Lectures 2-3

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I. DEFINITIONS, AFFINE SETS AND SUBSPACES

• Euclidean norm

• Open set

• Examples: $\|x\| < a \text{ and } \|x\| > a \text{; } (0,1)$

 Closed set
• Closed se

• Intersection and union of open sets

• Closure of a set, cl(A)

• Boundary of a set, bd(A)

• Affine set

• Example: 1-dim affine set

• Example: 2-dim affine set

• Example: $\{x : Ax = b\}$

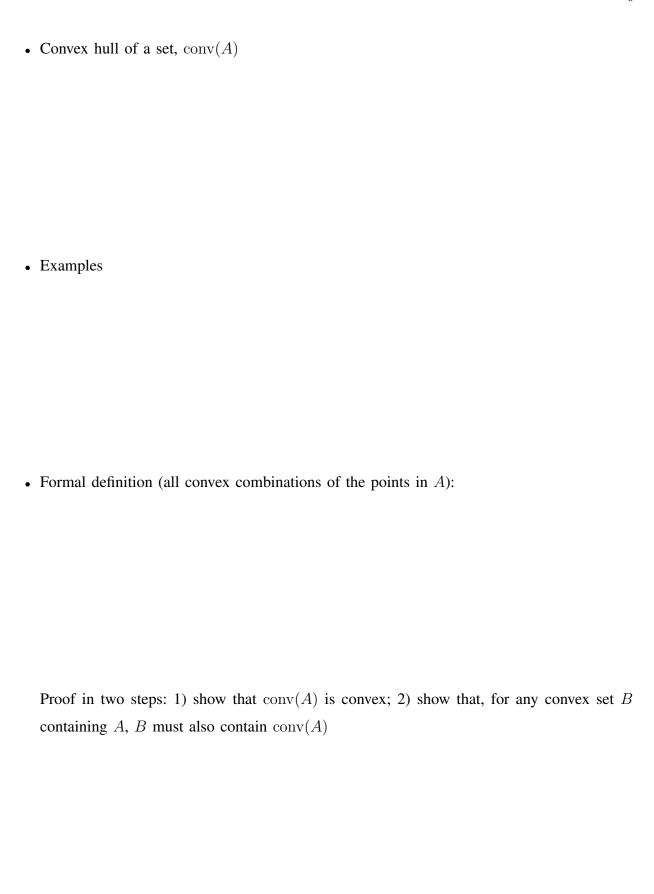
• Affine hull of a set, $\operatorname{aff}(A)$		
• Subspace		
Difference between affine set and subspace	e	

• Summary:

- Any affine set is a subspace + an offset
- The subspace associated with he affine set C does not depend on the choice of x_0
- Often a subspace is the set of x such that Ax=0, and the affine set is the set of x such that Ax=b.

II. CONVEX SETS

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•	Convex set							
	A set is convex if you can go from every point to another point via a straight line contained							
	in the set							
•	Difference with affine sets							
	Examples:							
•	Examples.							
•	Q: Are affine sets convex? Can a convex set be an open set? Or a closed set?							
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• Observation: the beauty of the theory of convexity is that every notion has both an algebraic definition and a geometric interpretation. It is important to be able to go back and forth between the two.

III. IMPORTANT EXAMPLES OF CONVEX SETS

• Expectation of a random variable X

- \emptyset , \mathbb{R}^n
- Any affine set/subspace

• Lines, rays, line segments

- Hyperplanes $\{x|a^Tx=b\}$
- Halfspaces $\{x|a^Tx \leq b\}$ or $\{x|a^Tx < b\}$

• Balls $\{x|\|x-x_0\| \le r$ (under any norm satisfying the triangular inequality)

• Ellipsoids $\{x|(x-x_0)^TP(x-x_0)\leq 1 \text{ where } P \text{ is positive semi-definite (convexity preserved under linear transformation)}$

• Polyhedra (convexity preserved under intersection)

• Note: any convex set is the intersection of (possibly, infinitely many) hyperplanes and halfspaces

IV. OPERATIONS THAT PRESERVE CONVEXITY

• Intersection

Useful when we have many constraints to be satisfied simultaneously

Q: what about union?

• Example: polyhedra

• Example: set of positive (semi-)definite matrices

• Example: set $S = \{x \in \mathbb{R}^n | |\sum_{k=1}^n x_k \cos(kt)| \le 1, \ \forall |t| \le \delta \}$

- Affine mappings: $f: \mathbb{R}^n \mapsto \mathbb{R}^m$, f(x) = Ax + b
- Image of affine mapping

• Example: balls and ellipsoids

- Perspective function: $f: \mathbb{R}^{n+1} \mapsto \mathbb{R}^n$, p(x) = p(z,t) = z/t where $x = (z,t), t \in \mathbb{R}^{++}$.
- Image of perspective function

Example: pinhole camera

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• Example: linear functional $f(x) = \frac{Ax+b}{c^Tx+d}$

• Summary:

- Key examples of convex sets: affine sets, subspaces, balls, ellipsoids, hyperplanes, halfspaces, polyhedra
- Operations preserving convexity: intersections, affine mappings, perspectives (and inverses)