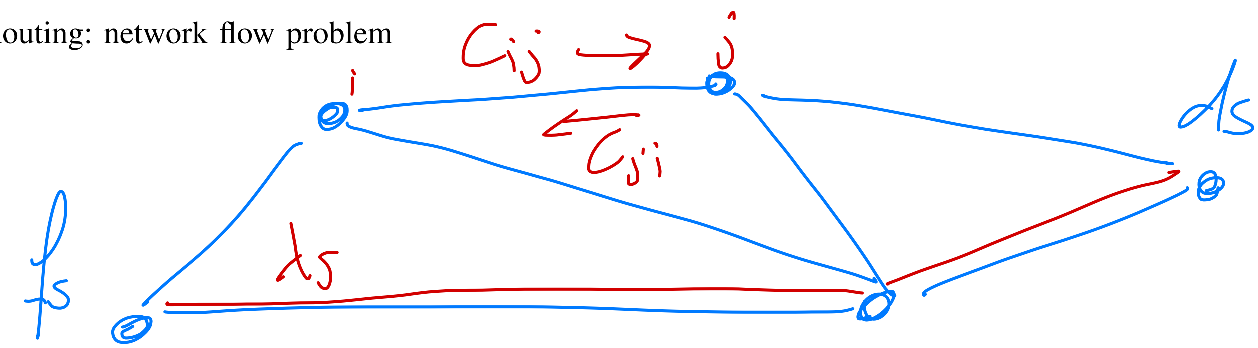
Solution for two users case (more when we study duality): there exists $v$ such that

$$\begin{cases} Q_{1,s} = 1 & U_{1,s} + v > U_{2,s} \\ Q_{2,s} = 1 & U_{1,s} + v < U_{2,s} \\ \text{any} & otherwise \end{cases}$$

- Other types of fairness constraints: utility constraint, fraction of utility constraint, etc.

- Reading: Xin Liu, E. K. P. Chong and N. B. Shroff, "Transmission scheduling for efficient wireless utilization," Proceedings IEEE INFOCOM 2001.

- Routing: network flow problem




- commodity $s$ need to be routed from $f_s$ to $d_s$ with rate $\lambda_s$

- capacity of link $(i, j)$ is $C_{i,j}$

- Q: how to route the flows?

- Node-balance equations: let $r_{i,j}^{(s)}$ be the amount of traffic of flow $s$ routed through $(i, j)$

If node n is not a source or destination of flow s:

$$\sum_i r_{i,n}^{(s)} = \sum_j r_{n,j}^{(s)}, \quad \forall n \notin \{f_s, d_s\}$$ traffic in = traffic out

If $n = f_s$ (n is source of flow s)

$$\sum_i r_{n,i}^{(s)} = \lambda_s$$ all traffic is forwarded

If $n = d_s$ (n is destination)

$$\sum_j r_{j,n}^{(s)} = \lambda_s$$ all traffic is received

- Capacity constraints

$$\sum_s r_{i,j}^{(s)} \le C_{i,j} \quad \forall i,j$$

- Objectives:
- Feasibility

$$\min 0$$

s.t. all above constraints $\Rightarrow$ LP

- Maximize throughput : Given $\lambda_s^{(0)}$ $\forall s$

$$\max \alpha$$
$$\text{s.t. } \lambda_s = \alpha \lambda_s^{(0)} \quad \Rightarrow \text{LP}$$

and

all above constraints

- Min congestion: define congestion measure for each link as $\beta_{i,j}(\sum_s r_{i,j}^{(s)})$

$$\min \sum_{i,j} \beta_{ij}\left(\sum_s r_{ij}^{(s)}\right) \quad \Rightarrow \text{convex if } \beta \text{ is concave}$$

s.t. all above constraints

- Reading: Bertsekas&Tsitsiklis, "Parallel and Distributed computation: numerical methods", Chapter 5

- Rate control and resource allocation
  - $R$: the amount of resource (e.g., bandwidth)
  - $N$ users
  - $x_i$: amount of resource scheduled to user $i$ $\Rightarrow \sum_i x_i \leq R$
  - Q: how to allocate resources among users?

  Utility maximization problem: let $U_i(x_i)$ be the utility to user $i$

$$\max \sum_i U_i(x_i)$$
$$\text{s.t. } \sum_i x_i \leq R$$
$$x_i \geq 0 \ \forall i$$

convex if $U_i$ concave $\forall i$

  Solution (more later): there exists $\lambda$ such that each user maximizes individually

$$\max_i U_i(x_i) - \lambda x_i$$

  $\lambda$ is chosen such that either $\lambda = 0$ or $\sum_i x_i = R$ and $\lambda > 0$.

- Rate allocation of the internet: multiple resources
  - $R_l$ the capacity of link $l$ (independent links)
  - $x_s$ rate allocated to user $s$, with utility $U_s(x_s)$
  - $H_{s,l}$: routing matrix (given)
  - Q: how to allocate the rates to maximize utility?

$$\max \sum_s U_s(x_s)$$
$$\text{s.t. } \sum_s H_{s,l} x_s \leq R_l \ \forall l$$

convex if $U_s$ is concave

- Example: $U_s(x_s) = \omega_s \frac{x_s^{1-\alpha}}{1-\alpha}$

For any optimal solution,

$$U_s'(x_s^*) = \frac{\omega_s}{x_s^{*,\alpha}}$$

and

$$\sum_s \omega_s \frac{x_s - x_s^*}{x_s^{*,\alpha}} \leq 0 \qquad \forall\ x \in \mathcal{F}$$

- Solution: there is a price $\lambda_l$ for each link such that each user maximizes:

$$\max_{x_s} U_s(x_s) - x_s \sum_l \lambda_l H_{s,l}$$

ans also

$$\sum_s H_{s,l} x_s = R_l \text{ if } \lambda_l > 0$$

and

$$\sum_s H_{s,l} x_s < R_l \text{ if } \lambda_l = 0$$

- Reading: J. Mo and J. Walrand, "Fair end-to-end window-based congestion control," in IEEE/ACM Transactions on Networking, vol. 8, no. 5, pp. 556-567, Oct. 2000.

- Multipath congestion control: joint routing and congestion control; we optimize both $H$ and $x$; this is an example of cross-layer control. Formulate into a convex problem:

Let $y_{s\ell} = x_s H_{s\ell}$ : portion of traffic through route $\ell$ of user $s$

$\Rightarrow x_s = \sum_\ell y_{s\ell}$

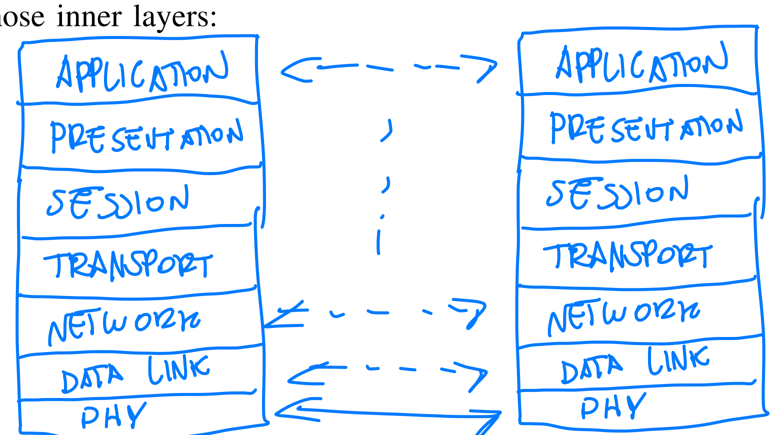
$$\max \sum_s U_s\Big(\sum_\ell y_{s\ell}\Big)$$

$$\text{s.t.} \quad \sum_s y_{s\ell} \le R_\ell$$

$$y_{s\ell} \geq 0 \quad \forall s,\ell$$

- Cross-layer control: in wireless networks, often protocols are classified into layers to provide modularity

  - the higher layer uses the services provided by the lower layers, but it does not need to know the workings of those inner layers:




  - Modularity is easy to understand and to change, but does not provide optimal performance, especially in wireless networks

  - For example, in routing we tend to minimize the number of links, but this may not be optimal in wireless networks since longer links suffer from lower SNR. In this case, we would prefer to capture physical layer characteristics into the design of routing algorithms

- Convex optimization provides a tool to make this design considerations. In an optimization approach, it is not difficult to incorporate controls at multiple layers into a unified optimization problem.

– physical layer: power control, water-filling, rate-power function

– MAC: scheduling

– Network layer: multipath routing, node-balance equations

– Transport layer: utility maximization

- We thus obtain various combinations. Key considerations are: convexity, distibuted/parallel solutions

- One way of putting all together

$$\max \sum_s U_s(x_s)$$ utility/congestion-control

$$\text{s.t.} \quad x_s = \sum_e y_{sj}$$ routing/load-balancing

→ traffic of user s on path j

$$\sum_{sj} H_{sj} x_{sj} \le r_e$$ rate constraint on link e

$$r_e \le \sum_k \alpha_e^k r_e^k$$ scheduling on link e

$$\sum_k \alpha_e^k = 1$$

$$r_e^k \le g_e(\bar{p}_k)$$ power control adaptive coding modulation

rate power function

power allocation

- Reading: Xiaojun Lin, N. B. Shroff and R. Srikant, "A tutorial on cross-layer optimization in wireless networks," in IEEE Journal on Selected Areas in Communications, vol. 24, no. 8, pp. 1452-1463, Aug. 2006.

We may include other constraints, such as node balance

- Special cases: $g_l(\mathbf{p}) = \log_2\left(1 + \frac{p_l}{\sum_{j\neq l} p_j + N_l}\right) \simeq \log_2\left(\frac{p_l}{\sum_{j\neq l} p_j + N_l}\right)$ (high SINR approximation)

  - becomes convex with the change of variables $p_l = e^{\theta_l}$

$$\gamma_e^u \leq \theta_e \log_2(e) - \log_2\left(\sum_{j\neq e} e^{\theta_j} + N\right) \Longrightarrow \text{convex const.}$$

  - Reading: Mung Chiang, "Balancing transport and physical Layers in wireless multihop networks: jointly optimal congestion control and power control," in IEEE Journal on Selected Areas in Communications, vol. 23, no. 1, pp. 104-116, Jan. 2005.

  - Drawback: solution might not lead to high SINR; no scheduling or time-interleaving: if noise is small, might be better the use time-interleaving:

  Example: two users transmit at power P simultaneously

  TOT rate: $2\log_2\left(1+\frac{P}{P+N}\right)$

  On the other hand, by using TDMA:

  TOT rate: $\log_2\left(1+\frac{P}{N}\right) \gg 2\log_2\left(1+\frac{P}{P+N}\right)$ at high SINR!

- Other example: Aloha random access; each link attempts transmission with probability $p_l$

  - Let $N(l)$ be the set of links that interfere with link $l$, then the probability of success is

$$p_e \cdot \prod_{i\in N(e)} (1-p_i)$$

  - Let $r_l$ be the peak capacity of link $l$, then the problem can be cast as:

$$\max \sum_s U(x_s)$$

$$\text{s.t.} \quad \sum_s H_s^e x_s \leq r_e p_e \prod_{i\in N(e)} (1-p_i)$$

- Use a transformation of constraints and the change of variables $x_s = e^{y_s}$ to get

$$\max \sum_s U_s(e^{y_s})$$

$$\text{s.t. } \ln\left(\sum_s H_s^e e^{y_s}\right) \leq \ln r_e + \ln p_e + \sum_i \ln(1-p_i)$$

- Can be made convex by a proper choice of utility function

- Reading: Xin Wang and Koushik Kar. 2005. Cross-layer rate control for end-to-end proportional fairness in wireless networks with random access. In Proceedings of the 6th ACM international symposium on Mobile ad hoc networking and computing (MobiHoc '05). ACM, New York, NY, USA, 157-168. DOI=http://dx.doi.org/10.1145/1062689.1062710