Lectures 2-3

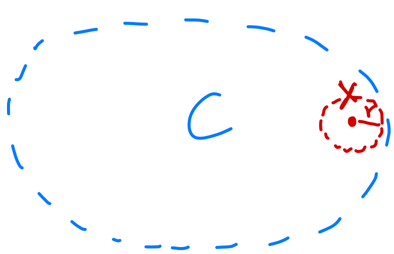
Nicolo Michelusi

I. Definitions, Affine sets and subspaces

- Euclidean norm

$$\|x\|_2 = \sqrt{x^T x}, \quad x \in \mathbb{R}^n$$

- Open set

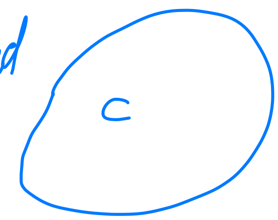


C open if $\forall x \in C$, $\exists r > 0$ s.t. $y \in C$, $\forall y : \|y - x\|_2 \leq r$

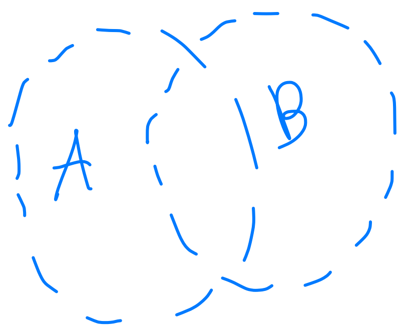
- Examples: $\|x\| < a$ and $\|x\| > a$; $(0, 1)$

- Closed set

C is closed if $\overline{C} \equiv \mathbb{R}^n / C$ is closed



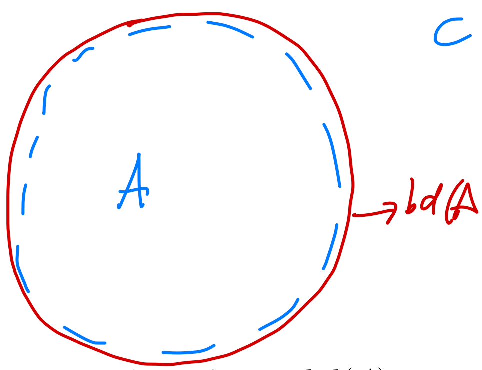
- Intersection and union of open sets



$A \cap B$ is open

$A \cup B$ is open

- Closure of a set, $\mathrm{cl}(A)$



$cl(A)$: smallest closed set containing A

$int(A)$ (interior of A): biggest open set contained in A

- Boundary of a set, $\mathrm{bd}(A)$

$$bd(A) = cl(A) / int(A)$$

- Affine set

- $\{x : Ax = b\} \subseteq \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$

- C affine $\Leftrightarrow$ $\forall\, x_1, x_2 \in C$ and $\forall \theta$ $\Rightarrow$ $\theta x_1 + (1-\theta) x_2 \in C$

- Example: 1-dim affine set

- Example: 2-dim affine set

- Example: $\{x : Ax = b\}$

- Affine hull of a set, $\mathrm{aff}(A)$

$\mathrm{aff}(A)$: smallest affine set containing A

$$\mathrm{aff}(A) \equiv \{ y : \exists\, x_1 \ldots x_N \in A,\ \theta_i \text{ s.t. } \sum_i \theta_i = 1 \text{ with } \sum_i \theta_i x_i = y \}$$

- Subspace

S subspace $\Leftrightarrow$ $\forall\, x_1 \ldots x_N \in S$ and $\theta_1 \ldots \theta_N$

$$\Rightarrow \sum_i \theta_i x_i \in S$$

- Difference between affine set and subspace

Affine set: $C = \{ x : Ax = b \}$ $\Rightarrow$ let x_0 s.t. $Ax_0 = b$

$\Downarrow$

Subspace: $S = \{ x : Ax = 0 \}$ $\qquad$ $S = C - x_0 = \{ x - x_0 : x \in C \}$

- Summary:
  - Any affine set is a subspace + an offset
  - The subspace associated with he affine set C does not depend on the choice of x_0
  - Often a subspace is the set of x such that $Ax = 0$, and the affine set is the set of x such that $Ax = b$.

II. Convex sets

- Convex set

A convex $\iff \forall\, x, y \in A$ and $\theta \in [0,1]$

$$\theta x + \bar{\theta} y \in A$$

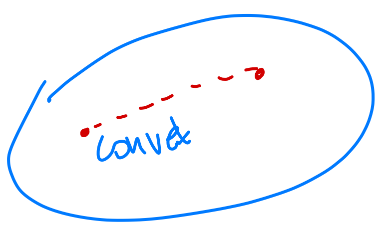
A set is convex if you can go from every point to another point via a straight line contained in the set

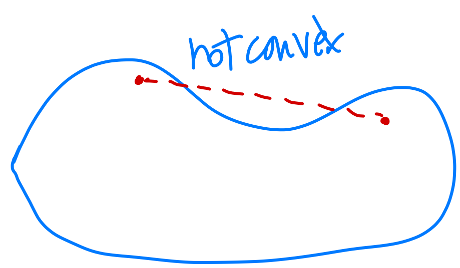
- Difference with affine sets

In affine set, there is no restriction $\theta \in [0,1]$

$$A \text{ affine} \iff \forall x, y \in A, \forall \theta \Rightarrow \theta x + \bar{\theta} y \in A$$

- Examples:





- **Q:** Are affine sets convex? Can a convex set be an open set? Or a closed set?

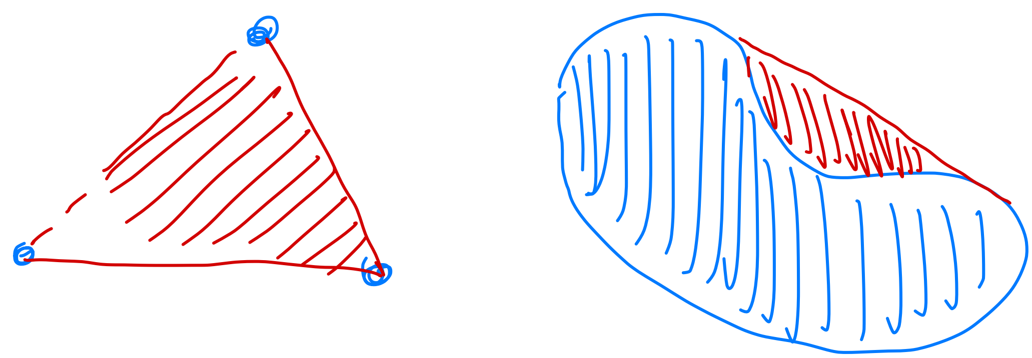
YES YES YES

- Convex hull of a set, $\mathrm{conv}(A)$

conv(A): smallest convex set containing A

$\mathrm{conv}(A) = \{ y : \exists\, x_1 \ldots x_N \in A,\ \theta_i \geq 0 \text{ with } \sum_i \theta_i = 1, \text{ s.t. } y = \sum_i \theta_i x_i \}$

- Examples



- Formal definition (all convex combinations of the points in A):

Proof in two steps: 1) show that $\mathrm{conv}(A)$ is convex; 2) show that, for any convex set B containing A, B must also contain $\mathrm{conv}(A)$

.

- Observation: the beauty of the theory of convexity is that every notion has both an algebraic definition and a geometric interpretation. It is important to be able to go back and forth between the two.

III. Important examples of convex sets

- Expectation of a random variable X

Let $X \in C$ with prob. distribution $p(x), x \in C$

$\Rightarrow \mathbb{E}(x) = \int p(x) \cdot x \, dx \in \mathrm{conv}(C)$

- $\emptyset$, $\mathbb{R}^n$
- Any affine set/subspace

$\{x : Ax = b\}$ is convex

- Lines, rays, line segments

- Hyperplanes $\{x | a^T x = b\}$
- Halfspaces $\{x | a^T x \leq b\}$ or $\{x | a^T x < b\}$

- Balls $\{x|\|x-x_0\| \leq r\}$ (under any norm satisfying the triangular inequality)

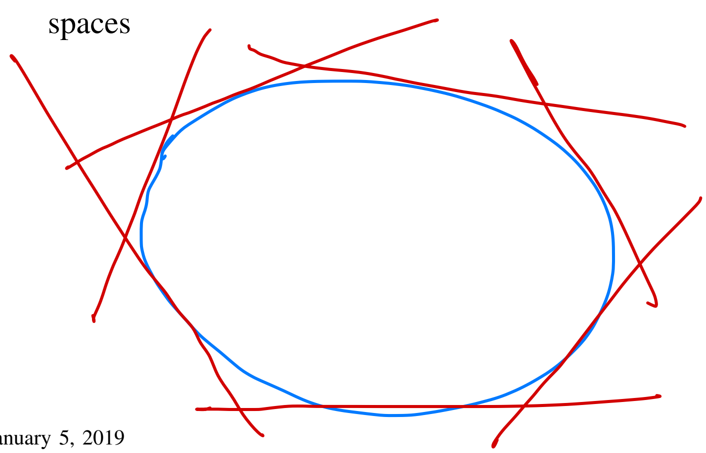
$$B = \{x : \|x - x_0\| \leq r\}$$

- Ellipsoids $\{x|(x-x_0)^T P(x-x_0) \leq 1$ where P is positive semi-definite (convexity preserved under linear transformation)

- Polyhedra (convexity preserved under intersection)

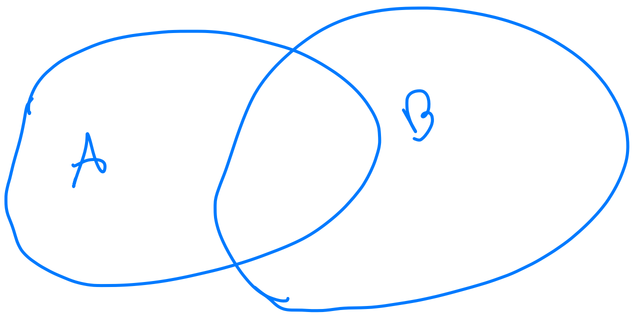
$$\{x : a_i^T x \leq b_i ;\ \forall i ;\ c_j^T x = d_j ;\ \forall j\}$$

- Note: any convex set is the intersection of (possibly, infinitely many) hyperplanes and half-spaces



IV. Operations that preserve convexity

- Intersection



A, B convex

$\Rightarrow A \cap B$ convex

Useful when we have many constraints to be satisfied simultaneously

Q: what about union?

$A \cup B$ may <u>not</u> be convex

- Example: polyhedra

- Example: set of positive (semi-)definite matrices

$$S = \{P : x^T P x \geq 0 \ \forall x ;\ P^T = P\}$$

Let $S_x = \{P : x^T P x \geq 0\}$ is an halfspace

$$\Rightarrow S = \bigcap_{x \in \mathbb{R}^n} S_x \cap \underbrace{\{P : P^T = P\}}_{\text{hyperplane}} \Rightarrow S \text{ is convex}$$

- Example: set $S = \{x \in \mathbb{R}^n | |\sum_{k=1}^n x_k \cos(kt)| \leq 1,\ \forall |t| \leq \delta\}$

Let $S_t = \{x \in \mathbb{R}^n ; -1 \leq \sum_k x_k \cos(kt) \leq 1\}$ (intersection of halfspaces)

$$\Rightarrow S = \bigcap_{t \in [-\delta, \delta]} S_t \Rightarrow \text{convex}$$

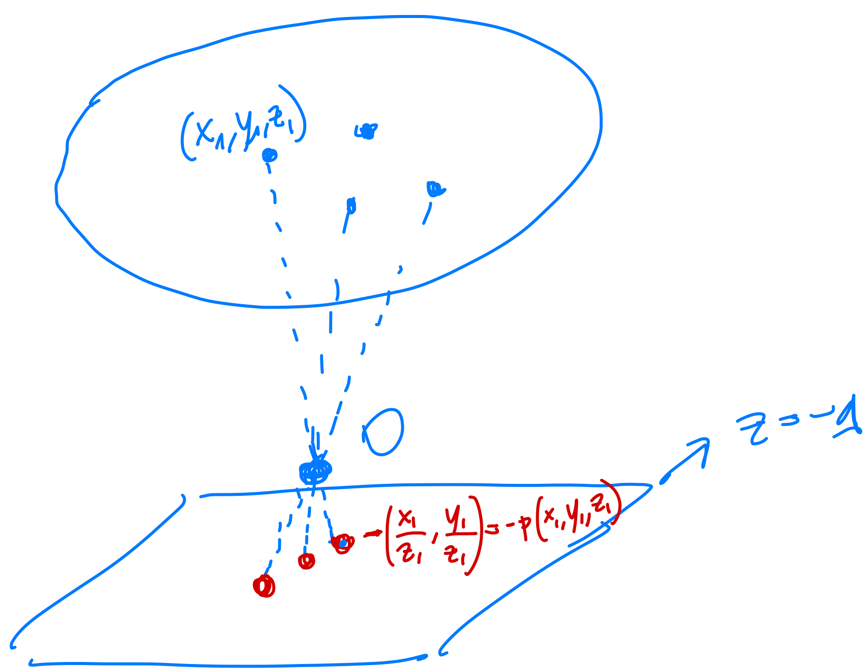
- Affine mappings: $f : \mathbb{R}^n \mapsto \mathbb{R}^m$, $f(x) = Ax + b$
- Image of affine mapping $f(x)=Ax+b$

$\text{im}(f) = \{Ax+b,\ x\in\mathbb{R}^n\}$: lines map to line $\Rightarrow$ convex

- Example: balls and ellipsoids

- Perspective function: $f : \mathbb{R}^{n+1} \mapsto \mathbb{R}^n$, $p(x) = p(z,t) = z/t$ where $x = (z,t), t \in \mathbb{R}^{++}$.
- Image of perspective function

  Example: pinhole camera



- Inverse is also true:

- Example: linear functional $f(x) = \frac{Ax+b}{c^T x+d}$, $c^T x + d > 0$

affine transformation and perspective

- Summary:
  - Key examples of convex sets: affine sets, subspaces, balls, ellipsoids, hyperplanes, half-spaces, polyhedra
  - Operations preserving convexity: intersections, affine mappings, perspectives (and inverses)