ECE59500NL Lecture 27: Automatic Differentiation—II

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The Essence of Forward-Mode AD

Taylor expansion:

$$f(c+\varepsilon) = \frac{f(c)}{0!} + \frac{f'(c)}{1!}\varepsilon + \frac{f''(c)}{2!}\varepsilon^2 + \cdots + \frac{f^{(i)}(c)}{i!}\varepsilon^i + \cdots$$

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- evaluate f

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(Analogous to complex numbers $a+bi$ represented as $\langle a, b\rangle$.)

Complex Numbers

$\mathrm{i}^2 = -1$

$$(a + b\mathrm{i}) + (c + d\mathrm{i}) = (a + c) + (b + d)\mathrm{i}$$
$$(a + b\mathrm{i})(c + d\mathrm{i}) = ac + (ad + bc)\mathrm{i} + bd\mathrm{i}^2 = (ac - bd) + (ad + bc)\mathrm{i}$$

Dual Numbers

$\varepsilon^2 = 0$, but $\varepsilon \neq 0$

$$\begin{aligned}(a + b\varepsilon) + (c + d\varepsilon) &= (a + c) + (b + d)\varepsilon\\(a + b\varepsilon)(c + d\varepsilon) &= ac + (ad + bc)\varepsilon + bd\varepsilon^2 = ac + (ad + bc)\varepsilon\end{aligned}$$

Arithmetic on Truncated Power Series (i.e. Dual Numbers)

$$(x_0 + x_1\varepsilon + \mathcal{O}(\varepsilon^2)) + (y_0 + y_1\varepsilon + \mathcal{O}(\varepsilon^2)) = (x_0 + y_0) + (x_1 + y_1)\varepsilon + \mathcal{O}(\varepsilon^2)$$

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$$\begin{aligned}(x_0 + x_1\varepsilon + \mathcal{O}(\varepsilon^2)) &\times (y_0 + y_1\varepsilon + \mathcal{O}(\varepsilon^2)) \\ &= (x_0 \times y_0) + (x_0 \times y_1 + x_1 \times y_0)\varepsilon + \mathcal{O}(\varepsilon^2)\end{aligned}$$

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$$u\ (x_0 + x_1\varepsilon + \mathcal{O}(\varepsilon^2)) = (u\ x_0) + (x_1 \times (u'\ x_0))\varepsilon + \mathcal{O}(\varepsilon^2)$$

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$$b\ ((x_0 + x_1\varepsilon + \mathcal{O}(\varepsilon^2)), (y_0 + y_1\varepsilon + \mathcal{O}(\varepsilon^2)))$$
$$= (b\ (x_0, y_0)) + (x_1 \times (b^{(1,0)}\ (x_0, y_0)) + y_1 \times (b^{(0,1)}\ (x_0, y_0)))\varepsilon + \mathcal{O}(\varepsilon^2)$$