Week 3

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I. Convex optimization problems

- Standard form:

- Domain of the problem:

- Feasible point/set

- Optimal value

- Optimal point (or set of)

- More general form:

Q: why must h_i be affine?

II. Basic properties of convex problems

- Local/global optimum

- what if we try to maximize convex function on a convex set?

- what if we try to minimize convex function on a non-convex set? (local minimum at the boundary might not be global minimum)

III. STANDARD FORM

- We will use the standard from, where f_i are convex and h_i are linear. Many problems, even if they don't appear convex or in standard form at first glance, can be converted into convex problems in standard form with some tricks.
- Change of variables
- Example: $\min_{x,y>0,z>0} x^2 + y$ s.t. $\ln(y+z) \leq x, y^2 + z^2 \leq 1$ (use $y = e^t, z = e^w$)
- Example: $\min_{x,y} \sqrt{x} - \sqrt{y}$ s.t. $x + y \leq 1$ (use $\sqrt{x} = z$,$\sqrt{y} = w$) (use $y = e^t, z = e^w$)
- Transformation of functions
- Example: $\frac{x_1}{1+x_2^2} \leq 1$

- Example: $\min \sqrt{x_1 + x_2}$

- Converting equality constraints into inequality constraints (if you know additional structure of the problem)
- Example: $\min x_1 + x_2$ s.t. $x_1^2 + x_2^2 = 1$

- Implicit constraints can be made explicit
- Example: $\min f(x)$ with $f(x) = x^T x$ for $Ax = b$ and $f(x) = +\infty$ otherwise

- Simplify objective/constraint functions by introducing additional constraints
- Example: $\min f_0(A_0 x + b_0)$ s.t. $f_i(A_i x + b_i) \leq 0$

- Sometimes we can do so with non-linear mappings, e.g. $\min f_0(g(x))$

- Conversely, equality constraints can be absorbed into the function
- Example: $\min f_0(x)$ s.t. $f_i(x) \leq 0, Ax = b$

- Epigraph form

- Slack variables to convert inequality constraints into equality ones

- All of these techniques may help to transform an otherwise non-convex problem into a convex one.

IV. Conditions for optimality

- When is a point x optimal? Roughly speaking, when the function is non-decreasing in any direction pointing towards the interior of the feasible set:

- Directional derivative $f'(x, d)$

- If $f'(x, d) = a^T d$ for all directions d, then the function is said to be differentiable at x, with gradient $\nabla f(x) = a$
- Existence of directional derivatives for convex function (possibly, $= \pm\infty$):

- May not exist for arbitrary functions, e.g. $x \sin(1/x)$ (does not exist in 0)

V. Necessary conditions for optimality

- Necessary condition for optimality: let $\bar{x}$ be a local minimizer of $f(x)$ in $\mathcal{C}$ convex; then, $f'(\bar{x}, x - \bar{x}) \geq 0$, $\forall x \in \mathcal{C}$
  (holds for any local minimum)
- What if the problem is convex? (local min is also global)
- Necessary condition for f differentiable
- To further understand these conditions, define the normal cone $N_C(\bar{x})$ to a convex set C at point $\bar{x}$:

$$N_C(\bar{x}) = \{d : d^T(x - \bar{x}) \leq 0, \forall x \in C\}$$

- With this definition, necessary condition of optimality for f differentiable becomes

$$-\nabla f(\bar{x}) \in N_C(\bar{x})$$

VI. SUFFICIENT CONDITIONS FOR OPTIMALITY

- First order sufficient condition: let f convex and C convex; let $\bar{x} \in C$. Then, if $f'(\bar{x}; x-\bar{x}) \geq 0$ for all $x \in C$, then $\bar{x}$ is a global minimizer of f in C.
- If in addition f is differentiable, then the sufficient condition becomes $-\nabla f(\bar{x}) \in N_C(\bar{x})$
- (in general, this statement does not hold for non-convex functions!)
- Proof: