Lectures 2-3

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I. Definitions, Affine sets and subspaces

- Euclidean norm

- Open set

- Examples: $\|x\| < a$ and $\|x\| > a$; $(0, 1)$

- Closed set

- Intersection and union of open sets

- Closure of a set, $\mathrm{cl}(A)$

- Boundary of a set, $\mathrm{bd}(A)$

- Affine set

- Example: 1-dim affine set

- Example: 2-dim affine set

- Example: $\{x : Ax = b\}$

- Affine hull of a set, $\mathrm{aff}(A)$

- Subspace

- Difference between affine set and subspace

- Summary:
  - Any affine set is a subspace + an offset
  - The subspace associated with he affine set C does not depend on the choice of x_0
  - Often a subspace is the set of x such that $Ax = 0$, and the affine set is the set of x such that $Ax = b$.

II. Convex sets

- Convex set

  A set is convex if you can go from every point to another point via a straight line contained in the set
- Difference with affine sets
- Examples:
- **Q:** Are affine sets convex? Can a convex set be an open set? Or a closed set?

- Convex hull of a set, $\mathrm{conv}(A)$

- Examples

- Formal definition (all convex combinations of the points in A):

Proof in two steps: 1) show that $\mathrm{conv}(A)$ is convex; 2) show that, for any convex set B containing A, B must also contain $\mathrm{conv}(A)$

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- Observation: the beauty of the theory of convexity is that every notion has both an algebraic definition and a geometric interpretation. It is important to be able to go back and forth between the two.

III. Important examples of convex sets

- Expectation of a random variable X

- $\emptyset$, $\mathbb{R}^n$
- Any affine set/subspace

- Lines, rays, line segments

- Hyperplanes $\{x|a^T x = b\}$
- Halfspaces $\{x|a^T x \leq b\}$ or $\{x|a^T x < b\}$

- Balls $\{x|\|x - x_0\| \leq r$ (under any norm satisfying the triangular inequality)

- Ellipsoids $\{x|(x-x_0)^T P(x-x_0) \leq 1$ where P is positive semi-definite (convexity preserved under linear transformation)

- Polyhedra (convexity preserved under intersection)

- Note: any convex set is the intersection of (possibly, infinitely many) hyperplanes and half-spaces

IV. Operations that preserve convexity

- Intersection

  Useful when we have many constraints to be satisfied simultaneously

  Q: what about union?

- Example: polyhedra

- Example: set of positive (semi-)definite matrices

- Example: set $S = \{x \in \mathbb{R}^n || \sum_{k=1}^{n} x_k \cos(kt)| \leq 1, \ \forall |t| \leq \delta\}$

- Affine mappings: $f : \mathbb{R}^n \mapsto \mathbb{R}^m,\ f(x) = Ax + b$
- Image of affine mapping

- Example: balls and ellipsoids

- Perspective function: $f : \mathbb{R}^{n+1} \mapsto \mathbb{R}^n,\ p(x) = p(z, t) = z/t$ where $x = (z, t), t \in \mathbb{R}^{++}$.
- Image of perspective function

  Example: pinhole camera

- Inverse is also true:

- Example: linear functional $f(x) = \frac{Ax+b}{c^T x+d}$

- Summary:
  - Key examples of convex sets: affine sets, subspaces, balls, ellipsoids, hyperplanes, half-spaces, polyhedra
  - Operations preserving convexity: intersections, affine mappings, perspectives (and inverses)